COMPACTNESS IN L^1 OF A VECTOR MEASURE

J.M. CALABUIG, S. LAJARA, J. RODRÍGUEZ, AND E.A. SÁNCHEZ-PÉREZ

Abstract. We study compactness and related topological properties in the space $L^1(m)$ of a Banach space valued measure m when the natural topologies associated to the convergence of the vector valued integrals are considered. The resulting topological spaces are shown to be angelic and the relationship of compactness and equi-integrability is explored. A natural norming subset of the dual unit ball of $L^1(m)$ appears in our discussion and we study when it is a boundary. The (almost) complete continuity of the integration operator is analyzed in relation with the positive Schur property of $L^1(m)$. The strong weakly compact generation of $L^1(m)$ is discussed as well.

1. Introduction

In recent years, a remarkable effort has been made in order to improve the knowledge of the topological properties of the Banach lattices $L^1(m)$ of integrable functions with respect to Banach space valued measures m. One of the main topological components of these spaces is the so called τ_m topology, that provides the information regarding the norm convergence of the integrals. The so called $\sigma(L^1(m),\Gamma)$ topology is weaker than the weak topology and is also relevant for the analysis of the spaces $L^1(m)$: it is the topology of the weak convergence of the integrals. Actually, in these spaces the most interesting summability properties involve in a certain sense the convergence of the integrals, and this was in fact the topic that motivated the original study of integration with respect to vector measures. The aim of this paper is to prove some fundamental facts regarding compact sets for these topologies, that can clarify the general theory, and also show some applications in Banach lattice theory and operator theory. It must be noted here that the spaces $L^1(m)$ represent all the order continuous Banach lattices with weak unit.

The structure of the paper is the following. After the preliminary Section 1, we start in Section 2 the analysis of the topological properties of the locally convex spaces $(L^1(m),\tau_m)$ and $(L^1(m),\sigma(L^1(m),\Gamma))$, showing that

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these spaces are angelic (Proposition 2.2). Technically, these results will allow us to work with the sequential characterization of compactness for these topologies.

In Section 3 we approach the main question regarding compact sets in $L^1(m)$. In particular, Theorem 3.7 gives a complete characterization of the relatively τ_m-compact sets for vector measures of relatively norm compact range as the sets that are bounded and equi-integrable. Proposition 3.5 states that τ_m-compactness of $B_{L^\infty(m)}$ when considered as a subset of $L^1(m)$ is equivalent to relative norm compactness of the range of m. We finish Section 3 with a characterization of $\sigma(L^1(m), \Gamma)$-precompact sets (Theorem 3.13).

In Section 4 we analyze when the set Γ of all functionals on $L^1(m)$ with an integral representation given by $f \rightsquigarrow \int fh\, d\langle m, x^*\rangle$, $x^* \in B_{X^*}$ and $h \in B_{L^\infty(m)}$, defines a boundary (we write X to denote the Banach space in which m takes values). The property of m having relatively norm compact range appears again and it is shown that, in this case, Γ is a boundary. Far from being a technical matter for specialists, this result has some nice consequences on the structure of $L^1(m)$. In Theorem 4.3 we prove that, for vector measures of relatively norm compact range, the extreme points of the dual unit ball are included in Γ.

Section 5 will present a detailed discussion of when $L^1(m)$ is strongly weakly compactly generated (shortly SWCG), thus showing some improvements of the known results of weak generation of this space. The positive Schur property (shortly PSP) of $L^1(m)$ implies that this space is SWCG. By the way, the following new characterization of completely continuous integration operators $I_m : L^1(m) \to X$ is given: I_m is completely continuous if and only if $L^1(m)$ has the PSP and $m(\Sigma)$ is relatively norm compact (Theorem 5.8). We also prove that $L^1(m)$ has the PSP if and only if I_m is almost Dunford-Pettis (Theorem 5.12).

Some papers that are closely connected with our results have appeared recently. The natural topologies associated to the convergence of the integrals studied here have been analyzed for the case of $L^p(m)$, $1 \leq p < \infty$, in [37, 38], where some applications on factorization of homogeneous maps are shown. As a consequence of the analysis of the τ_m-compactness of the unit ball in the spaces $L^p(m)$, a generalized Dvoretsky-Rogers type theorem is proved in [39]. Also related to compactness in $L^1(m)$, the properties of the integration operators fixing a copy of c_0 have been intensively studied in [35].

Notation. Our topological spaces are Hausdorff and our Banach spaces are real. By an "operator" between Banach spaces we mean a linear continuous mapping. By a "subspace" of a Banach space we mean a closed linear subspace. Given a Banach space Y, its norm is denoted by $\|\cdot\|_Y$ (or simply $\|\cdot\|$) if needed explicitly. We write B_Y to denote the closed unit ball of Y. The convex (resp. absolutely convex) hull of a set C is denoted by $\mathrm{co}(C)$ (resp. $\mathrm{aco}(C)$), and its closure by $\overline{\mathrm{co}}(C)$ (resp. $\overline{\mathrm{aco}}(C)$). The topological dual of Y is denoted by Y^* and the evaluation of $y^* \in Y^*$ at $y \in Y$ is denoted by either $y^*(y)$, $\langle y, y^*\rangle$ or $\langle y^*, y\rangle$. A set $B \subseteq B_{Y^*}$ is said to be *norming* if for every $y \in Y$ we have $\|y\| = \sup_{y^* \in B} |y^*(y)|$. In this case, the topology on Y of pointwise convergence on B, denoted by $\sigma(Y, B)$, is locally convex, Hausdorff and weaker than the weak topology of Y.

Spaces of integrable functions with respect to a vector measure. Throughout the paper, we will assume that X is a Banach space, (Ω, Σ) is a measurable space and $m : \Sigma \to X$ is a countably additive vector measure. The characteristic function of a set $A \in \Sigma$ is denoted by 1_A. By a "scalar measure" we mean a real-valued countably additive measure. For any $x^* \in X^*$ we write $\langle m, x^*\rangle$ to denote the scalar measure given by $\langle m, x^*\rangle(A) := \langle m(A), x^*\rangle$, $A \in \Sigma$. A Rybakov control measure of m is a scalar measure of the form $\mu = |\langle m, x_0^*\rangle|$ (for some $x_0^* \in B_{X^*}$) such that m is μ-absolutely continuous, i.e. $m(A) = 0$ whenever $\mu(A) = 0$. Throughout the paper the symbol μ will denote such a measure (see e.g. [14, p. 268, Theorem 2] for a proof of its existence).

A measurable function $f : \Omega \to \mathbb{R}$ is said to be *m-integrable* if it is integrable with respect to all the scalar measures of the form $|\langle m, x^*\rangle|$ and, for each $A \in \Sigma$, there exists an element $\int_A f\, dm \in X$ such that $\langle \int_A f\, dm, x^*\rangle = \int_A f\, d\langle m, x^*\rangle$ for every $x^* \in X^*$. The space $L^1(m)$ is defined as the Banach lattice of all (μ-equivalence classes of) m-integrable functions when the μ-a.e. order and the norm

$$\|f\|_{L^1(m)} := \sup_{x^* \in B_{X^*}} \int |f| d|\langle m, x^*\rangle|, \quad f \in L^1(m),$$

are considered. $L^1(m)$ is an order continuous Banach function space over μ with weak unit. We will write $I_m : L^1(m) \to X$ for the integration operator, that is, the operator given by $I_m(f) := \int_\Omega f dm$ for all $f \in L^1(m)$. It is well-known that $L^1(m)^*$ can be identified with the Köthe dual of $L^1(m)$, defined by

$$L^1(m)^\times := \{h \in L^1(\mu) : hf \in L^1(\mu) \text{ for every } f \in L^1(m)\}.$$

After such identification, the duality between $L^1(m)^*$ and $L^1(m)$ is given by the formula $\langle h, f\rangle = \int_\Omega hf\,d\mu$.

