On nonsingularity of combinations of two group invertible matrices and two tripotent matrices

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Abstract

Let $T_1$ and $T_2$ be two $n \times n$ tripotent matrices and $c_1$, $c_2$ two nonzero complex numbers. We mainly study the nonsingularity of combinations $T = c_1T_1 + c_2T_2 - c_3T_1T_2$ of two tripotent matrices $T_1$ and $T_2$, and give some formulae for the inverse of $c_1T_1 + c_2T_2 - c_3T_1T_2$ under some conditions. Some of these results are given in terms of group invertible matrices.

Key words: Tripotent matrix, Linear combination, Diagonalization, Nonsingularity, Group invertible matrix.

1 Introduction

Let $\mathbb{C}$ be the field of complex numbers, and let the symbols $\mathbb{C}^*$ and $\mathbb{C}^{n\times n}$ denote the set of nonzero complex numbers and $n \times n$ complex matrices respectively. Moreover, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $A^*$ stand for the column space, null space and conjugate transpose of $A \in \mathbb{C}^{n\times n}$, respectively. A matrix $A \in \mathbb{C}^{n\times n}$ is said to be idempotent if $A^2 = A$, and tripotent if $A^3 = A$.

Recall that $A \in \mathbb{C}^{n\times n}$ is nonsingular if and only if $\mathcal{N}(A) = \{0\}$. Also notice that if $T \in \mathbb{C}^{n\times n}$ and $k$ is a natural number greater than 1, then $T$ satisfies $T^k = T$ if and only if $T$ is diagonalizable and the spectrum of $T$ is contained in $k\sqrt{k} \cup 0$, which have been proved in [5].

Recently, the nonsingularity of linear combinations of idempotent matrices, projectors, tripotent matrices and $k$-potent matrices (see, for example, [1, 2, 4, 6, 7]) have
been extensively investigated. In [9], Sarduvan and Özdemir have considered the non-singularity of linear combinations $T = c_1T_1 + c_2T_2$, where $T_1, T_2$ are two commuting $n \times n$ tripotent matrices and $c_1, c_2 \in \mathbb{C}^\ast$. In [10], Zuo studied the nonsingularity of combinations $c_1P + c_2Q - c_3PQ$ of two idempotent matrices $P$ and $Q$, and the same author generalized the results in [11].

In this paper, we discuss the nonsingularity of combinations $c_1T_1 + c_2T_2 - c_3T_1T_2$ of two tripotent matrices and give some formulae for the inverse of $c_1T_1 + c_2T_2 - c_3T_1T_2$ under some conditions. We point out that the main results of this article are similar to the ones obtained in [10]. Notice that an idempotent matrix is always a tripotent matrix, but a tripotent matrix may not be idempotent. Special types of matrices, such as idempotents, tripotents, etc, are very useful in many contexts and they have been extensively studied in the literature. For example, quadratic forms with idempotent matrices are used extensively in statistical theory. So it is worth to stress and spread these kinds of results.

A matrix $A \in \mathbb{C}^{n \times n}$ is said to be group invertible if there exists $X \in \mathbb{C}^{n \times n}$ such that

$$AXA = A, \quad XAX = X, \quad AX =XA. \quad (1.1)$$

See [3, Chapter 4] for more information on this kind of generalized inverse. It can be proved that the set of matrices $X$ satisfying (1.1) is or empty or a singleton and when is a singleton, it is customary to denote its unique element by $A^\#$. Also, it is known [8, Exercise 5.10.12] that a matrix $A \in \mathbb{C}^{n \times n}$ is group invertible if and only if there exist nonsingular $S \in \mathbb{C}^{n \times n}, C \in \mathbb{C}^{r \times r}$ such that $A = S(C \oplus 0)S^{-1}$, being $r$ the rank of $A$. In this situation, one has $A^\# = S(C^{-1} \oplus 0)S^{-1}$.

Evidently, if $T$ is a tripotent matrix, then $T$ is group invertible and $T^\# = T$. Many of the results given in this paper will be given in terms of group invertible matrices.

2 Main results

In [10, Corollary 2.6], it had been proved that $P - Q$ is nonsingular if and only if $aP + bQ - cPQ$ and $I_n - PQ$ are nonsingular for any two idempotent matrices $P, Q \in \mathbb{C}^{n \times n}$, $a, b \in \mathbb{C}^\ast$. In the following theorem, a similar result is established for tripotent matrices.

