On nonsingularity of combinations of three group invertible matrices and three tripotent matrices

Abstract

Let $\mathbf{T} = c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 - c_4\left(\mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_3\mathbf{T}_1 + \mathbf{T}_2\mathbf{T}_3\right)$, where $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$ are three $n \times n$ tripotent matrices and c_1, c_2, c_3, c_4 are complex numbers with c_1, c_2, c_3 nonzero. In this paper, it is mainly established necessary and sufficient conditions for the nonsingularity of such combinations and obtained some formulae for the inverses of them. Some of these results are given in terms of group invertible matrices.

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1 Introduction and Preliminaries

Let $\mathbb{C}$ be the field of complex numbers and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For a positive integer n, let $\mathcal{M}_n$ be the set of all $n \times n$ complex matrices over $\mathbb{C}$. The symbols $\text{rank}(\mathbf{A})$, $\mathbf{A}^*$, $\mathcal{R}(\mathbf{A})$, and $\mathcal{N}(\mathbf{A})$ stands for the rank, conjugate transpose, the range space, and the null space of $\mathbf{A} \in \mathcal{M}_n$, respectively. Recall that a matrix $\mathbf{A} \in \mathcal{M}_n$ is *idempotent* if $\mathbf{A}^2 = \mathbf{A}$ and *tripotent* if $\mathbf{A}^3 = \mathbf{A}$.

The nonsingularity of linear combinations of idempotent matrices and k-potent matrices was studied in, for example, [1, 2, 4, 6, 9, 15]. The nonsingularities of the combinations $c_1\mathbf{P} + c_2\mathbf{Q} - c_3\mathbf{PQ}$ and $c_1\mathbf{P} + c_2\mathbf{Q} - c_3\mathbf{PQ} - c_4\mathbf{QP} - c_5\mathbf{PQP}$ of two idempotent matrices $\mathbf{P}$, $\mathbf{Q}$ were investigated in [16] and [17], respectively. The considerations of this paper are inspired by Liu et al.[10]. They established necessary and sufficient conditions for the nonsingularity of combinations $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 - c_3\mathbf{T}_1\mathbf{T}_2$ of two trioptent matrices and gave some formulae for the inverse of $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 - c_3\mathbf{T}_1\mathbf{T}_2$ under the some conditions.

Consider a combination of the form

$$\mathbf{T} = c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 - c_4\left(\mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_3\mathbf{T}_1 + \mathbf{T}_2\mathbf{T}_3\right) \tag{1.1}$$

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where c_1, c_2, $c_3 \in \mathbb{C}^*$, $c_4 \in \mathbb{C}$ and $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3 \in \mathcal{M}_n$ are three tripotent matrices. The purpose of this paper is mainly twofold: first, to establish necessary and sufficient conditions for the nonsingularity of combinations of the form (1.1) and then to give some formulae for the inverse of them.

Now, let us give the following additional concepts and properties. For a given matrix $\mathbf{A} \in \mathcal{M}_n$ is said to be *group invertible* if there exists a matrix $\mathbf{X} \in \mathcal{M}_n$ such that

$$\mathbf{AXA} = \mathbf{A}, \qquad \mathbf{XAX} = \mathbf{X}, \qquad \mathbf{AX} = \mathbf{XA}$$

hold. If such an $\mathbf{X} \in \mathcal{M}_n$ exists, then it is unique, customarily denoted by $\mathbf{A}^{\#}$[3]. A matrix $\mathbf{A} \in \mathcal{M}_n$ is group invertible if and only if there exist nonsingular $\mathbf{S} \in \mathcal{M}_n$, $\mathbf{C} \in \mathcal{M}_r$ such that $\mathbf{A} = \mathbf{S}\left(\mathbf{C} \oplus \mathbf{0}\right)\mathbf{S}^{-1}$, r being the rank of $\mathbf{A}$ [12, Exercise 5.10.12]. In this situation, one has $\mathbf{A}^{\#} = \mathbf{S}\left(\mathbf{C}^{-1} \oplus \mathbf{0}\right)\mathbf{S}^{-1}$. This latter representation implies that any diagonalizable matrix is group invertible. Moreover, it is well known that $\mathbf{A} \in \mathcal{M}_n$ is nonsingular if and only if $\mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$. Furthermore, if $\mathbf{A} \in \mathcal{M}_n$ and k is a natural number greater than 1, then $\mathbf{A}$ satisfies $\mathbf{A}^k = \mathbf{A}$ if and only if $\mathbf{A}$ is diagonalizable and the spectrum of $\mathbf{A}$ is contained in $\sqrt[k-1]{1} \cup \{0\}$[5].

Special types of matrices, such as idempotents, tripotents, etc., are very useful in many contexts and they have been extensively studied in the literature. For example, quadratic forms with idempotent matrices are used extensively in statistical theory. So it is worth to stress and spread these kinds of results. Evidently, if $\mathbf{T}$ is a tripotent matrix, then $\mathbf{T}$ is group invertible and $\mathbf{T}^{\#}= \mathbf{T}$. Many of the results given in this work will be given in terms of group invertible matrices.