The topologies $\sigma(L^1(m),\Gamma)$ and τ_m. Given $h \in L^\infty(m)$, let $Q_h : L^1(m) \to X$ be the operator defined by

$$Q_h(f) := I_m(fh) = \int_\Omega fh\,dm, \quad f \in L^1(m),$$

and, for any $x^* \in X^*$, consider the functional $\gamma_{h,x^*} := x^* \circ Q_h \in L^1(m)^*$, i.e.

$$\langle \gamma_{h,x^*}, f\rangle = \int_\Omega fh\,d\langle m, x^*\rangle, \quad f \in L^1(m).$$

For every $f \in L^1(m)$ we have

$$\|f\|_{L^1(m)} = \sup_{h \in B_{L^\infty(m)}} \|Q_h(f)\|_X$$

(see e.g. [34, Lemma 3.11]), and so the set

$$\Gamma := \{\gamma_{h,x^*} : \, h \in B_{L^\infty(m)}, \, x^* \in B_{X^*}\} \subseteq B_{L^1(m)}$$

is norming. Note that a net (f_α) in $L^1(m)$ is $\sigma(L^1(m),\Gamma)$-convergent to $f \in L^1(m)$ if and only if for every $h \in L^\infty(m)$ we have

$$\int_\Omega f_\alpha h\,dm \to \int_\Omega fh\,dm \quad \text{weakly.}$$

The family of seminorms $\{\|Q_h(\cdot)\|_X : h \in L^\infty(m)\}$ induces another locally convex Hausdorff topology on $L^1(m)$ which we denote by τ_m. That is, a net (f_α) in $L^1(m)$ is τ_m-convergent to $f \in L^1(m)$ if and only if for every $h \in L^\infty(m)$ we have

$$\int_\Omega f_\alpha h\,dm \to \int_\Omega fh\,dm \quad \text{in norm.}$$

Observe that τ_m is weaker than the norm topology and stronger than $\sigma(L^1(m),\Gamma)$. Bearing in mind the density of simple functions in $L^\infty(m)$, it is clear that a bounded net (f_α) in $L^1(m)$ converges to $f \in L^1(m)$ with respect to $\sigma(L^1(m),\Gamma)$ (resp. τ_m) if and only if for every $A \in \Sigma$ we have

$$\int_A f_\alpha\,dm \to \int_A f\,dm \quad \text{weakly (resp. in norm).}$$

2. Angelicity of τ_m and $\sigma(L^1(m),\Gamma)$

The natural topologies τ_m and $\sigma(L^1(m),\Gamma)$ do not coincide in general with the usual ones –the norm and the weak topologies–, but they share some properties with them. In this section we analyze the sequential characterization of compactness in the spaces $(L^1(m),\tau_m)$ and $(L^1(m),\sigma(L^1(m),\Gamma))$.

Let us start by recalling some topological notions. Let T be a topological space. A set $A \subseteq T$ is said to be

(i) (relatively) *countably compact* if every sequence in A has a cluster point in A (resp. in T);
(ii) (relatively) *sequentially compact* if every sequence in A has a subsequence converging to a point in A (resp. in T).

Following Fremlin's terminology (see [19, 3.3]), T is said to be *angelic* if every relatively countably compact set $A \subseteq T$ satisfies the following properties:

- A is relatively compact;
- for every $x \in \overline{A}$ there is a sequence in A converging to x.

If T is angelic, then for any set $A \subseteq T$ the following equivalences hold:

$$\text{compact} \Leftrightarrow \text{countably compact} \Leftrightarrow \text{sequentially compact}$$

and the same happens for the corresponding "relative" properties, see [19, 3.3]. Of course, all metric spaces are angelic. Beyond the metrizable case, all Banach spaces equipped with their weak topology are angelic, see [19, 3.10].

The aim of this section is to prove that $L^1(m)$ is angelic when endowed with the topologies $\sigma(L^1(m), \Gamma)$ and τ_m. Up to this moment, the main argument for the use of sequential characterizations of compactness in $L^p(m)$ spaces, $1 \leq p < \infty$, has been the assumption of metrizability of the spaces involved (see for instance [38, Corollary 8]). The techniques explained here can also be extended to the general case of $L^p(m)$ spaces without the metrizability requirement. This could be relevant also for applications; for example, in [39], τ_m-compactness and sequential τ_m-compactness are treated as different properties, which seems not to be necessary.

Lemma 2.1. *Let $x^* \in X^*$. Then the identity operator $L^1(m) \to L^1(\langle m, x^* \rangle)$ satisfies the following properties:*

(i) *it is $\sigma(L^1(m), \Gamma)$-weak continuous on bounded sets;*
(ii) *it is $\sigma(L^1(m), \Gamma)$-weak continuous on $L^1(m)$ whenever $|\langle m, x^* \rangle|$ is a Rybakov control measure of m.*

Proof. (i). Let (f_α) be a bounded net in $L^1(m)$ which converges to $f \in L^1(m)$ with respect to $\sigma(L^1(m), \Gamma)$. Then we have

$$\lim_\alpha \int_A f_\alpha \, d\langle m, x^* \rangle = \int_A f \, d\langle m, x^* \rangle \quad \text{for every } A \in \Sigma. \tag{2.1}$$

Since (f_α) is bounded in $L^1(\langle m, x^* \rangle)$, (2.1) is equivalent to saying that (f_α) is weakly convergent to f in $L^1(\langle m, x^* \rangle)$. This proves the first statement.

(ii). Repeat the argument of (i) without the assumption of boundedness on (f_α) and replace (2.1) by

$$\lim_\alpha \int_\Omega f_\alpha h \, d\langle m, x^*\rangle = \int_\Omega f h \, d\langle m, x^*\rangle \quad \text{for every } h \in L^\infty(m). \tag{2.2}$$

Since $|\langle m, x^*\rangle|$ is a Rybakov control measure of m, condition (2.2) is equivalent to saying that (f_α) is weakly convergent to f in $L^1(\langle m, x^*\rangle)$. □

Proposition 2.2. *$(L^1(m), \sigma(L^1(m), \Gamma))$ and $(L^1(m), \tau_m)$ are angelic.*

Proof. Since the identity operator $i : L^1(m) \to L^1(\mu)$ is one-to-one and $\sigma(L^1(m), \Gamma)$-weak continuous (Lemma 2.1(ii)) and $L^1(\mu)$ is angelic when equipped with its weak topology, we can apply the so called "angelic lemma" [19, 3.3.(2)] to conclude that $(L^1(m), \sigma(L^1(m), \Gamma))$ is angelic as well. Finally, since $\sigma(L^1(m), \Gamma)$ is weaker than τ_m, another appeal to [19, 3.3.(2)] ensures that $(L^1(m), \tau_m)$ is angelic. □

We state the following straightforward corollary for future reference.

Corollary 2.3. *Let $C \subseteq L^1(m)$. The following statements are equivalent:*

(i) *C is (relatively) τ_m-countably compact.*
(ii) *C is (relatively) τ_m-sequentially compact.*
(iii) *C is (relatively) τ_m-compact.*

A compact topological space is said to be *Eberlein* (resp. *uniform Eberlein*) if it is homeomorphic to a weakly compact subset of a Banach (resp. Hilbert) space. For instance, any compact metric space is uniform Eberlein. A result of Argyros and Farmaki [3] (cf. [20, Corollary 6.47]) states that every weakly compact subset of the L^1 space of a scalar measure is uniform Eberlein. We next extend that result to the setting of L^1 spaces of vector measures.

Proposition 2.4. *Every $\sigma(L^1(m), \Gamma)$-compact subset of $L^1(m)$ is uniform Eberlein.*

Proof. Let K be a $\sigma(L^1(m), \Gamma)$-compact subset of $L^1(m)$. Since the identity operator $i : L^1(m) \to L^1(\mu)$ is $\sigma(L^1(m), \Gamma)$-weak continuous (Lemma 2.1(ii)) and one-to-one, its restriction to K is a $\sigma(L^1(m), \Gamma)$-weak homeomorphism between K and $i(K)$. Since every weakly compact subset of $L^1(\mu)$ is uniform Eberlein (by the aforementioned result in [3]), $i(K)$ is uniform Eberlein and so is K. □

3. τ_m-COMPACTNESS AND $\sigma(L^1(m),\Gamma)$-PRECOMPACTNESS

3.1. τ_m-compactness and equi-integrability. A set $C \subseteq L^1(m)$ is called *equi-integrable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|f1_A\|_{L^1(m)} \le \varepsilon$ for every $A \in \Sigma$ with $\mu(A) \le \delta$ and every $f \in C$. The classical Dunford-Pettis criterion states that a subset of the L^1 space of a scalar measure is relatively weakly compact if and only if it is bounded and equi-integrable (see e.g. [13, p. 93]). In general:

- Every bounded and equi-integrable subset of $L^1(m)$ is relatively weakly compact, but the converse might fail.
- Every relatively norm compact subset of $L^1(m)$ is equi-integrable, and the converse holds true for bounded sets whenever m is purely atomic.
- A subset of $L^1(m)$ is bounded and equi-integrable if and only if it is *L-weakly compact*, i.e. every disjoint sequence in its solid hull is norm convergent to 0.