**Theorem 2.1.** Let $T_1, T_2 \in \mathbb{C}^{n \times n}$ be two commuting tripotent matrices. Then $T_1 - T_2$ is nonsingular if and only if $I_n - T_1T_2$ and $T_1^2 + (I_n - T_1^2)T_2$ are nonsingular.

**Proof.** By a suitable simultaneous diagonalization, there exists $S \in \mathbb{C}^{n \times n}$ such that $T_1 = S\text{diag}(\lambda_1, \ldots, \lambda_n)S^{-1}$ and $T_2 = S\text{diag}(\mu_1, \ldots, \mu_n)S^{-1}$ being $\{\lambda_i\}_{i=1}^n$ and $\{\mu_i\}_{i=1}^n$, the sets of eigenvalues of $T_1$ and $T_2$, respectively. Observe that $\lambda_i, \mu_j \in \{-1, 0, 1\}$ for all $1 \leq i, j \leq n$ since $T_1$ and $T_2$ are tripotent. Moreover,

$$T_1 - T_2 = S\text{diag}(\lambda_1 - \mu_1, \ldots, \lambda_n - \mu_n)S^{-1},$$

$$I_n - T_1T_2 = S\text{diag}(1 - \lambda_1\mu_1, \ldots, 1 - \lambda_n\mu_n)S^{-1},$$

$$T_1^2 + (I_n - T_1^2)T_2 = S\text{diag}(\lambda_1^2 + \lambda_1\mu_1, \ldots, \lambda_n^2 + \lambda_n\mu_n)S^{-1}.$$
and\[ T_1^2 + (I_n - T_1^2)T_2 = S \text{diag}(\lambda_1^2 + (1 - \lambda_1^2)\mu_1, \ldots, \lambda_n^2 + (1 - \lambda_n^2)\mu_n)S^{-1}. \tag{2.1} \]

Assume that $T_1 - T_2$ is nonsingular. Then $\lambda_i \neq \mu_i$ for all $i \in \{1, \ldots, n\}$. Hence
\[
(\lambda_i, \mu_i) \in \{(1, -1), (1, 0), (-1, 1), (-1, 0), (0, 1), (0, -1)\} \quad \text{for all } i = 1, \ldots, n.
\]

Easily we have that $1 - \lambda_i\mu_i \neq 0$ and $\lambda_i^2 + (1 - \lambda_i^2)\mu_i \neq 0$ for all $1 \leq i \leq n$. Therefore, $I_n - T_1T_2$ and $T_1^2 + (I_n - T_1^2)T_2$ are nonsingular.

Assume that $I_n - T_1T_2$ and $T_1^2 + (I_n - T_1^2)T_2$ are nonsingular. Since $I_n - T_1T_2$ is nonsingular, then $\lambda_i\mu_i \neq 1$ for all $1 \leq i \leq n$. If $T_1 - T_2$ were singular, then there would exist $j \in \{1, \ldots, n\}$ such that $\lambda_j = \mu_j$. Taking $1 \leq j \leq n$ and $\lambda_j \mu_j = 1$, we would get $\lambda_j = \mu_j = 0$. But now, $\lambda_i^2 + (1 - \lambda_i^2)\mu_i = 0$, which would yield the singularity of $T_1^2 + (I_n - T_1^2)T_2$. \hfill \square

**Remark:** Let $p : \mathbb{C}^2 \to \mathbb{C}$ be the following complex polynomial:
\[ p(z, w) = a_{1,0}z + a_{2,0}z^2 + a_{0,1}w + a_{1,1}zw + a_{2,1}z^2w + a_{0,2}w^2 + a_{1,2}zw^2 + a_{2,2}z^2w^2, \tag{2.2} \]

where $a_{i,j}$ are complex numbers. We have
\[ p(T_1, T_2) = S \text{diag}(p(\lambda_1, \mu_1), \ldots, p(\lambda_n, \mu_n))S^{-1}. \]

Now, if $T_1^2 + (I_n - T_1^2)T_2$ were singular, then by (2.1) there would exist $j \in \{1, \ldots, n\}$ such that
\[ \lambda_j^2 + (1 - \lambda_j^2)\mu_j = 0. \tag{2.3} \]

This expression is contradicted by $\lambda_j = \pm 1$. Therefore, $\lambda_j = 0$ and by using again (2.3) we get $\mu_j = 0$. Therefore, $p(T_1, T_2)$ is singular because $p(0, 0) = 0$. Hence we can formulate the following corollary:

**Corollary 2.1.** Let $T_1, T_2 \in \mathbb{C}^{n \times n}$ be two commuting tripotent matrices. If $I_n - T_1T_2$ is nonsingular and there exists a polynomial as in (2.2) such that $p(T_1, T_2)$ is nonsingular, then $T_1 - T_2$ is nonsingular.