2 Main Results

Baksalary and Baksalary [1] proved that the nonsingularity of $\mathbf{P}_1 + \mathbf{P}_2$, where $\mathbf{P}_1$ and $\mathbf{P}_2$ are idempotent matrices, is equivalent to the nonsingularity of any linear combinations $c_1\mathbf{P}_1 + c_2\mathbf{P}_2$, where $c_1, c_2 \in \mathbb{C}\backslash\{0\}$ and $c_1 + c_2 \neq 0$. This result was further generalized in [8], where it was proved the stability of the nullity and rank of $c_1\mathbf{P}_1 + c_2\mathbf{P}_2$ for any $c_1, c_2 \in \mathbb{C}\backslash\{0\}$. In the forthcoming results, we give similar results for two and three commuting tripotent matrices. For another related paper concerning this topic, the reader is referred to [15]. We need the following simple lemma whose proof is left to the reader

Lemma 2.1. *Let $\mathbf{A}, \mathbf{B} \in \mathcal{M}_n$ be two group invertible matrices such that there exist nonsingular matrices $\mathbf{S} \in \mathcal{M}_n$, $\mathbf{A}_1, \mathbf{B}_1 \in \mathcal{M}_r$ satisfying $\mathbf{A} = \mathbf{S}(\mathbf{A}_1 \oplus \mathbf{0})\mathbf{S}^{-1}$ and $\mathbf{B} = \mathbf{S}(\mathbf{B}_1 \oplus \mathbf{0})\mathbf{S}^{-1}$. Then $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{B})$ and $\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{B})$.*

Theorem 2.1. *Let $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{M}_n \backslash \{\mathbf{0}\}$ be two commuting tripotent matrices and $c_1, c_2 \in \mathbb{C}^*$ such that $c_1^2 - c_2^2 \neq 0$. Then $\mathcal{R}(\mathbf{T}_1^2 + \mathbf{T}_2^2) = \mathcal{R}(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)$, $\mathcal{N}(\mathbf{T}_1^2 + \mathbf{T}_2^2) = \mathcal{N}(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)$, $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ is group invertible and*

$$(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)^{\#} = \frac{c_2^2}{c_1(c_1^2 - c_2^2)}\mathbf{T}_1\mathbf{T}_2^2 + \frac{c_1^2}{c_2(c_2^2 - c_1^2)}\mathbf{T}_2\mathbf{T}_1^2 + \frac{1}{c_1}\mathbf{T}_1 + \frac{1}{c_2}\mathbf{T}_2. \tag{2.1}$$

In particular, If $\mathbf{T}_1^2 + \mathbf{T}_2^2$ is nonsingular, then $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ is nonsingular and $(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)^{-1}$ is given by (2.1).

Proof. Let $p = \operatorname{rank}(\mathbf{T}_1\mathbf{T}_2)$, $q = \operatorname{rank}(\mathbf{T}_1)$, and $r = \operatorname{rank}(\mathbf{T}_2)$. Since $\mathbf{T}_1$ and $\mathbf{T}_2$ are diagonalizable and commuting, there exists a nonsingular $\mathbf{S} \in \mathcal{M}_n$ such that

$$\mathbf{T}_1 = \mathbf{S}(\mathbf{A}_1 \oplus \mathbf{B}_1 \oplus \mathbf{0} \oplus \mathbf{0})\mathbf{S}^{-1}, \qquad \mathbf{T}_2 = \mathbf{S}(\mathbf{A}_2 \oplus \mathbf{0} \oplus \mathbf{B}_2 \oplus \mathbf{0})\mathbf{S}^{-1}, \tag{2.2}$$

being $\mathbf{A}_1, \mathbf{A}_2 \in \mathcal{M}_p$, $\mathbf{B}_1 = \mathcal{M}_{q-p}$, $\mathbf{B}_2 \in \mathcal{M}_{r-p}$, and $\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_1, \mathbf{B}_2$ nonsingular. By using $\mathbf{T}_1^3 = \mathbf{T}_1$ and $\mathbf{T}_2^3 = \mathbf{T}_2$ one gets $\mathbf{A}_1^2 = \mathbf{A}_2^2 = \mathbf{I}_p$, $\mathbf{B}_1^2 = \mathbf{I}_{q-p}$, and $\mathbf{B}_2^2 = \mathbf{I}_{r-p}$. Therefore,

$$\mathbf{T}_1^2 + \mathbf{T}_2^2 = \mathbf{S}(2\mathbf{I}_p \oplus \mathbf{I}_{q-p} \oplus \mathbf{I}_{r-p} \oplus \mathbf{0})\mathbf{S}^{-1}.$$

By considering the equality

$$(c_1\mathbf{A}_1 + c_2\mathbf{A}_2)(c_1\mathbf{A}_1 - c_2\mathbf{A}_2) = (c_1^2 - c_2^2)\mathbf{I}_p, \tag{2.3}$$

we get the nonsingularity of $c_1\mathbf{A}_1 + c_2\mathbf{A}_2$. Since

$$c_1\mathbf{T}_1 + c_2\mathbf{T}_2 = \mathbf{S}\left[(c_1\mathbf{A}_1 + c_2\mathbf{A}_2) \oplus c_1\mathbf{B}_1 \oplus c_2\mathbf{B}_2 \oplus \mathbf{0}\right]\mathbf{S}^{-1}, \tag{2.4}$$

and by applying Lemma 2.1 to matrices $\mathbf{T}_1^2 + \mathbf{T}_2^2$ and $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ we obtain the equality of the range spaces and null spaces of this theorem. Also, $\mathbf{B}_1^2 = \mathbf{I}_{q-p}$, $\mathbf{B}_2^2 = \mathbf{I}_{r-p}$, the expression (2.4), and [12, Exercise 5.10.12] permit assure that $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ is group invertible and

$$(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)^{\#} = \mathbf{S}\left[(c_1\mathbf{A}_1 + c_2\mathbf{A}_2)^{-1} \oplus c_1^{-1}\mathbf{B}_1 \oplus c_2^{-1}\mathbf{B}_2 \oplus \mathbf{0}\right]\mathbf{S}^{-1}$$