See for instance [34, Lemma 2.37] and [26, §3.6]. In this subsection we discuss the link between equi-integrability and τ_m-compactness.

The set $B_{L^\infty(m)}$ is equi-integrable and weakly compact in $L^1(m)$. In particular, $B_{L^\infty(m)}$ is $\sigma(L^1(m),\Gamma)$-closed and so it is τ_m-closed as well. This set plays a basic role in the approximation of equi-integrable sets, as the next lemma shows. The equivalence between (i) and (ii) is well-known (see e.g. [34, Lemma 2.37]).

Lemma 3.1. *Let $C \subseteq L^1(m)$. The following statements are equivalent:*

(i) *C is bounded and equi-integrable.*

(ii) *For every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that*

$$C \subseteq nB_{L^\infty(m)} + \varepsilon B_{L^1(m)} \quad \textit{in } L^1(m).$$

(iii) *For every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that for every $x^* \in B_{X^*}$ we have*

$$C \subseteq nB_{L^\infty(m)} + \varepsilon B_{L^1(\langle m,x^*\rangle)} \quad \textit{in } L^1(\langle m,x^*\rangle).$$

Proof. (i)⇒(ii). Fix $\varepsilon > 0$ and choose $\delta > 0$ such that

$$\|f1_A\|_{L^1(m)} \le \varepsilon \quad \text{for every } A \in \Sigma \text{ with } \mu(A) \le \delta. \tag{3.1}$$

Since C is bounded, we can find $n \in \mathbb{N}$ such that

$$\sup_{f\in C} \|f\|_{L^1(m)} \le \delta n. \tag{3.2}$$

We claim that $C \subseteq nB_{L^\infty(m)} + \varepsilon B_{L^1(m)}$. Indeed, pick $f \in C$ and consider the set $A := \{\omega \in \Omega : |f(\omega)| \geq n\} \in \Sigma$. Since

$$\mu(A)n \leq \int_A |f|\, d\mu \leq \|f\|_{L^1(\mu)} \leq \|f\|_{L^1(m)} \overset{(3.2)}{\leq} \delta n,$$

we get $\|f1_A\|_{L^1(m)} \leq \varepsilon$ (by (3.1)). Thus, $f = f1_{\Omega\setminus A} + f1_A \in nB_{L^\infty(m)} + \varepsilon B_{L^1(m)}$.

(ii)⇒(iii) is obvious.

(iii)⇒(i). Fix $\varepsilon > 0$ and take $n \in \mathbb{N}$ as in (ii). For every $f \in C$ and $x^* \in B_{X^*}$, we fix $g_{f,x^*} \in nB_{L^\infty(m)}$ such that

$$\int_\Omega |f - g_{f,x^*}|\, d|\langle m, x^*\rangle| \leq \varepsilon.$$

For each $f \in C$ and $A \in \Sigma$, we have

$$\begin{aligned}\|f1_A\|_{L^1(m)} = \sup_{x^* \in B_{X^*}} \int_A |f|\, d|\langle m, x^*\rangle| &\leq \sup_{x^* \in B_{X^*}} \int_A |g_{f,x^*}|\, d|\langle m, x^*\rangle| + \varepsilon \leq \\ &\leq \sup_{x^* \in B_{X^*}} n|\langle m, x^*\rangle|(A) + \varepsilon = n\|m\|(A) + \varepsilon,\end{aligned}$$

where $\|m\|$ stands for the semivariation of m. This implies that C is bounded (just take $A = \Omega$) and that

$$\sup_{f \in C} \|f1_A\|_{L^1(m)} \leq 2\varepsilon \quad \text{whenever } \|m\|(A) \leq \frac{\varepsilon}{n}.$$

As $\varepsilon > 0$ is arbitrary, C is equi-integrable. $\square$

Lemma 3.2. *The following statements hold:*

(i) *I_m is $\sigma(L^1(m), \Gamma)$-weak continuous.*

(ii) *Every $\sigma(L^1(m), \Gamma)$-bounded subset of $L^1(m)$ is norm bounded.*

Proof. (i) follows from the equality $\langle \gamma_{1_\Omega, x^*}, f\rangle = \langle x^*, I_m(f)\rangle$, which is valid for all $f \in L^1(m)$ and $x^* \in X^*$.

(ii). Let $C \subseteq L^1(m)$ be a $\sigma(L^1(m), \Gamma)$-bounded set. Fix $A \in \Sigma$. Since the linear mapping $f \mapsto f1_A$ is $\sigma(L^1(m), \Gamma)$-$\sigma(L^1(m), \Gamma)$ continuous on $L^1(m)$, the set $C1_A := \{f1_A : f \in C\}$ is $\sigma(L^1(m), \Gamma)$-bounded. From (i) and the Uniform Boundedness Principle it follows that $I_m(C1_A) = \{\int_A f\, dm : f \in C\}$ is bounded. Nikodým's boundedness theorem (see e.g. [14, p. 14, Theorem 1]) applied to the family of X-valued measures

$$A \mapsto \int_A f\, dm, \quad f \in C,$$

ensures that C is norm bounded. $\square$

Statement (ii) of Lemma 3.2 is equivalent to saying that Γ is *w^*-thick*, see [30, Theorem 3.5] (cf. [31, Theorem 1.5]).

Proposition 3.3. *Let $C \subseteq L^1(m)$ be τ_m-compact. Then:*

(i) *C is bounded and equi-integrable.*
(ii) *τ_m, $\sigma(L^1(m),\Gamma)$ and the weak topology coincide on C.*

Proof. (i). Since C is $\sigma(L^1(m),\Gamma)$-compact, it is bounded (Lemma 3.2(ii)). In order to prove that C is equi-integrable it suffices to check that every sequence (f_n) in C admits an equi-integrable subsequence. Let (f_{n_k}) be a τ_m-convergent subsequence (we apply Corollary 2.3). Since $(\int_A f_{n_k}\,dm)$ is norm convergent for every $A \in \Sigma$, the Vitali-Hahn-Saks theorem (see e.g. [14, p. 24, Corollary 10]) applied to the sequence of μ-absolutely continuous measures $A \mapsto \int_A f_{n_k}\,dm$ yields

$$\lim_{\mu(A)\to 0} \sup_{k\in\mathbb{N}} \left\| \int_A f_{n_k}\,dm \right\|_X = 0,$$

which is equivalent to saying that (f_{n_k}) is equi-integrable, because

$$\|f\|_{L^1(m)} \leq 2 \sup_{A\in\Sigma} \left\| \int_A f\,dm \right\| \quad \text{for all } f \in L^1(m).$$

(ii). Since $\sigma(L^1(m),\Gamma)$ is weaker than τ_m, both topologies coincide on the τ_m-compact set C. On the other hand, C is relatively weakly compact (by (i)). Since C is $\sigma(L^1(m),\Gamma)$-compact, it is also $\sigma(L^1(m),\Gamma)$-closed and so weakly closed. Therefore, the weak topology and $\sigma(L^1(m),\Gamma)$ coincide on the weakly compact set C. $\square$

We next characterize when $B_{L^\infty(m)}$ is τ_m-compact. To this end, we need the following known lemma; see e.g. the proof of [28, Lemma 9.1].

Lemma 3.4. *$B_{L^\infty(m)} \subseteq 2\,\overline{\mathrm{aco}}(\{1_A : A \in \Sigma\})$ in $L^1(m)$.*

Proposition 3.5. *The following statements are equivalent:*

(i) *$B_{L^\infty(m)}$ is τ_m-compact.*
(ii) *$m(\Sigma)$ is relatively norm compact.*

Proof. By Lemma 3.4, we have

$$m(\Sigma) \subseteq I_m(B_{L^\infty(m)}) \subseteq 2\,\overline{\mathrm{aco}}(m(\Sigma)). \tag{3.3}$$

Hence (i)⇒(ii) follows at once from the τ_m-norm continuity of I_m.