We weaken the hypotheses of the commutativity and the tripotency in the following result.

**Theorem 2.2.** Let $T_1, T_2 \in \mathbb{C}^{n \times n}$ be two group invertible matrices such that $T_2T_1T_1^\# = T_1T_2^\#T_2$. If $I_n - T_1^\#T_2$ is nonsingular and there exists a polynomial $p$ in two noncommuting variables such that $p(0, 0) = 0$ and $p(T_1, T_2)$ is nonsingular, then $T_1 - T_2$ is nonsingular.

**Proof.** Let $x \in \mathcal{N}(T_1 - T_2)$, i.e., $T_1x = T_2x$. Premultiplying by $T_1T_1^\#$ we get
\[ T_1x = T_1T_1^\#T_2x. \tag{2.4} \]
Premultiplying $T_1 x = T_2 x$ by $T_2 T_1^\#$ and using $T_2 T_1 T_1^\# = T_1 T_2 T_2$ yield $T_1 T_1^\# T_2 x = T_2 T_1^\# T_2 x$; but having in mind that $T_1 T_1^\# T_2 x = T_1 x = T_2 x$ we get

$$T_2 T_1^\# T_2 x = T_2 x.$$  \hspace{1cm} (2.5)

From (2.4) and (2.5) we have

$$T_1 (I_n - T_1^\# T_2) x = 0,$$

$$T_2 (I_n - T_1^\# T_2) x = 0.$$ \hspace{1cm} (2.6)

Since $p(0,0) = 0$, there exist two polynomials in noncommuting variables, say $p_1$ and $p_2$ such that $p(T_1, T_2) = p_1(T_1, T_2) T_1 + p_2(T_1, T_2) T_2$. Thus from (2.6)

$$p(T_1, T_2) (I_n - T_1^\# T_2) x = (p_1(T_1, T_2) T_1 + p_2(T_1, T_2) T_2) (I_n - T_1^\# T_1) x$$

$$= p_1(T_1, T_2) T_1 (I_n - T_1^\# T_2) x + p_2(T_1, T_2) T_2 (I_n - T_1^\# T_2) x$$

$$= 0.$$  

Under the assumption that $I_n - T_1^\# T_2$ and $p(T_1, T_2)$ are nonsingular, the above computation yields $x = 0$, which means that $T_1 - T_2$ is nonsingular. \hfill \Box

**Remark.** Let $T_1, T_2 \in \mathbb{C}^{n \times n}$ be group invertible and $r$ the rank of $T_1$. If $T_1 T_2 = T_2 T_1$, then by writing $T_1 = S(C \oplus 0) S^{-1}$, where $S \in \mathbb{C}^{n \times n}$ and $C \in \mathbb{C}^{r \times r}$ are nonsingular we get that $T_2$ can be written as $T_2 = S(D \oplus E) S^{-1}$, where $CD = DC$ and $D \in \mathbb{C}^{r \times r}$. Hence, $T_1 T_1^\# T_2 = S(D \oplus 0) S^{-1} = T_2 T_1 T_1^\#$. However, observe that the condition $T_1 T_1^\# T_2 = T_2 T_1 T_1^\#$ is more general than $T_1 T_2 = T_2 T_1$ (it is enough to consider the case when $T_1$ is nonsingular and $T_1 T_2 \neq T_2 T_1$).