Now we use the equality (2.3):

$$\begin{aligned}
\left[(c_1\mathbf{A}_1 + c_2\mathbf{A}_2)^{-1} \oplus \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{0}\right] &= \frac{1}{c_1^2 - c_2^2}\left[(c_1\mathbf{A}_1 - c_2\mathbf{A}_2) \oplus \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{0}\right] \\
&= \frac{1}{c_1^2 - c_2^2}\left[c_1(\mathbf{A}_1 \oplus \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{0}) - c_2(\mathbf{A}_2 \oplus \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{0})\right] \\
&= \frac{1}{c_1^2 - c_2^2}\left[c_1\mathbf{S}^{-1}\mathbf{T}_1\mathbf{T}_2^2\mathbf{S} - c_2\mathbf{S}^{-1}\mathbf{T}_1^2\mathbf{T}_2\mathbf{S}\right].
\end{aligned}$$

In addition we have $\mathbf{S}(\mathbf{0} \oplus \mathbf{B}_1 \oplus \mathbf{0} \oplus \mathbf{0})\mathbf{S}^{-1} = \mathbf{T}_1(\mathbf{I}_n - \mathbf{T}_2^2)$ and $\mathbf{S}(\mathbf{0} \oplus \mathbf{0} \oplus \mathbf{B}_2 \oplus \mathbf{0})\mathbf{S}^{-1} = \mathbf{T}_2(\mathbf{I}_n - \mathbf{T}_1^2)$. Therefore

$$(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)^{\#} = \frac{1}{c_1^2 - c_2^2}\left[c_1\mathbf{T}_1\mathbf{T}_2^2 - c_2\mathbf{T}_1^2\mathbf{T}_2\right] + \frac{1}{c_1}\mathbf{T}_1(\mathbf{I}_n - \mathbf{T}_2^2) + \frac{1}{c_2}\mathbf{T}_2(\mathbf{I}_n - \mathbf{T}_1^2).$$

By simplifying this last equality, one can gets (2.1). □

The proof of Theorem 2.1 permits affirm that if $\mathbf{T}_1\mathbf{T}_2 = \mathbf{0}$, then the first summand in the two direct sums appearing in (2.2) are absent and hence we can deduce the following corollary:

Corollary 2.1. *Let $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{M}_n \setminus \{\mathbf{0}\}$ be two commuting tripotent matrices satisfying $\mathbf{T}_1\mathbf{T}_2 = \mathbf{0}$ and let $c_1, c_2 \in \mathbb{C}^*$. Then $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ is group invertible and*

$$(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)^{\#} = \frac{1}{c_1}\mathbf{T}_1 + \frac{1}{c_2}\mathbf{T}_2.$$

Remark 2.1. Observe that $\mathbf{T}_1^2 + \mathbf{T}_2^2$ is nonsingular if and only if $\operatorname{rank}(\mathbf{T}_1) + \operatorname{rank}(\mathbf{T}_2) = n + \operatorname{rank}(\mathbf{T}_1\mathbf{T}_2)$. In fact, from the representation (2.2) we have

$\mathbf{T}_1^2 + \mathbf{T}_2^2$ is nonsingular $\Leftrightarrow p + (q-p) + (r-p) = n \Leftrightarrow \operatorname{rank}(\mathbf{T}_1) + \operatorname{rank}(\mathbf{T}_2) = n + \operatorname{rank}(\mathbf{T}_1\mathbf{T}_2)$.

The following simple pair of equalities will be useful to prove next result: If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C} \in \mathcal{M}_n$ satisfy $\mathbf{A}^2 = \mathbf{B}^2 = \mathbf{C}^2 = \mathbf{I}_n$ and they are mutually commuting, then

$$(a\mathbf{A} + b\mathbf{B} + c\mathbf{C})(x\mathbf{A} + y\mathbf{B} + z\mathbf{C} + w\mathbf{ABC}) = (a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2)\mathbf{I}_n, \quad (2.5)$$

where $x = a^3 - ab^2 - ac^2$, $y = b^3 - bc^2 - ba^2$, $z = c^3 - ca^2 - cb^2$, $w = 2abc$, and a, b, c are arbitrary nonzero complex numbers. Furthermore,

$$a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2 = (a + b + c)(a + b - c)(a - b + c)(a - b - c)$$

holds. In addition, the following simple lemma (whose proof is left to the reader) will help us to prove Theorem 2.2 below

Lemma 2.2. *Let $\mathbf{B}_i \in M_{n_i}$ for $i = 1, \ldots, m$, $n = n_1 + \cdots + n_m$, a nonsingular $\mathbf{S} \in M_n$. If we define $\mathbf{A}_i = \mathbf{S}(\mathbf{0} \oplus \cdots \oplus \mathbf{0} \oplus \mathbf{B}_i \oplus \mathbf{0} \oplus \cdots \oplus \mathbf{0})\mathbf{S}^{-1}$, where the summand $\mathbf{B}_i$ is on the ith position, and $\mathbf{A} = \mathbf{S}(\mathbf{B}_1 \oplus \cdots \oplus \mathbf{B}_m)\mathbf{S}^{-1}$, then*

$$\bigcap_{i=1}^{m} \mathcal{N}(\mathbf{A}_i) = \mathcal{N}(\mathbf{A}) \quad \text{and} \quad \sum_{i=1}^{m} \mathcal{R}(\mathbf{A}_i) = \mathcal{R}(\mathbf{A}).$$

In addition, if $\mathbf{B}_1, \ldots, \mathbf{B}_m$ are group invertible, then $\mathbf{A}$ is also group invertible and $\mathbf{A}^{\#} = \mathbf{S}(\mathbf{B}_1^{\#} \oplus \cdots \oplus \mathbf{B}_m^{\#})\mathbf{S}^{-1}$.