(ii)⇒(i). Let (f_α) be a net in $B_{L^\infty(m)}$. Since $K := \overline{I_m(B_{L^\infty(m)})} \subseteq X$ is norm compact (by (3.3) and Mazur's theorem, [14, p. 51, Theorem 12]), the product K^Σ is compact when equipped with the product topology induced by the norm topology. Define $y_\alpha := (\int_A f_\alpha\,dm)_{A\in\Sigma} \in K^\Sigma$ for all α. Since the net (y_α) admits a convergent subnet, we can assume without loss of generality that for every $A \in \Sigma$ the limit $\nu(A) := \lim_\alpha \int_A f_\alpha \, dm$ exists in the norm topology. Note that for every $x^* \in X^*$ and $A \in \Sigma$ we have

$$|\langle \nu(A), x^* \rangle| = \lim_\alpha \left| \int_A f_\alpha \, d\langle m, x^* \rangle \right| \leq |\langle m, x^* \rangle|(A).$$

By the Radon-Nikodým theorem for couples of vector measures [29] (cf. [12, Theorem 3.1]), there is $f \in B_{L^\infty(m)}$ such that $\nu(A) = \int_A f \, dm$ for every $A \in \Sigma$. Hence $f = \tau_m - \lim_\alpha f_\alpha$. This proves that $B_{L^\infty(m)}$ is τ_m-compact. $\square$

A weaker version of the former result can be found in [38, Theorem 10] in the setting of $L^p(m)$ spaces, $1 \leq p < \infty$.

Our next lemma is the "τ_m-version" of a well-known characterization of relative weak compactness due to Grothendieck (see e.g. [13, p. 227, Lemma 2]).

Lemma 3.6. *Let $C \subseteq L^1(m)$ be a set such that for every $\varepsilon > 0$ there is a τ_m-compact set $K \subseteq L^1(m)$ such that $C \subseteq K + \varepsilon B_{L^1(m)}$. Then C is relatively τ_m-compact.*

Proof. For each $k \in \mathbb{N}$ we choose a τ_m-compact set $K_k \subseteq L^1(m)$ in such a way that $C \subseteq K_k + \frac{1}{k} B_{L^1(m)}$. Let (f_α) be a net in C. For each α and $k \in \mathbb{N}$ we can write $f_\alpha = f_{\alpha,k} + g_{\alpha,k}$, where $f_{\alpha,k} \in K_k$ and $g_{\alpha,k} \in \frac{1}{k} B_{L^1(m)}$. Since $\prod_{k \in \mathbb{N}} K_k$ is compact with the product topology induced by τ_m, we can find a sequence (h_k) in $L^1(m)$ and a subnet of (f_α), not relabeled, such that $h_k = \tau_m - \lim_\alpha f_{\alpha,k}$ for all $k \in \mathbb{N}$.

Claim 1: For each $A \in \Sigma$, the net $(\int_A f_\alpha \, dm)$ is norm convergent. Indeed, fix $\varepsilon > 0$ and choose $k \in \mathbb{N}$ such that $\frac{1}{k} \leq \varepsilon$. Now, take α_0 such that

$$\left\| \int_A f_{\alpha,k} \, dm - \int_A h_k \, dm \right\| \leq \varepsilon \quad \text{ for all } \alpha \geq \alpha_0.$$

Then for every $\alpha, \alpha' \geq \alpha_0$ we have

$$\begin{aligned} \left\| \int_A f_\alpha \, dm - \int_A f_{\alpha'} \, dm \right\| &\leq \left\| \int_A g_{\alpha,k} \, dm \right\| + \left\| \int_A f_{\alpha,k} \, dm - \int_A h_k \, dm \right\| + \\ &+ \left\| \int_A f_{\alpha',k} \, dm - \int_A h_k \, dm \right\| + \left\| \int_A g_{\alpha',k} \, dm \right\| \leq \\ &\leq \|g_{\alpha,k}\|_{L^1(m)} + 2\varepsilon + \|g_{\alpha',k}\|_{L^1(m)} \leq 4\varepsilon. \end{aligned}$$

This shows that the net $(\int_A f_\alpha \, dm)$ is norm Cauchy, hence norm convergent.

Write $\nu(A) := \lim_\alpha \int_A f_\alpha \, dm$ for all $A \in \Sigma$.

Claim 2: The sequence (h_k) is norm convergent in $L^1(m)$. Indeed, for each $k \in \mathbb{N}$ and $A \in \Sigma$, we have

$$\left\| \int_A f_\alpha \, dm - \int_A f_{\alpha,k} \, dm \right\| \leq \|g_{\alpha,k}\|_{L^1(m)} \leq \frac{1}{k} \quad \text{ for all } \alpha,$$

hence

$$\left\|\nu(A) - \int_A h_k \, dm\right\| \leq \frac{1}{k}. \tag{3.4}$$

Therefore

$$\|h_k - h_{k'}\|_{L^1(m)} \leq 2 \sup_{A \in \Sigma} \left\|\int_A h_k \, dm - \int_A h_{k'} \, dm\right\| \leq \frac{1}{k} + \frac{1}{k'}$$

for every $k, k' \in \mathbb{N}$, which shows that (h_k) is Cauchy.

Finally, observe that if $h \in L^1(m)$ is the limit of (h_k), then inequality (3.4) yields $\int_A h \, dm = \nu(A) = \lim_\alpha \int_A f_\alpha \, dm$ for all $A \in \Sigma$, that is, (f_α) converges to h with respect to τ_m. This shows that C is relatively τ_m-compact. □

Theorem 3.7. *Let $C \subseteq L^1(m)$ and consider the following statements:*

(i) *C is relatively τ_m-compact.*

(ii) *C is bounded and equi-integrable.*

Then (i)⇒(ii). If $m(\Sigma)$ is relatively norm compact, then (i)⇔(ii).

Proof. (i)⇒(ii) follows from Proposition 3.3(i).

Suppose now that $m(\Sigma)$ is relatively norm compact. Then $B_{L^\infty(m)}$ is τ_m-compact (Proposition 3.5). If C is bounded and equi-integrable, then for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $C \subseteq nB_{L^\infty(m)} + \varepsilon B_{L^1(m)}$ (Lemma 3.1). An appeal to Lemma 3.6 ensures that C is relatively τ_m-compact. □

3.2. **A Dunford-Pettis type property.** Concerning compactness properties of operators defined on $L^1(m)$, the aim of this subsection is to analyze when they send bounded equi-integrable sets to relatively norm compact sets. Due to the relation between equi-integrability and weak compactness, this can be understood as a Dunford-Pettis type property. In particular, we provide an alternative proof of the following result from [10].

Theorem 3.8 (Curbera)**.** *Suppose m has σ-finite variation. Let Y be a Banach space and $T : L^1(m) \to Y$ a weakly compact operator. If $C \subseteq L^1(m)$ is bounded and equi-integrable, then $T(C)$ is relatively norm compact.*

Our approach to Theorem 3.8 is based on Proposition 3.9 below and the Davis-Figiel-Johnson-Pelczynski factorization theorem. Recall first that a Banach space Y is said to have the *Compact Range Property* (shortly *CRP*) if every Y-valued countably additive measure with σ-finite variation has relatively norm compact range. For instance, every Banach space with the Radon-Nikodým property has the CRP. A result of Rybakov (cf. [27, Corollary 10]) states that Y^* has the CRP if and only if Y contains no subspace isomorphic to ℓ^1.