Assume in this paragraph that $T_1, T_2 \in \mathbb{C}^{n \times n}$ are two commuting group invertible matrices. Now, if we use the condition $T_1^2 T_2 = T_2^2 T_1$, then we can give some kind of the converse of Theorem 2.2. From $T_1^2 T_2 = T_2^2 T_1$ and $T_1 T_2 = T_2 T_1$, we have $(T_1 - T_2) T_1 T_2 = 0$, hence the invertibility of $T_1 - T_2$ leads to $T_1 T_2 = 0$. Thus, $c_1 T_1 + c_2 T_2 - c_3 T_1 T_2 = c_1 T_1 + c_2 T_2$, and we will give the explicit expression of $(c_1 T_1 + c_2 T_2)^{-1}$ in terms of $(T_1 - T_2)^{-1}$ under mild conditions.

**Theorem 2.3.** Let $T_1, T_2 \in \mathbb{C}^{n \times n}$ be two group invertible matrices and $c_1, c_2 \in \mathbb{C}^*$. If $T_2 T_1 = 0$ and $T_1 - T_2$ is nonsingular. Then $c_1 T_1 + c_2 T_2$ is nonsingular and

$$(c_1 T_1 + c_2 T_2)^{-1} = ((c_1^{-1} + c_2^{-1}) T_1 T_1^\# - c_2^{-1} I_n) (T_1 - T_2)^{-1}.$$  

**Proof.** It follows from the following computations:

$$(c_1 T_1 + c_2 T_2) \left[ (c_1^{-1} + c_2^{-1}) T_1 T_1^\# - c_2^{-1} I_n \right]$$

$$= (1 + c_1 c_2^{-1}) T_1 - c_1 c_2^{-1} T_1 + (c_2 c_1^{-1} + 1) T_2 T_1 T_1^\# - T_2 = T_1 - T_2.$$  

\hfill \Box
Remark. In Theorem 2.3, the situation $c_1 + c_2 = 0$ is trivial and it does not need the hypothesis $T_2 T_1 = 0$.

In [10], it was showed that for two idempotent matrices $P, Q \in \mathbb{C}^{n \times n}$, and $a, b \in \mathbb{C}^*$, one has that matrix $aP + bQ - (a + b)PQ$ is nonsingular if and only if $\mathbb{C}^n = \mathcal{R}(P(I_n - Q)) \oplus \mathcal{R}((I_n - P)Q)$ if and only if $\mathbb{C}^n = \mathcal{N}(P(I_n - Q)) \oplus \mathcal{N}((I_n - P)Q)$. We give similar results in the following theorem.

**Theorem 2.4.** Let $T_1, T_2 \in \mathbb{C}^{n \times n}$ be two matrices and $c_1, c_2, r_1, r_2 \in \mathbb{C}$. If $c_1 T_1 + c_2 T_2 + (r_1 c_1 + r_2 c_2) T_1 T_2$ is nonsingular, then we have

$$\mathcal{N}[T_1(I_n + r_1 T_2)] \cap \mathcal{N}[(I_n + r_2 T_1) T_2] = 0,$$

and

$$\mathcal{R}(T_1(I_n + r_1 T_2)) + \mathcal{R}((I_n + r_2 T_1) T_2) = \mathbb{C}^n.$$  

**Proof.** Denote $\alpha = r_1 c_1 + r_2 c_2$. Let $x \in \mathcal{N}(T_1(I_n + r_1 T_2)) \cap \mathcal{N}((I_n + r_2 T_1) T_2)$. Since $T_1(I_n + r_1 T_2)x = 0$ and $(I_n + r_2 T_1) T_2 x = 0$, we have

$$[c_1 T_1 + c_2 T_2 + \alpha T_1 T_2] x = (c_1 T_1 + c_2 T_2 + r_1 c_1 T_1 T_2 + r_2 c_2 T_1 T_2)x$$

$$= c_1 T_1(I_n + r_1 T_2)x + c_2 (I_n + r_2 T_1) T_2 x = 0.$$

The nonsingularity of $c_1 T_1 + c_2 T_2 + \alpha T_1 T_2$ leads to $x = 0$. Hence (2.7) holds.

Since $c_1 T_1 + c_2 T_2 + \alpha T_1 T_2$ is nonsingular, then $\overline{c_1} T_1^* + \overline{c_2} T_2^* + \alpha T_1^* T_2^*$ is nonsingular. Applying the first part of the proof we have

$$\mathcal{N}[(I_n + r_2 T_1^*)] \cap \mathcal{N}((I_n + \sigma_1 T_2^*) T_1^*) = 0.$$  

Recalling that $[\mathcal{N}(X^*)]^\perp = \mathcal{R}(X)$ holds for any matrix $X$, by taking perp in (2.9) we get that (2.8) holds. \hfill \qed

In the following result, it is given an expression of the inverse of $c_1 T_1 + c_2 T_2 - c_3 T_1 T_2$ under some condition.