Theorem 2.2. *Let $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3 \in \mathcal{M}_n \setminus \{\mathbf{0}\}$ be three mutually commuting tripotent matrices and $c_1, c_2, c_3 \in \mathbb{C}^*$ such that $c_2^2 - c_3^3, c_1^2 - c_3^2, c_1^2 - c_2^2, c_1 + c_2 + c_3, c_1 + c_2 - c_3, c_1 - c_2 +$*

$c_3, c_1 - c_2 - c_3 \neq 0$. Then $\mathcal{R}(\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2) = \mathcal{R}(c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3)$, $\mathcal{N}(\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2) = \mathcal{N}(c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3)$, $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3$ is group invertible, and

$$\begin{aligned}(c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3)^{\#} &= q(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)\mathbf{T}_1^2\mathbf{T}_2^2\mathbf{T}_3^2 + p_{c_1,c_2}(\mathbf{T}_1, \mathbf{T}_2)\mathbf{T}_1^2\mathbf{T}_2^2(\mathbf{I}_n - \mathbf{T}_3^2) \\ &+ p_{c_1,c_3}(\mathbf{T}_1, \mathbf{T}_3)\mathbf{T}_1^2(\mathbf{I}_n - \mathbf{T}_2^2) + p_{c_2,c_3}(\mathbf{T}_2, \mathbf{T}_3)(\mathbf{I}_n - \mathbf{T}_3^2),\end{aligned} \tag{2.6}$$

where $p_{a,b} : \mathbb{C}^2 \to \mathbb{C}$ and $q : \mathbb{C}^3 \to \mathbb{C}$ are the following complex polynomials,

$$p_{a,b}(z,w) = \frac{b^2}{a(a^2-b^2)}zw^2 + \frac{a^2}{b(a^2-b^2)}z^2w + \frac{1}{a}z + \frac{1}{b}w, \qquad (a, b \in \mathbb{C}, a^2 \neq b^2),$$

$$q(z,w,u) = \frac{(c_1^3 - c_1c_2^2 - c_1c_3^2)z + (c_2^3 - c_2c_3^2 - c_2c_1^2)w + (c_3^3 - c_3c_1^2 - c_3c_2^2)u}{(c_1+c_2+c_3)(c_1+c_2-c_3)(c_1-c_2+c_3)(c_1-c_2-c_3)}.$$

In particular, if $\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2$ is nonsingular, then $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3$ is nonsingular and $(c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3)^{-1}$ is given by (2.6).

Proof. By [12, Exercise 5.10.12], there exist nonsingular matrices $\mathbf{S}_1 \in \mathcal{M}_n$ and $\mathbf{X}_1 \in \mathcal{M}_{n-t}$ such that $\mathbf{T}_1 = \mathbf{S}_1(\mathbf{X}_1 \oplus \mathbf{0})\mathbf{S}_1^{-1}$. The tripotency of $\mathbf{T}_1$ and the nonsingularity of $\mathbf{X}_1$ leads to $\mathbf{X}_1^2 = \mathbf{I}_{n-t}$. As $\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_2\mathbf{T}_1$ and $\mathbf{T}_1\mathbf{T}_3 = \mathbf{T}_3\mathbf{T}_1$, we can write matrices $\mathbf{T}_2$ and $\mathbf{T}_3$ as follows

$$\mathbf{T}_2 = \mathbf{S}_1 \begin{pmatrix} \mathbf{X}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_2 \end{pmatrix} \mathbf{S}_1^{-1}, \qquad \mathbf{T}_3 = \mathbf{S}_1 \begin{pmatrix} \mathbf{X}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_3 \end{pmatrix} \mathbf{S}_1^{-1}, \qquad \mathbf{D}_2, \mathbf{D}_3 \in \mathcal{M}_t,$$

with

$$\mathbf{X}_1\mathbf{X}_2 = \mathbf{X}_2\mathbf{X}_1, \qquad \mathbf{X}_1\mathbf{X}_3 = \mathbf{X}_3\mathbf{X}_1. \tag{2.7}$$

Let us notice that matrices $\mathbf{X}_2, \mathbf{X}_3, \mathbf{D}_2, \mathbf{D}_3$ are tripotent because $\mathbf{T}_2$ and $\mathbf{T}_3$ are tripotent. By applying again exercise [12, Exercise 5.10.12], there exist nonsingular matrices $\mathbf{S}_2 \in \mathcal{M}_{n-t}$ and $\mathbf{Y}_2 \in \mathcal{M}_{n-t-s}$ such that $\mathbf{X}_2 = \mathbf{S}_2(\mathbf{Y}_2 \oplus \mathbf{0})\mathbf{S}_2^{-1}$. From (2.7) we can write

$$\mathbf{X}_1 = \mathbf{S}_2 \begin{pmatrix} \mathbf{Y}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_1 \end{pmatrix} \mathbf{S}_2^{-1}, \qquad \mathbf{X}_3 = \mathbf{S}_2 \begin{pmatrix} \mathbf{Y}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_3 \end{pmatrix} \mathbf{S}_2^{-1}.$$

Observe that $\mathbf{Y}_1^2 = I_{n-t-s}$, $\mathbf{C}_1^2 = I_s$, $\mathbf{Y}_3^3 = \mathbf{Y}_3$, and $\mathbf{C}_3^3 = \mathbf{C}_3$.