Proposition 3.9. *Suppose m has σ-finite variation. Let Y be a Banach space with the CRP and let $T : L^1(m) \to Y$ be an operator. If $C \subseteq L^1(m)$ is bounded and equi-integrable, then $T(C)$ is relatively norm compact.*

Proof. In view of Lemma 3.1, it suffices to prove that $T(B_{L^\infty(m)})$ is relatively norm compact. To this end, define a countably additive measure $\tilde{m} : \Sigma \to Y$ by $\tilde{m}(A) := T(1_A)$. By Lemma 3.4, we have $T(B_{L^\infty(m)}) \subseteq 2\,\overline{\mathrm{aco}}(\tilde{m}(\Sigma))$. Since $\tilde{m}$ has σ-finite variation and Y has the CRP, the set $\tilde{m}(\Sigma)$ is relatively norm compact, hence $\overline{\mathrm{aco}}(\tilde{m}(\Sigma))$ is norm compact (thanks to Mazur's theorem, see e.g. [14, p. 51, Theorem 12]). It follows that $T(B_{L^\infty(m)})$ is relatively norm compact. □

The appearance of the CRP for that kind of result in somehow unavoidable, as we can observe by considering the integration operator of any Y-valued measure with σ-finite variation:

Corollary 3.10. *Let Y be a Banach space. The following statements are equivalent:*

(i) *Y has the CRP.*
(ii) *For every Y-valued countably additive measure ν with σ-finite variation, the set $I_\nu(B_{L^\infty(\nu)})$ is relatively norm compact.*

Proof of Theorem 3.8. By the Davis-Figiel-Johnson-Pelczynski factorization theorem (see e.g. [16, Theorem 13.33]), there exist a reflexive Banach space Z and operators $T_1 : L^1(m) \to Z$ and $T_2 : Z \to Y$ such that $T = T_2 \circ T_1$. Since Y has the Radon-Nikodým property, it also has the CRP. Proposition 3.9 applied to T_1 yields the desired conclusion. □

Corollary 3.11. *If m has σ-finite variation and I_m is weakly compact, then $m(\Sigma)$ is relatively norm compact.*

Proof. Just apply Theorem 3.8 to $Y := X$, $T := I_m$ and $C := B_{L^\infty(m)}$. □

There exist vector measures with finite variation and relatively norm compact range whose integration operator is not weakly compact, like the Volterra measure for $r \in \{1, \infty\}$, see [34, Example 3.49(iv)].

3.3. $\sigma(L^1(m), \Gamma)$**-precompactness.** We study here precompactness with respect to the topology $\sigma(L^1(m), \Gamma)$.

Let Y be a Banach space and $B \subseteq B_{Y^*}$ a norming set. A set $C \subseteq Y$ is said to be $\sigma(Y, B)$-*precompact* if every sequence (y_n) in C admits a $\sigma(Y, B)$-Cauchy subsequence (y_{n_k}), i.e. the sequence $(y^*(y_{n_k}))$ is convergent for every

$y^* \in B$. By taking $B = B_{Y^*}$ we obtain the usual notion of *weak* precompactness. Clearly, if Y is weakly sequentially complete (shortly WSC), then a set $C \subseteq Y$ is weakly precompact if and only if it is relatively weakly compact. On the other hand, Rosenthal's ℓ^1-theorem (see e.g. [16, Theorem 5.37]) states that Y does not contain subspaces isomorphic to ℓ^1 if and only if every bounded subset of Y is weakly precompact.

A classical result due to Dieudonné [15], when applied to our particular setting, says that a set $C \subseteq L^1(m)$ is *weakly* precompact if and only if it is bounded and, for every $h \in L^1(m)^\times$, the set $hC := \{hf : f \in C\}$ is equi-integrable in $L^1(\mu)$. Theorem 3.13 below shows that if we restrict our attention to h's of the form $\frac{d\langle m,x^*\rangle}{d\mu}$ (the Radon-Nikodým derivative of $\langle m, x^*\rangle$ with respect to μ), then we get a characterization of $\sigma(L^1(m),\Gamma)$-precompact subsets of $L^1(m)$. This characterization should be compared with Lemma 3.1 and the statement of Corollary 5.14 (for the convex weakly compact set $K = B_{L^\infty(m)}$).

Lemma 3.12. *For every $x^* \in X^*$ the identity operator $L^1(m) \to L^1(\langle m, x^*\rangle)$ maps $\sigma(L^1(m),\Gamma)$-Cauchy sequences to weakly convergent sequences.*

Proof. Let (f_n) be a $\sigma(L^1(m),\Gamma)$-Cauchy sequence in $L^1(m)$. In particular, it is $\sigma(L^1(m),\Gamma)$-bounded, hence norm bounded (Lemma 3.2(ii)). Now Lemma 2.1(i) applies to conclude that (f_n) is weakly Cauchy, hence weakly convergent, in the WSC space $L^1(\langle m, x^*\rangle)$. □

Theorem 3.13. *Let $C \subseteq L^1(m)$. The following statements are equivalent:*

(i) *C is relatively weakly compact in $L^1(\langle m, x^*\rangle)$ for every $x^* \in X^*$.*

(ii) *For every $x^* \in X^*$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that*

$$C \subseteq nB_{L^\infty(m)} + \varepsilon B_{L^1(\langle m,x^*\rangle)} \quad \text{in } L^1(\langle m, x^*\rangle).$$

(iii) *C is bounded in $L^1(\mu)$ and equi-integrable in $L^1(\langle m, x^*\rangle)$ for every $x^* \in X^*$.*

(iv) *C is bounded in $L^1(\mu)$ and the set $\frac{d\langle m,x^*\rangle}{d\mu}C$ is equi-integrable in $L^1(\mu)$ for every $x^* \in X^*$.*

(v) *C is $\sigma(L^1(m),\Gamma)$-precompact.*

Proof. (i)⇒(ii) follows as in Lemma 3.1.

(ii)⇒(iii) and (iii)⇒(iv) are clear.

(iv)⇒(v). Let (f_n) be a sequence in C. Since C is relatively weakly compact in $L^1(\mu)$ (apply the hypothesis to $x_0^* \in B_{X^*}$ such that $\mu = |\langle m, x_0^*\rangle|$), there is a subsequence (f_{n_k}) converging in the weak topology of $L^1(\mu)$. To finish the proof of the implication (iv)⇒(v) it suffices to prove the following claim.

Claim: The sequence $(\langle \gamma, f_{n_k}\rangle)$ converges for every $\gamma \in \Gamma$. Indeed, let us write $\gamma = \gamma_{h,x^*}$ for some $h \in B_{L^\infty(m)}$ and $x^* \in B_{X^*}$. Let $g = \frac{d\langle m,x^*\rangle}{d\mu} \in L^1(\mu)$. Fix $\varepsilon > 0$. Since by assumption the set gC is equi-integrable in $L^1(\mu)$, the same holds for hgC and so there is $\delta > 0$ such that

$$\sup_{k\in\mathbb{N}} \int_A |f_{n_k} hg| \, d\mu \le \varepsilon \tag{3.5}$$

for every $A \in \Sigma$ with $\mu(A) \le \delta$. For each $p \in \mathbb{N}$, set

$$Z_p := \{\omega \in \Omega : |g(\omega)| \le p\} \in \Sigma.$$

Since the sequence (Z_p) is increasing and $\Omega = \bigcup_{p\in\mathbb{N}} Z_p$, we can find $p \in \mathbb{N}$ large enough such that $\mu(\Omega \setminus Z_p) \le \delta$, so (3.5) yields

$$\sup_{k\in\mathbb{N}} \int_{\Omega\setminus Z_p} |f_{n_k} hg| \, d\mu \le \varepsilon. \tag{3.6}$$

On the other hand, since (f_{n_k}) converges weakly in $L^1(\mu)$ and $hg1_{Z_p} \in L^\infty(m)$, there is $k_0 \in \mathbb{N}$ such that

$$\left| \int_{Z_p} f_{n_k} hg \, d\mu - \int_{Z_p} f_{n_{k'}} hg \, d\mu \right| \le \varepsilon \quad \text{for every } k, k' \ge k_0. \tag{3.7}$$

By putting together (3.6) and (3.7), we get

$$\big|\langle \gamma_{h,x^*}, f_{n_k}\rangle - \langle \gamma_{h,x^*}, f_{n_{k'}}\rangle\big| = \left| \int_\Omega f_{n_k} hg \, d\mu - \int_\Omega f_{n_{k'}} hg \, d\mu \right| \le 3\varepsilon$$

for every $k, k' \ge k_0$. This proves the claim.