**Theorem 2.5.** Let $T_1, T_2 \in \mathbb{C}^{n \times n}$ be two nonzero tripotent matrices such that $T_2 T_1 = T_2 T_1$ and $c_1, c_2, c_3 \in \mathbb{C}^*$. Assume that $T_1$ or $T_2$ are nonsingular. If $(c_1 + c_2)^2 = c_3^2$, then $c_1 T_1 + c_2 T_2 - c_3 T_1 T_2$ or $c_1 T_1 + c_2 T_2 + c_3 T_1 T_2$ is singular. If $(c_1 + c_2)^2 \neq c_3^2$, then $c_1 T_1 + c_2 T_2 - c_3 T_1 T_2$ is nonsingular; and in this case,

(i) If $T_1$ is nonsingular, then

$$[(c_1 + c_2)^2 - c_3^2] (c_1 T_1 + c_2 T_2 - c_3 T_1 T_2)^{-1}$$

$$= (c_1 + c_2) T_1 + c_3 T_2^2 + c_1^{-1} c_2 c_3 (T_2^2 - T_1 T_2) + c_1^{-1} c_3 (T_2 - T_1 T_2^2)$$

$$+ c_1^{-1} (c_2^2 + c_1 c_2 - c_3^2) (T_1 - T_1 T_2^2).$$

(ii) If $T_2$ is nonsingular, then

$$[(c_1 + c_2)^2 - c_3^2] (c_1 T_1 + c_2 T_2 - c_3 T_1 T_2)^{-1}$$

$$= (c_1 + c_2) T_2 - c_3 (2 T_1^2 - T_2 T_1) + c_2^{-1} (c_1^2 + c_1 c_2 - c_3^2) (T_2 - T_2 T_1^2).$$ 

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Proof. We split the proof depending if $T_1$ or $T_2$ is nonsingular.

(i) Let us assume that $T_1$ is nonsingular. First, let us prove the first part of the Theorem. The nonsingularity of $T_1$ implies $T_1^2 = I_n$, hence $T_2^2 T_1 = T_2^2 T_1$ reduces to $T_2^2 T_1 = T_2$. Since $T_2$ is tripotent, there exists a nonsingular $S \in \mathbb{C}^{n \times n}$ such that

$$T_2 = S \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} S^{-1}, \quad A \in \mathbb{C}^{r \times r},$$

being $r$ the rank of $T_2$. Since $A$ is nonsingular and $T_2^2 = T_2$, then $A^2 = I_r$. Let us write

$$T_1 = S \begin{pmatrix} B & C \\ D & E \end{pmatrix} S^{-1}, \quad B \in \mathbb{C}^{r \times r}.$$

From $T_2^2 T_1 = T_2$ we conclude $B = A$ and $C = 0$. Therefore

$$T_1 = S \begin{pmatrix} A & 0 \\ D & E \end{pmatrix} S^{-1}$$

and

$$c_1 T_1 + c_2 T_2 - c_3 T_1 T_2 = S \begin{pmatrix} (c_1 + c_2) A - c_3 I_r & 0 \\ c_1 D - c_3 DA & c_1 E \end{pmatrix} S^{-1}.$$ (2.13)

Since $T_1^2 = I_n$, we obtain that $E$ is nonsingular and $E^2 = I_{n-r}$. From (2.13) we get that $c_1 T_1 + c_2 T_2 - c_3 T_1 T_2$ is nonsingular if and only if $(c_1 + c_2) A - c_3 I_r$ is nonsingular (recall that the first row in the block matrix appearing in (2.13) must be present, since otherwise, $T_2 = 0$).

Since we have

$$[(c_1 + c_2) A - c_3 I_r][c_1 + c_2] A + c_3 I_r] = [(c_1 + c_2)^2 - c_3^2] I_r,$$ (2.14)

we get that if $(c_1 + c_2)^2 - c_3^2 = 0$, then $(c_1 + c_2) A - c_3 I_r$ or $(c_1 + c_2) A + c_3 I_r$ is singular, which in view of (2.13) we get that $c_1 T_1 + c_2 T_2 - c_3 T_1 T_2$ or $c_1 T_1 + c_2 T_2 + c_3 T_1 T_2$ is singular.