Finally, utilize again [12, Exercise 5.10.12] to matrix $\mathbf{Y}_3$ to obtain nonsingular matrices $\mathbf{S}_3 \in \mathcal{M}_{n-t-s}$ and $\mathbf{A}_3 \in \mathcal{M}_{n-t-s-r}$ such that $\mathbf{Y}_3 = \mathbf{S}_3(\mathbf{A}_3 \oplus \mathbf{0})\mathbf{S}_3^{-1}$. By carrying out the same routine as before, we can write

$$\mathbf{Y}_1 = \mathbf{S}_3 \begin{pmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_1 \end{pmatrix} \mathbf{S}_3^{-1}, \qquad \mathbf{Y}_2 = \mathbf{S}_3 \begin{pmatrix} \mathbf{A}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{pmatrix} \mathbf{S}_3^{-1}.$$

Let us define $m = n - t - s - r$. By setting $\mathbf{S} = \mathbf{S}_1(\mathbf{S}_2 \oplus \mathbf{I}_t)(\mathbf{S}_3 \oplus \mathbf{I}_s \oplus \mathbf{I}_t)$, one easily has

$$\mathbf{T}_1 = \mathbf{S}(\mathbf{A}_1 \oplus \mathbf{B}_1 \oplus \mathbf{C}_1 \oplus \mathbf{0})\mathbf{S}^{-1}, \qquad \mathbf{T}_2 = \mathbf{S}(\mathbf{A}_2 \oplus \mathbf{B}_2 \oplus \mathbf{0} \oplus \mathbf{D}_2)\mathbf{S}^{-1},$$

$$\mathbf{T}_3 = \mathbf{S}(\mathbf{A}_3 \oplus \mathbf{0} \oplus \mathbf{C}_3 \oplus \mathbf{D}_3)\mathbf{S}^{-1}.$$

and the matrices $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{B}_1, \mathbf{B}_2$, and $\mathbf{C}_1$ are nonsingular. Observe that the tripotency of $\mathbf{T}_i$ leads to the tripotency of these matrices $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$, and $\mathbf{D}_i$. Furthermore, since $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{B}_1, \mathbf{B}_2$, and $\mathbf{C}_1$ are nonsingular, then $\mathbf{A}_i^2 = \mathbf{I}_m$ (for $i = 1, 2, 3$), $\mathbf{B}_i^2 = \mathbf{I}_r$ (for $i = 1, 2$) and $\mathbf{C}_1^2 = \mathbf{I}_s$. In addition, the families $\{\mathbf{A}_i\}_{i=1,2,3}$, $\{\mathbf{B}_i\}_{i=1,2}$, $\{\mathbf{C}_i\}_{i=1,3}$, and $\{\mathbf{D}_i\}_{i=2,3}$ are commutative.

Observe that

$$\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2 = \mathbf{S}\left(3\mathbf{I}_m \oplus (\mathbf{B}_1^2 + \mathbf{B}_2^2) \oplus (\mathbf{C}_1^2 + \mathbf{C}_3^2) \oplus (\mathbf{D}_2^2 + \mathbf{D}_3^2)\right)\mathbf{S}^{-1} \tag{2.8}$$

and

$$\begin{aligned} & c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 \\ & \quad = \mathbf{S}\left((c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + c_3\mathbf{A}_3) \oplus (c_1\mathbf{B}_1 + c_2\mathbf{B}_2) \oplus (c_1\mathbf{C}_1 + c_3\mathbf{C}_3) \oplus (c_2\mathbf{D}_2 + c_3\mathbf{D}_3)\right)\mathbf{S}^{-1}. \end{aligned} \tag{2.9}$$

By the equality given in (2.5) we have that $c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + c_3\mathbf{A}_3$ is nonsingular and

$$(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + c_3\mathbf{A}_3)^{-1} = q(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3).$$

Since $c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + c_3\mathbf{A}_3$ is nonsingular, then $\mathcal{N}(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + c_3\mathbf{A}_3) = \mathcal{N}(3\mathbf{I}_m)$ and $\mathcal{R}(c_1\mathbf{A}_1 + c_2\mathbf{A}_2 + c_3\mathbf{A}_3) = \mathcal{R}(3\mathbf{I}_m)$. Theorem 2.1 leads to $\mathcal{N}(c_1\mathbf{B}_1 + c_2\mathbf{B}_2) = \mathcal{N}(\mathbf{B}_1^2 + \mathbf{B}_2^2)$, $\mathcal{N}(c_1\mathbf{C}_1 + c_3\mathbf{C}_3) = \mathcal{N}(\mathbf{C}_1^2 + \mathbf{C}_3^2)$, $\mathcal{N}(c_2\mathbf{D}_2 + c_3\mathbf{D}_3) = \mathcal{N}(\mathbf{D}_2^2 + \mathbf{D}_3^2)$, and analogous identities for the range space. By considering (2.8), (2.9), and the first part of Lemma 2.2 we get that the null space (range space) of $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3$ equals to the null space (range space) $\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2$.