(v)⇒(i). For every $x^* \in X^*$ the identity operator $L^1(m) \to L^1(\langle m, x^*\rangle)$ maps $\sigma(L^1(m), \Gamma)$-Cauchy sequences to weakly convergent sequences (Lemma 3.12). □

4. When is Γ a boundary?

Motivated in part by our previous results, in this section we analyze a norming type property (being a boundary, see below for the definition) of the set Γ and its applications to the study of compactness in $L^1(m)$. Some other norming properties of Γ have been discussed in [40].

Curbera [9] and Okada [32] showed that $\sigma(L^1(m), \Gamma)$ and the weak topology coincide on bounded sets whenever $L^1(m)$ contains no subspace isomorphic to ℓ^1. As observed in [32], a result of Lewis (see [23, Corollary 3.3]) implies that every (necessarily bounded) $\sigma(L^1(m), \Gamma)$-convergent *sequence* in $L^1(m)$ is weakly convergent whenever $m(\Sigma)$ is relatively norm compact, but not in general (see [10, Section 6]). But the relative norm compactness of $m(\Sigma)$ does not imply, in general, that $\sigma(L^1(m),\Gamma)$ and the weak topology coincide on bounded sets, see [9, Example 3]. Manjabacas (see [24, Section 4.7]) discussed this type of questions by using a new approach based on the notion of boundary, as follows.

Given an arbitrary Banach space Y, a set $B \subseteq B_{Y^*}$ is called a *boundary* (or a *James boundary*) if for every $y \in Y$ there is $y^* \in B$ such that $\|y\| = y^*(y)$. A typical example of boundary is the set $\mathrm{ext}(B_{Y^*})$ of extreme points of B_{Y^*}. If $B \subseteq B_{Y^*}$ is a boundary, the Rainwater-Simons theorem (see e.g. [16, Theorem 3.134]) states that every norm bounded $\sigma(Y,B)$-convergent sequence in Y is weakly convergent. More generally, a striking result of Pfitzner [36] states that, if $B \subseteq B_{Y^*}$ is a boundary, then every norm bounded $\sigma(Y,B)$-compact subset of Y is weakly compact. This was previously known in particular classes of Banach spaces like, for instance, weakly compactly generated (shortly WCG) spaces, cf. [6, Corollary 2.2].

Thus, Manjabacas (see [24, Proposition 4.38]) showed that Γ is a boundary whenever $m(\Sigma)$ is relatively norm compact. This has also been proved (without using that terminology) in [33, Lemma 3.3]. The aim of this section is to improve Manjabacas' result by showing that, in fact, the relative norm compactness of $m(\Sigma)$ implies that $\Gamma \supseteq \mathrm{ext}(B_{L^1(m)^*})$ (Theorem 4.3 below).

Given any $f \in L^1(m)$, we consider the mapping $M_f : B_{L^\infty(m)} \to X$ defined by

$$M_f(h) := I_m(fh) = \int_\Omega fh\,dm.$$

Lemma 4.1. *Let $f \in L^1(m)$. Then:*

(i) *M_f is τ_m-norm continuous.*
(ii) *M_f is $\sigma(L^1(m),\Gamma)$-weak continuous.*

Proof. Both statements are clear whenever f is a simple function. In the general case, let (f_n) be a sequence of simple functions such that $\|f_n - f\|_{L^1(m)} \to 0$. Then (M_{f_n}) converges to M_f uniformly on $B_{L^\infty(m)}$, hence M_f is τ_m-norm continuous and $\sigma(L^1(m),\Gamma)$-weak continuous. □

Part (i) of the following corollary appears in [33, Lemma 3.3].

Corollary 4.2. *Suppose $m(\Sigma)$ is relatively norm compact. Then:*

(i) *For every $f \in L^1(m)$ the set $I_m(fB_{L^\infty(m)})$ is norm compact.*
(ii) *Γ is a boundary.*

Proof. (i). Bearing in mind that $B_{L^\infty(m)}$ is τ_m-compact (because $m(\Sigma)$ is relatively norm compact, see Proposition 3.5) and the τ_m-norm continuity of M_f (Lemma 4.1(i)), we deduce that $M_f(B_{L^\infty(m)}) = I_m(fB_{L^\infty(m)})$ is norm compact.

Now, (ii) follows from (i) and the equality

$$\|f\|_{L^1(m)} = \sup_{h\in B_{L^\infty(m)}} \|I_m(fh)\|,$$

which is valid for all $f \in L^1(m)$ (see e.g. [34, Lemma 3.11]). $\square$

The proof of the next result is based on ideas from [18, Theorem 3.9].

Theorem 4.3. *Suppose $m(\Sigma)$ is relatively norm compact. Then:*

(i) Γ *is w^*-compact.*
(ii) $\mathrm{ext}(B_{L^1(m)^*}) \subseteq \Gamma$.

Proof. We have $B_{L^1(m)^*} = \overline{\mathrm{co}(\Gamma)}^{w^*}$ by the Hahn-Banach separation theorem (bear in mind that Γ is norming and symmetric). Thus, the "converse" of the Krein-Milman theorem (see e.g. [16, Theorem 3.66]) yields the inclusion

$$\mathrm{ext}(B_{L^1(m)^*}) \subseteq \overline{\Gamma}^{w^*}.$$

Hence (ii) follows immediately from (i). To prove (i), let us consider the mapping

$$\Phi : B_{L^\infty(m)} \times B_{X^*} \to L^1(m)^*, \quad \Phi(h, x^*) := \gamma_{h,x^*}.$$

We shall check that Φ is continuous when $L^1(m)^*$ is equipped with its w^*-topology and the set $P := B_{L^\infty(m)} \times B_{X^*}$ is equipped with the product topology $\mathfrak{T}$ induced by τ_m and the w^*-topology of X^*. Since $m(\Sigma)$ is relatively norm compact, $B_{L^\infty(m)}$ is τ_m-compact (Proposition 3.5) and so P is $\mathfrak{T}$-compact. Therefore, statement (i) will follow at once from the $\mathfrak{T}$-w^* continuity of Φ.

Let (h_α, x^*_α) be a net in P which $\mathfrak{T}$-converges to some $(h, x^*) \in P$. In order to prove that $\Phi(h_\alpha, x^*_\alpha) \to \Phi(h, x^*)$ in the w^*-topology, fix $f \in L^1(m)$ and set

$$x_\alpha := I_m(fh_\alpha) = \int_\Omega fh_\alpha \, dm \in X \quad \text{for every } \alpha.$$

Since the set $\{x_\alpha\}$ is relatively norm compact (by Corollary 4.2(i)), and (x^*_α) is a bounded net which w^*-converges to x^*, we have

$$|\gamma_{h_\alpha,x^*_\alpha}(f) - x^*(x_\alpha)| = |x^*_\alpha(x_\alpha) - x^*(x_\alpha)| \to 0. \tag{4.1}$$

On the other hand, as a consequence of Lemma 4.1(i) we also have

$$x^*(x_\alpha) = \int_\Omega fh_\alpha \; d\langle m, x^*\rangle \to \int_\Omega fh \; d\langle m, x^*\rangle = \gamma_{h,x^*}(f). \tag{4.2}$$

From (4.1) and (4.2) it follows that $|\gamma_{h_\alpha,x^*_\alpha}(f) - \gamma_{h,x^*}(f)| \to 0$. As $f \in L^1(m)$ is arbitrary, we conclude that $\Phi(h_\alpha, x^*_\alpha) \to \Phi(h, x^*)$ in the w^*-topology. $\square$

Remark 4.4. In another direction, it is worth mentioning that if X is a Banach lattice and m is positive (meaning that $m(A) \geq 0$ for all $A \in \Sigma$), then Γ is a boundary. Indeed, in this case the norm of any $f \in L^1(m)$ can be computed as

$$\|f\|_{L^1(m)} = \Big\| \int_\Omega |f| \, dm \Big\|_X$$

(see e.g. [34, Lemma 3.13]).

We finish this section by pointing out two specialized versions of Theorem 3.13 when Γ is assumed to be a boundary.