Now, let us prove the second part of the Theorem, i.e., we shall prove (2.10) for any $c_1, c_2 \in \mathbb{C}^*$ satisfying $(c_1 + c_2)^2 \neq c_3^2$. From (2.14) we get that $(c_1 + c_2)^2 - c_3^2 \neq 0$ leads to the nonsingularity of $(c_1 + c_2) A - c_3 I_r$ and

$$[(c_1 + c_2) A - c_3 I_r]^{-1} = \frac{1}{(c_1 + c_2)^2 - c_3^2} [(c_1 + c_2) A + c_3 I_r].$$

Since $T_1^2 = I_n$, we have $T_1^{-1} = T_1$ and from (2.12) we have

$$T_1^{-1} = S \begin{pmatrix} A & 0 \\ -E DA & E \end{pmatrix} S^{-1},$$ (2.15)

from them we can conclude that $D = -E DA$, i.e., $-DA = ED$. 

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If \( c_1, c_2 \in \mathbb{C}^* \) satisfy \( (c_1 + c_2)^2 \neq c_3^2 \), then by using (2.13)

\[
(c_1 T_1 + c_2 T_2 - c_3 T_1 T_2)^{-1} =
\begin{bmatrix}
\left[ (c_1 + c_2) A - c_3 I_r \right]^{-1} & 0 \\
-c_1^{-1} E (c_1 D - c_3 DA) & c_1^{-1} E
\end{bmatrix} S^{-1}
\]

\[
= S \begin{bmatrix}
-c_1^{-1} E (c_1 D - c_3 DA) [(c_1 + c_2) A - c_3 I_r]^{-1} & 0 \\
[(c_1 + c_2)^2 - c_3^2]^{-1} [(c_1 + c_2) A + c_3 I_r] & c_1^{-1} E
\end{bmatrix} S^{-1}.
\]

Hence

\[
[(c_1 + c_2)^2 - c_3^2][(c_1 T_1 + c_2 T_2 - c_3 T_1 T_2)^{-1} =
\]

\[
= (c_1 + c_2) S \begin{bmatrix}
A & 0 \\
-E D A & E
\end{bmatrix} S^{-1} + c_3 S \begin{bmatrix}
I_r & 0 \\
0 & 0
\end{bmatrix} S^{-1} + c_1^{-1} c_2 c_3 S \begin{bmatrix}
0 & 0 \\
-2 E D A & 0
\end{bmatrix} S^{-1} + c_1^{-1} c_2 c_3 S \begin{bmatrix}
0 & 0 \\
0 & E
\end{bmatrix} S^{-1}.
\]

(2.16)

On the other hand, we have

\[
T_2^2 = S \begin{bmatrix}
I_r & 0 \\
0 & 0
\end{bmatrix} S^{-1},
\]

(2.17)

\[
T_2^2 - T_1 T_2 = S \begin{bmatrix}
0 & 0 \\
E D & 0
\end{bmatrix} S^{-1},
\]

(2.18)

\[
T_2 - T_1 T_2^2 = S \begin{bmatrix}
0 & 0 \\
E D A & 0
\end{bmatrix} S^{-1},
\]

(2.19)

\[
T_1 - T_1 T_2^2 = S \begin{bmatrix}
0 & 0 \\
0 & E
\end{bmatrix} S^{-1}.
\]

(2.20)

From (2.16)-(2.20), we get that (2.10) holds.

(ii) Assume that \( T_2 \) is nonsingular. First, let us prove the first part of the Theorem. As in (i), but now interchanging the roles of \( T_1 \) and \( T_2 \), we can write

\[
T_1 = S \begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix} S^{-1}, \quad T_2 = S \begin{bmatrix}
A & 0 \\
D & E
\end{bmatrix} S^{-1}, \quad A \in \mathbb{C}^{r \times r},
\]

being \( r \) the rank of \( T_1 \). Since \( T_2^2 = I_n \) we have \( A^2 = I_r \) and \( E^2 = I_{n-r} \). The difference from (i) is

\[
c_1 T_1 + c_2 T_2 \pm c_3 T_1 T_2 = S \begin{bmatrix}
(c_1 + c_2) A & c_3 I_r \\
0 & c_2 E
\end{bmatrix} S^{-1}.
\]