By Theorem 2.1 we have the group invertibility of $c_1\mathbf{B}_1 + c_2\mathbf{B}_2$, $c_1\mathbf{C}_1 + c_3\mathbf{C}_3$, and $c_2\mathbf{D}_2 + c_3\mathbf{D}_3$. Also we get

$$(c_1\mathbf{B}_1 + c_2\mathbf{B}_2)^{\#} = p_{c_1,c_2}(\mathbf{B}_1, \mathbf{B}_2), \qquad (c_1\mathbf{C}_1 + c_3\mathbf{C}_3)^{\#} = p_{c_1,c_3}(\mathbf{C}_1, \mathbf{C}_3),$$

and

$$(c_2\mathbf{D}_2 + c_3\mathbf{D}_3)^{\#} = p_{c_2,c_3}(\mathbf{D}_2, \mathbf{D}_3).$$

The second part of Lemma 2.2 leads to the group invertibility of $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3$ and

$$\begin{aligned} & (c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3)^{\#} \\ & \quad = \mathbf{S}\left[q(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) \oplus p_{c_1,c_2}(\mathbf{B}_1, \mathbf{B}_2) \oplus p_{c_1,c_3}(\mathbf{C}_1, \mathbf{C}_3) \oplus p_{c_2,c_3}(\mathbf{D}_2, \mathbf{D}_3)\right]\mathbf{S}^{-1}. \end{aligned} \tag{2.10}$$

Now, observe that

$$\begin{aligned} \mathbf{S}\left[q(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) \oplus \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{0}\right]\mathbf{S}^{-1} &= q(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)\mathbf{S}(\mathbf{I}_m \oplus \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{0})\mathbf{S}^{-1} \\ &= q(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)\mathbf{T}_1^2\mathbf{T}_2^2\mathbf{T}_3^3. \end{aligned} \tag{2.11}$$

Since $\mathbf{S}(\mathbf{0} \oplus \mathbf{I}_r \oplus \mathbf{0} \oplus \mathbf{0})\mathbf{S}^{-1} = \mathbf{T}_1^2\mathbf{T}_2^2 - \mathbf{T}_1^2\mathbf{T}_2^2\mathbf{T}_3^2 = \mathbf{T}_1^2\mathbf{T}_2^2(\mathbf{I}_n - \mathbf{T}_3^2)$, we have

$$\mathbf{S}\left[\mathbf{0} \oplus p_{c_1,c_2}(\mathbf{B}_1, \mathbf{B}_2) \oplus \mathbf{0} \oplus \mathbf{0}\right]\mathbf{S}^{-1} = p_{c_1,c_2}(\mathbf{T}_1, \mathbf{T}_2)\mathbf{T}_1^2\mathbf{T}_2^2(\mathbf{I}_n - \mathbf{T}_3^2). \tag{2.12}$$

Another two useful idempotents are the following two matrices: $\mathbf{S}(\mathbf{0} \oplus \mathbf{0} \oplus \mathbf{I}_s \oplus \mathbf{0})\mathbf{S}^{-1} = \mathbf{T}_1^2 - \mathbf{T}_1^2\mathbf{T}_1^2 = \mathbf{T}_1^2(\mathbf{I}_n - \mathbf{T}_2^2)$ and $\mathbf{S}(\mathbf{0} \oplus \mathbf{0} \oplus \mathbf{0} \oplus \mathbf{I}_t)\mathbf{S}^{-1} = \mathbf{I}_n - \mathbf{T}_1^2$. Thus we have

$$\mathbf{S}\left[\mathbf{0} \oplus \mathbf{0} \oplus p_{c_1,c_3}(\mathbf{C}_1, \mathbf{C}_3) \oplus \mathbf{0}\right]\mathbf{S}^{-1} = p_{c_1,c_3}(\mathbf{T}_1, \mathbf{T}_3)\mathbf{T}_1^2(\mathbf{I}_n - \mathbf{T}_2^2) \tag{2.13}$$

and

$$\mathbf{S}\left[\mathbf{0} \oplus \mathbf{0} \oplus \mathbf{0} \oplus p_{c_2,c_3}(\mathbf{D}_2, \mathbf{D}_3)\right]\mathbf{S}^{-1} = p_{c_2,c_3}(\mathbf{T}_2, \mathbf{T}_3)(\mathbf{I}_n - \mathbf{T}_1^2). \tag{2.14}$$

Considering (2.10)–(2.14) finishes the proof. □

As we already pointed out, in this paper, similar results to the ones obtained in [10] are established for three tripotent or group invertible matrices.

Theorem 2.3. *Let* $\mathbf{T}_1$, $\mathbf{T}_2$, *and* $\mathbf{T}_3 \in \mathcal{M}_n$ *be three mutually commuting tripotent matrices. Then* $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$ *is nonsingular if and only if* $\mathbf{I}_n + \mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_2\mathbf{T}_3 + \mathbf{T}_3\mathbf{T}_1 + \mathbf{T}_1\mathbf{T}_2\mathbf{T}_3$ *and* $\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2$ *are nonsingular.*

Proof. Since $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$ are tripotent and mutually commutating, they are simultaneously diagonalizable (see, e.g., [7, page 52]). Hence there is a single similarity matrix $\mathbf{S} \in \mathcal{M}_n$ such that $\mathbf{T}_1 = \mathbf{S}\,\mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n)\,\mathbf{S}^{-1}$, $\mathbf{T}_2 = \mathbf{S}\,\mathrm{diag}\,(\mu_1, \mu_2, \ldots, \mu_n)\,\mathbf{S}^{-1}$ and

$\mathbf{T}_3 = \mathbf{S}\,\mathrm{diag}\,(\gamma_1, \gamma_2, \ldots, \gamma_n)\,\mathbf{S}^{-1}$ being $\{\lambda_i\}_{i=1}^n$, $\{\mu_i\}_{i=1}^n$ and $\{\gamma_i\}_{i=1}^n$ the sets of eigenvalues of $\mathbf{T}_1, \mathbf{T}_2$ and $\mathbf{T}_3$, with proper multiplicities, respectively. On the other hand,