Corollary 4.5. *Suppose Γ is a boundary. Then a subset of $L^1(m)$ is weakly precompact if and only if it is relatively weakly compact in $L^1(\langle m, x^* \rangle)$ for every $x^* \in X^*$.*

Proof. Bearing in mind Lemma 3.2(ii), the Rainwater-Simons theorem (see e.g. [16, Theorem 3.134]) implies that the identity mapping on $L^1(m)$ is $\sigma(L^1(m), \Gamma)$-weak sequentially continuous, and so every $\sigma(L^1(m), \Gamma)$-Cauchy sequence in $L^1(m)$ is weakly Cauchy. The result now follows from Theorem 3.13. $\square$

Corollary 4.6. *Suppose $L^1(m)$ is WSC and Γ is a boundary. Then a subset of $L^1(m)$ is relatively weakly compact if and only if it is relatively weakly compact in $L^1(\langle m, x^* \rangle)$ for every $x^* \in X^*$.*

In general, the assumption that Γ is a boundary cannot be removed from the previous statements. Indeed, in [10, Section 6] there is an example of an ℓ^2-valued measure m and a $\sigma(L^1(m), \Gamma)$-null sequence in $L^1(m)$ which is equivalent to the usual basis of ℓ^1 (and so it does not have any weakly Cauchy subsequence).

5. When is $L^1(m)$ a SWCG space?

It is well-known that the space $L^1(m)$ is WCG (see [8, Theorem 2], cf. [34, Theorem 3.7]). As an application of the results obtained before, in this section we analyze the property of being strongly weakly compactly generated (defined below), that does not hold for all spaces $L^1(m)$. By the way, we will prove some new results regarding the integration operator, after introducing the so called positive Schur property for Banach lattices in our discussion.

Following [41], a Banach space Y is called *strongly weakly compactly generated* (shortly SWCG) if there is a weakly compact set $K \subset Y$ such that for every weakly compact set $L \subset Y$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$

such that $L \subset nK + \varepsilon B_Y$ (in this case, we say that K *strongly generates* Y). Every SWCG space is both WCG and WSC [41] (cf. [20, Theorem 6.38]). Typical examples of spaces in this class are the reflexive spaces, separable spaces with the Schur property and the L^1 space of any scalar measure. For more information on SWCG spaces and related classes of Banach spaces, we refer the reader to [20, Section 6.4] and [17, 21, 22, 25].

We stress that, being a Banach lattice, $L^1(m)$ is WSC if and only if it does not contain subspaces isomorphic to c_0 (see e.g. [1, Theorem 4.60]). Curbera proved in [8, Theorem 3] that $L^1(m)$ is WSC whenever X does not contain subspaces isomorphic to c_0 (cf. [35]).

The following well-known general construction will be helpful to exhibit concrete examples.

Remark 5.1. Let X be a Banach space having an unconditional Schauder basis (e_n). Fix a sequence (α_n) of strictly positive real numbers such that the series $\sum_n \alpha_n e_n$ is unconditionally convergent. Define a countably additive measure $m : \mathcal{P}(\mathbb{N}) \to X$ by $m(A) := \sum_{n\in A} \alpha_n e_n$. Then:

(i) m is purely atomic.
(ii) m has relatively norm compact range.
(iii) m has finite variation if and only if $\sum_n \alpha_n e_n$ is absolutely convergent.
(iv) I_m is an order isomorphism between $L^1(m)$ and X.

Example 5.2. c_0 is an L^1 space of a vector measure which is not WSC, hence it is not SWCG.

Example 5.3. Mercourakis and Stamati [25] constructed a subspace of $L^1[0,1]$ having unconditional Schauder basis which is not SWCG. This is an L^1 space of a vector measure which is WSC (because $L^1[0,1]$ is WSC) but non-SWCG.

Example 5.4. $\ell^2(\ell^1)$ is an L^1 space of a vector measure which is WSC (because it is the ℓ^2-sum of countably many WSC spaces) but does not embed isomorphically into any SWCG space (see [22, Corollary 2.29]).

The previous examples of non-SWCG spaces are based on vector measures taking values in non-SWCG spaces. Thus, one might wonder whether $L^1(m)$ is SWCG whenever X is. It turns out that this is not the case even for reflexive X.

Example 5.5. $\ell^2(L^1[0,1])$ is the L^1 space of some ℓ^2-valued measure, see [4, Example 3.7]. This space is WSC and does not embed isomorphically into any SWCG space (see [22, Corollary 2.29]).

5.1. **The positive Schur property in $L^1(m)$.** A Banach lattice is said to have the *positive Schur property* (shortly PSP) if every weakly null sequence of positive vectors is norm null. For instance, the L^1 space of any scalar measure has the PSP. This property is equivalent to saying that every relatively weakly compact set is L-weakly compact. Therefore, *$L^1(m)$ has the PSP if and only if every weakly compact subset of $L^1(m)$ is equi-integrable.* The reader can find information about these concepts and their relations in [34, Remark 2.40] and the references therein.

Proposition 5.6. *If $L^1(m)$ has the PSP, then it is SWCG.*

Proof. The set $K := B_{L^\infty(m)} \subseteq L^1(m)$ is weakly compact and strongly generates $L^1(m)$. Indeed, if $L \subseteq L^1(m)$ is weakly compact, then it is bounded and equi-integrable (according to the comments above), and so for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $L \subseteq nK + \varepsilon B_{L^1(m)}$ (Lemma 3.1). □

Similarly, bearing in mind Propositions 3.3 and 3.5, we have the following "strong generation" property with respect to the topology τ_m:

Remark 5.7. Suppose $m(\Sigma)$ is relatively norm compact. Then $B_{L^\infty(m)}$ is a τ_m-compact subset of $L^1(m)$ such that for every τ_m-compact set $L \subseteq L^1(m)$ there is $n \in \mathbb{N}$ such that $L \subseteq nB_{L^\infty(m)} + \varepsilon B_{L^1(m)}$.

It was pointed out in [10, Claim 1] that $L^1(m)$ has the PSP whenever X has the Schur property. As we show in Theorem 5.8 below, this is a consequence of the complete continuity of I_m when X has the Schur property. Recall that an operator between Banach spaces is called *completely continuous* (or *Dunford-Pettis*) if it maps weakly convergent sequences to norm convergent ones. The complete continuity of I_m has strong consequences on the structure of $L^1(m)$. Under some assumptions on X (namely, that X^* has the Radon-Nikodým property), it is known that if I_m is completely continuous, then m has finite variation and $L^1(m)$ is order isomorphic to $L^1(|m|)$ (via the identity mapping), see [5] (cf. [33, Theorem 1.2] for the particular case in which X is also assumed to have an unconditional Schauder basis).

Theorem 5.8. *The following statements are equivalent:*

(i) *I_m is completely continuous.*
(ii) *$L^1(m)$ has the PSP and $m(\Sigma)$ is relatively norm compact.*

Proof. Observe first that (i) is equivalent to

(i') the identity mapping on $L^1(m)$ is weak-τ_m sequentially continuous, because a sequence (f_n) in $L^1(m)$ is weakly convergent to $f \in L^1(m)$ if and only if $(f_n 1_A)$ is weakly convergent to $f 1_A$ for every $A \in \Sigma$.

(i')⇒(ii). If $C \subseteq L^1(m)$ is weakly compact, then it is weakly sequentially compact and so the assumption implies that C is τ_m-sequentially compact, hence equi-integrable (see the proof of Proposition 3.3). Therefore, $L^1(m)$ has the PSP.

On the other hand, it is well-known that $m(\Sigma)$ is relatively norm compact whenever I_m is completely continuous; see e.g. [34, p. 153]. (This fact can also be deduced by combining (i') and Proposition 3.5.)

(ii)⇒(i). Theorem 3.7 and (ii) ensure us that a subset of $L^1(m)$ is relatively τ_m-compact if and only if it is relatively weakly compact. By considering the topology $\sigma(L^1(m), \Gamma)$, which is weaker than both τ_m and the weak topology, it follows at once that a subset of $L^1(m)$ is τ_m-compact if and only if it is weakly compact. Therefore, I_m maps weakly compact sets to norm compact sets. $\square$

Example 5.9. Let m be the $L^1[0,1]$-valued measure defined by $m(A) := 1_A$ for every Borel set $A \subseteq [0,1]$. Then $L^1(m) = L^1[0,1]$ has the PSP, but the range of m is not relatively norm compact.