(2.21)

As in (i), if \( (c_1 + c_2)^2 = c_3^2 \), by (2.14) and (2.21), one has that \( c_1 T_1 + c_2 T_2 + c_3 T_1 T_2 \) or \( c_1 T_1 + c_2 T_2 - c_3 T_1 T_2 \) is singular.
Now, we shall prove (2.11) for any \(c_1, c_2 \in \mathbb{C}^*\) satisfying \((c_1 + c_2)^2 \neq c_3^2\). By using (2.21) one has

\[
(c_1T_1 + c_2T_2 - c_3T_1T_2)^{-1} = S \left( \begin{array}{cc}
[(c_1 + c_2)A - c_3I_r]^{-1} & 0 \\
-ED[(c_1 + c_2)A - c_3I_r] - c_2^{-1}E
\end{array} \right) S^{-1}
\]

Hence

\[
((c_1 + c_2)^2 - c_3^2)(c_1T_1 + c_2T_2 - c_3T_1T_2)^{-1} = S \left( \begin{array}{cc}
(c_1 + c_2)A + c_3I_r & 0 \\
-ED[(c_1 + c_2)A + c_3I_r] - c_2^{-1}[(c_1 + c_2)^2 - c_3^2]E
\end{array} \right) S^{-1}
\]

On the other hand, we easily have

\[
2T_1^2 - T_2T_1 = S \left( \begin{array}{cc}
I_r & 0 \\
ED & 0
\end{array} \right) S^{-1}
\]

and

\[
T_2 - T_2T_1^2 = S \left( \begin{array}{cc}
0 & 0 \\
0 & E
\end{array} \right) S^{-1}.
\]

The proof is completed. \(\square\)

If \(c_3 = 0\), then we have the following corollary.

**Corollary 2.2.** [4, Theorem 3.1] Let \(T_1, T_2 \in \mathbb{C}^{n \times n}\) be two nonzero tripotent matrices such that \(T_1^2T_2 = T_2^2T_1\) and \(c_1, c_2 \in \mathbb{C}^*\). If \(T_1\) or \(T_2\) are nonsingular, then \(c_1T_1 + c_2T_2\) is nonsingular if and only if \(c_1 + c_2 \neq 0\). In this case,

(i) If \(T_1\) is nonsingular, then

\[
(c_1 + c_2)(c_1T_1 + c_2T_2)^{-1} = T_1 + c_2c_1^{-1}T_1(I_n - T_2^2).
\]

(ii) If \(T_2\) is nonsingular, then

\[
(c_1 + c_2)(c_1T_1 + c_2T_2)^{-1} = T_2 + c_2c_1^{-1}T_2(I_n - T_1^2).
\]

The following theorem shows that the nonsingularity of \(c_1T_1 + c_2T_2 - c_3T_1T_2\) is also related to the nonsingularity of the same combination of \(T_1^2T_2\) and \(T_2^2T_1\) or \(T_2T_1^2\) and \(T_1T_2^2\).

**Theorem 2.6.** Let \(T_1, T_2 \in \mathbb{C}^{n \times n}\) be two tripotent matrices and any \(c_1, c_2 \in \mathbb{C}^*\). The following statements are equivalent:

(i) \(c_1T_2^2T_1 + c_2T_1^2T_2 - c_3T_2^2T_1T_2\) is nonsingular.
(ii) $c_1 T_1^2 + c_2 T_2^2 - c_3 T_1 T_2^3$ is nonsingular.

(iii) $c_1 T_1 + c_2 T_2 - c_3 T_1 T_2$ and $I_n - T_1^2 - T_2^2$ are nonsingular.

Proof. The results follow quite easily from the equalities

$$(I_n - T_1^2 - T_2^2)(c_1 T_1 + c_2 T_2 - c_3 T_1 T_2) = -(c_1 T_2^2 T_1 + c_2 T_1^2 T_2 - c_3 T_2^2 T_1 T_2)$$

and

$$(c_1 T_1 + c_2 T_2 - c_3 T_1 T_2)(I_n - T_1^2 - T_2^2) = -(c_1 T_1 T_2^2 + c_2 T_2 T_1^2 - c_3 T_1 T_2 T_1^2).$$

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References