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 = \mathbf{S}\,\mathrm{diag}\,(\lambda_1 + \mu_1 + \gamma_1, \ldots, \lambda_n + \mu_n + \gamma_n)\,\mathbf{S}^{-1}, \tag{2.15}$$

$$\mathbf{I}_n + \mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_2\mathbf{T}_3 + \mathbf{T}_3\mathbf{T}_1 + \mathbf{T}_1\mathbf{T}_2\mathbf{T}_3 = \mathbf{S}\,\mathrm{diag}\,(p(\lambda_1, \mu_1, \gamma_1), \ldots, p(\lambda_n, \mu_n, \gamma_n))\,\mathbf{S}^{-1}, \tag{2.16}$$

and

$$\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2 = \mathbf{S}\,\mathrm{diag}\left(\lambda_1^2 + \mu_1^2 + \gamma_1^2, \ldots, \lambda_n^2 + \mu_n^2 + \gamma_n^2\right)\mathbf{S}^{-1}, \tag{2.17}$$

where $p : \mathbb{C}^3 \to \mathbb{C}$ is given by $p(z, w, u) = 1 + zw + wu + uz + zwu$.

Assume that $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$ is nonsingular. From (2.15), we get $\lambda_i + \mu_i + \gamma_i \neq 0$ for any $i = 1, \ldots, n$ and hence

$$(\lambda_i, \mu_i, \gamma_i) \in \Phi^3 \setminus \{(-1,1,0), (0,-1,1), (-1,0,1), (0,0,0), (1,0,-1), (0,1,-1), (1,-1,0)\}$$

for all $i = 1, 2, \ldots, n$, where $\Phi = \{-1, 0, 1\}$. Therefore, it is obtained that $p(\lambda_i, \mu_i, \gamma_i) \neq 0$ and $\lambda_i^2 + \mu_i^2 + \gamma_i^2 \neq 0$ for all $i = 1, 2, \ldots, n$. In view of (2.16) and (2.17) it is seen that $\mathbf{I}_n + \mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_2\mathbf{T}_3 + \mathbf{T}_3\mathbf{T}_1 + \mathbf{T}_1\mathbf{T}_2\mathbf{T}_3$ and $\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2$ are nonsingular.

Now, assume that $\mathbf{I}_n + \mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_2\mathbf{T}_3 + \mathbf{T}_3\mathbf{T}_1 + \mathbf{T}_1\mathbf{T}_2\mathbf{T}_3$ and $\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2$ are nonsingular. From the nonsingularity of the first matrix we get

$$1 + \lambda_i\mu_i + \mu_i\gamma_i + \gamma_i\lambda_i + \lambda_i\mu_i\gamma_i \neq 0 \qquad \text{for all } i = 1, 2, \ldots, n.$$

If $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$ were singular, then there would exist some $j \in \{1, 2, \ldots, n\}$ such that $\lambda_j + \mu_j + \gamma_j = 0$. So, the unique solution satisfying simultaneously these two equations would be $(\lambda_j, \mu_j, \gamma_j) = (0, 0, 0)$. Hence, $\lambda_j^2 + \mu_j^2 + \gamma_j^2 = 0$ which would contradict to the assumption of the nonsingularity of $\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2$. So the proof is complete. □

Remark 2.2. It is evident that for a given $\mathbf{X} \in \mathcal{M}_n$, then $\mathbf{X}$ is tripotent if and only if $-\mathbf{X}$ is tripotent. Thus, by means of Theorem 2.3, we can characterize the nonsingularity of $\varepsilon_1\mathbf{T}_1 + \varepsilon_2\mathbf{T}_2 + \varepsilon_3\mathbf{T}_3$, where $\varepsilon_1, \varepsilon_1, \varepsilon_1 \in \{-1, 1\}$ and $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3 \in \mathcal{M}_n$ are tripotent matrices.

Remark 2.3. Let $p : \mathbb{C}^3 \longrightarrow \mathbb{C}$ be the following complex polynomial:

$$p\,(z, w, t) = \sum_{\substack{i,j,k=0 \\ (i,j,k)\neq(0,0,0)}}^{m} c_{i,j,k} z^i w^j t^k, \tag{2.18}$$

where $m \in \mathbb{Z}^+$, $c_{i,j,k} \in \mathbb{C}$. Let $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3 \in \mathcal{M}_n$ be three mutually commuting tripotent matrices. Then,

$$p\,(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) = \mathbf{S}\,\mathrm{diag}\,[p\,(\lambda_1, \mu_1, \gamma_1), \ldots, p\,(\lambda_n, \mu_n, \gamma_n)]\,\mathbf{S}^{-1}.$$

If $\mathbf{T}_1^2 + \mathbf{T}_2^2 + \mathbf{T}_3^2$ were singular, then there would exist $j \in \{1, \ldots, n\}$ satisfying $\lambda_j^2 + \mu_j^2 + \gamma_j^2 = 0$. Therefore, $\lambda_j = \mu_j = \gamma_j = 0$. So, $p(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)$ is singular because $p(0, 0, 0) = 0$.

Hence, the following corollary can be given.

Corollary 2.2. *Let $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3 \in \mathcal{M}_n$ be three mutually commuting tripotent matrices. If $\mathbf{I}_n + \mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_2\mathbf{T}_3 + \mathbf{T}_3\mathbf{T}_1 + \mathbf{T}_1\mathbf{T}_2\mathbf{T}_3$ is nonsingular and there exists a polynomial p as in (2.18) such that $p\,(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)$ is nonsingular, then $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$ is nonsingular.*

The next theorem is presented under weaker assumptions than the previous theorem.