It was known that $L^1(m)$ is WSC whenever I_m is completely continuous, [11, Theorem 3.6] (cf. [7, second proof of Theorem 2.2] and [35, Theorem 1.1]). The following consequence of Proposition 5.6 and Theorem 5.8 improves that result:

Corollary 5.10. *If I_m is completely continuous, then $L^1(m)$ is SWCG.*

Remark 5.11. A Banach space Y has the *Dunford-Pettis* property if every weakly compact operator from Y to another Banach space is completely continuous. For instance, any L^1 space of a scalar measure satisfies this property, as well as any $C(K)$ space of a compact topological space K. Within the setting of L^1 spaces of vector measures, Curbera [10] applied Theorem 3.8 to deduce that if m has σ-finite variation and $L^1(m)$ has the PSP, then it has the Dunford-Pettis property. The converse does not hold in general, as c_0 is order isomorphic to the L^1 space of a c_0-valued vector measure with finite variation (Remark 5.1).

We finish this subsection by characterizing the PSP of $L^1(m)$ in terms of I_m. An operator from a Banach lattice to a Banach space is called *almost Dunford-Pettis* if it maps weakly null disjoint sequences to norm null ones or, equivalently, if it maps weakly null *positive* sequences to norm null ones, see [2, Theorem 2.2].

Theorem 5.12. *The following statements are equivalent:*

(i) *$L^1(m)$ has the PSP.*
(ii) *I_m is almost Dunford-Pettis.*

Proof. (i)⇒(ii) is clear. In order to prove (ii)⇒(i), we first check that (ii) implies that $L^1(m)$ is WSC. Indeed, if $L^1(m)$ is not WSC, then it contains a sublattice which is order isomorphic to c_0 (see e.g. [1, Theorem 4.60]). Now, following the proof of [7, Theorem 2.2] (cf. [35, Theorem 1.1]), one can find a c_0-sequence (f_n) in $L^1(m)$ such that $f_n \geq 0$ for all $n \in \mathbb{N}$ and I_m is an isomorphism when restricted to $\overline{\mathrm{span}}(f_n)$. This contradicts (ii), and so $L^1(m)$ is WSC.

Let $C \subseteq L^1(m)$ be any relatively weakly compact set. We shall prove that C is L-weakly compact. The solid hull

$$\mathrm{Sol}(C) = \{g \in L^1(m) : |g| \leq |f| \text{ for some } f \in C\}$$

is relatively weakly compact, thanks to the weak sequential completeness of $L^1(m)$ (see e.g. [1, Theorems 4.39 and 4.60]). Let (f_n) be a disjoint sequence in $\mathrm{Sol}(C)$. Since $\mathrm{Sol}(C)$ is relatively weakly compact and the f_n's are pairwise disjoint, (f_n) is weakly null. Since I_m is almost Dunford-Pettis and each sequence of the form $(f_n 1_A)$, where $A \in \Sigma$, is weakly null and disjoint, we conclude that (f_n) is τ_m-convergent to 0. In particular, (f_n) is equi-integrable (Proposition 3.3). From this fact and the disjointness of (f_n) it follows that $\|f_n\| \to 0$. This proves that C is L-weakly compact.

Therefore, every relatively weakly compact subset of $L^1(m)$ is L-weakly compact, that is, $L^1(m)$ has the PSP. The proof is over. □

5.2. **A characterization of $L^1(m)$ spaces which are SWCG.** We finish this section by giving a characterization of SWCG spaces of integrable functions with respect to a vector measure. For each $h \in L^1(m)^* = L^1(m)^\times$, we can consider the scalar measure $\mu_h := h\, d\mu$ given by $\mu_h(A) := \int_A h\, d\mu$ for all $A \in \Sigma$, so that the identity mapping defines an operator from $L^1(m)$ to $L^1(\mu_h)$ with norm ≤ 1.

Proposition 5.13. *Let $K \subseteq L^1(m)$ be a convex weakly compact set, $L \subseteq L^1(m)$ and $\varepsilon > 0$. The following statements are equivalent:*

(i) $L \subseteq K + \varepsilon B_{L^1(m)}$.
(ii) *There is a convex w^*-dense set $\Delta \subseteq B_{L^1(m)^*}$ such that, for every $h \in \Delta$,*

$$L \subseteq K + \varepsilon B_{L^1(\mu_h)} \quad \textit{in } L^1(\mu_h).$$

Proof. (i)⇒(ii) is clear by taking $\Delta = B_{L^1(m)^*}$.

(ii)⇒(i). Our proof is by contradiction. Suppose there is $f \in L$ such that $f \notin K + \varepsilon B_{L^1(m)}$. Since $K + \varepsilon B_{L^1(m)}$ is convex and closed, the Hahn-Banach separation theorem ensures the existence of $\phi \in L^1(m)^*$ with norm one such that

$$\langle \phi, f \rangle > \sup_{g \in K + \varepsilon B_{L^1(m)}} \langle \phi, g \rangle = \sup_{g \in K} \langle \phi, g \rangle + \varepsilon. \tag{5.1}$$

Let $\mathfrak{T}$ denote the Mackey topology on $L^1(m)^*$, that is, the topology of uniform convergence on weakly compact subsets of $L^1(m)$. Then $\overline{C}^{w^*} = \overline{C}^{\mathfrak{T}}$ for every convex set $C \subseteq L^1(m)^*$ (see e.g. [16, Theorem 3.45]). In particular, we have

$$B_{L^1(m)^*} = \overline{\Delta}^{w^*} = \overline{\Delta}^{\mathfrak{T}}.$$

This equality and (5.1) imply that there is $h \in \Delta$ such that

$$\langle h, f \rangle > \sup_{g \in K} \langle h, g \rangle + \varepsilon. \tag{5.2}$$

Since $L \subseteq K + \varepsilon B_{L^1(\mu_h)}$ in the space $L^1(\mu_h)$, we can write $f = g + u$ for some $g \in K$ and $u \in L^1(\mu_h)$ with $\|u\|_{L^1(\mu_h)} \leq \varepsilon$. But inequality (5.2) implies that $\int_\Omega f \, d\mu_h > \int_\Omega g \, d\mu_h + \varepsilon$, hence $\int_\Omega u \, d\mu_h > \varepsilon$, a contradiction. □

Corollary 5.14. *Let $K \subseteq L^1(m)$ be a convex weakly compact set. The following statements are equivalent:*

(i) *K strongly generates $L^1(m)$.*

(ii) *For every weakly compact set $L \subseteq L^1(m)$ and every $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and a convex w^*-dense set $\Delta \subseteq B_{L^1(m)^*}$ such that*

$$L \subseteq nK + \varepsilon B_{L^1(\mu_h)} \quad \textit{in } L^1(\mu_h)$$

for every $h \in \Delta$.

(iii) *There exists a convex w^*-dense set $\Delta \subseteq B_{L^1(m)^*}$ such that, for every weakly compact set $L \subseteq L^1(m)$ and every $\varepsilon > 0$, there is $n \in \mathbb{N}$ such that*

$$L \subseteq nK + \varepsilon B_{L^1(\mu_h)} \quad \textit{in } L^1(\mu_h)$$

for every $h \in \Delta$.

Proof. The implications (i)⇒(iii) and (iii)⇒(ii) are clear (just take $\Delta = B_{L^1(m)^*}$). Proposition 5.13 yields (ii)⇒(i). □

Remark 5.15. In particular, we might apply the previous criterion by choosing the convex w^*-dense set $\Delta = \mathrm{co}(\Gamma)$. Under the identification of

$L^1(m)^*$ and $L^1(m)^\times$, Γ is precisely the set

$$\Big\{h\frac{d\langle m,x^*\rangle}{d\mu}:\, h\in B_{L^\infty(m)},\, x^*\in B_{X^*}\Big\},$$

where $\frac{d\langle m,x^*\rangle}{d\mu}$ denotes the Radon-Nikodým derivative of $\langle m,x^*\rangle$ with respect to μ.

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Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain
E-mail address: jmcalabu@mat.upv.es

Departamento de Matemáticas, Escuela de Ingenieros Industriales, Universidad de Castilla-La Mancha, 02071 Albacete, Spain
E-mail address: sebastian.lajara@uclm.es

Departamento de Matemática Aplicada, Facultad de Informática, Universidad de Murcia, 30100 Espinardo (Murcia), Spain
E-mail address: joserr@um.es

Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, Camino de Vera s/n, 46022 Valencia, Spain
E-mail address: easancpe@mat.upv.es