Theorem 2.4. *Let $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3 \in \mathcal{M}_n$ such that $\mathbf{T}_1$ is group invertible and $\mathbf{I}_n - \mathbf{T}_1^{\#}\mathbf{T}_2 - \mathbf{T}_1^{\#}\mathbf{T}_3$ is nonsingular. If one of the below conditions holds,*

(i) *if $\mathbf{T}_2\mathbf{T}_1\mathbf{T}_1^{\#} = \mathbf{T}_2$, $\mathbf{T}_3\mathbf{T}_1\mathbf{T}_1^{\#} = \mathbf{T}_3$, and there exists a polynomial p in three variables not necessarily commutatative such that $p(0, 0, 0) = 0$ and $p(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)$ is nonsingular,*

(ii) *if* $\mathbf{T}_2\mathbf{T}_1\mathbf{T}_1^{\#} = \mathbf{T}_1\mathbf{T}_1^{\#}\mathbf{T}_2$, $\mathbf{T}_3\mathbf{T}_1\mathbf{T}_1^{\#} = \mathbf{T}_3$, *and there exists a polynomial* p *in three variables not necessarily commutatative such that* $p(0,0,0) = 0$ *and* $p(\mathbf{T}_1, \mathbf{T}_1\mathbf{T}_2, \mathbf{T}_3)$ *is nonsingular,*

(iii) *if* $\mathbf{T}_2\mathbf{T}_1\mathbf{T}_1^{\#} = \mathbf{T}_1\mathbf{T}_1^{\#}\mathbf{T}_2$, $\mathbf{T}_3\mathbf{T}_1\mathbf{T}_1^{\#} = \mathbf{T}_1\mathbf{T}_1^{\#}\mathbf{T}_3$, *and there exists a polynomial* p *in three variables not necessarily commutatative such that* $p(0,0,0) = 0$ *and* $p(\mathbf{T}_1, \mathbf{T}_1\mathbf{T}_2, \mathbf{T}_1\mathbf{T}_3)$ *is nonsingular,*

then $\mathbf{T}_1 - \mathbf{T}_2 - \mathbf{T}_3$ *is nonsingular.*

Proof. Let $\mathbf{x} \in \mathcal{N}(\mathbf{T}_1 - \mathbf{T}_2 - \mathbf{T}_3)$, i.e.,

$$\mathbf{T}_1\mathbf{x} = (\mathbf{T}_2 + \mathbf{T}_3)\mathbf{x}. \tag{2.19}$$ [t1x]

(i) Assume that the conditions given in (i) are satisfied. Premultiplying (2.19) by $\mathbf{T}_1\mathbf{T}_1^{\#}$, $\mathbf{T}_2\mathbf{T}_1^{\#}$, $\mathbf{T}_3\mathbf{T}_1^{\#}$, it is obtained $\mathbf{T}_1\mathbf{x} = \mathbf{T}_1\mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3)\mathbf{x}$, $\mathbf{T}_2\mathbf{x} = \mathbf{T}_2\mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3)\mathbf{x}$, and $\mathbf{T}_3\mathbf{x} = \mathbf{T}_3\mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3)\mathbf{x}$, respectively. If these equations are reorganized, we get

$$\mathbf{T}_1\left(\mathbf{I}_n - \mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3\right)\mathbf{x} = \mathbf{T}_2\left(\mathbf{I}_n - \mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3\right)\mathbf{x} = \mathbf{T}_2\left(\mathbf{I}_n - \mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3\right)\mathbf{x} = \mathbf{0}. \tag{2.20}$$ [T1T3T1T2]

There exists three polynomials in three variables not necessarily commutative, say p_1, p_2, and p_3, such that $p(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3) = p_1(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)\mathbf{T}_1 + p_2(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)\mathbf{T}_2 + p_3(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)\mathbf{T}_3$. Thus from (2.20) it is obtained

$$\begin{aligned} &p(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)\left[\mathbf{I}_n - \mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3\right]\mathbf{x} \\ &\quad = [p_1(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)\mathbf{T}_1 + p_2(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)\mathbf{T}_2 + p_3(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)\mathbf{T}_3]\left[\mathbf{I}_n - \mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3)\right]\mathbf{x} \\ &\quad = \mathbf{0}. \end{aligned}$$

Under the assumption that $\mathbf{I}_n - \mathbf{T}_1^{\#}\mathbf{T}_2 - \mathbf{T}_1^{\#}\mathbf{T}_3$ and $p(\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3)$ are nonsingular, the above computation yields $\mathbf{x} = \mathbf{0}$, which means that $\mathbf{T}_1 - \mathbf{T}_2 - \mathbf{T}_3$ is nonsingular. So the proof of item (i) is complete.

(ii) By premultiplying (2.19) by $\mathbf{T}_1\mathbf{T}_1^{\#}$, $\mathbf{T}_1\mathbf{T}_2\mathbf{T}_1^{\#}$, and $\mathbf{T}_3\mathbf{T}_1^{\#}$ it follows that $\mathbf{T}_1\mathbf{x} = \mathbf{T}_1\mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3)\mathbf{x}$, $\mathbf{T}_1\mathbf{T}_2\mathbf{x} = \mathbf{T}_1\mathbf{T}_2\mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3)\mathbf{x}$, and $\mathbf{T}_3\mathbf{x} = \mathbf{T}_3\mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3)\mathbf{x}$, respectively. From these identities we obtain

$$\mathbf{T}_1\left(\mathbf{I}_n - \mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3\right)\mathbf{x} = \mathbf{T}_1\mathbf{T}_2\left(\mathbf{I}_n - \mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3\right)\mathbf{x} = \mathbf{T}_3\left(\mathbf{I}_n - \mathbf{T}_1^{\#}(\mathbf{T}_2 + \mathbf{T}_3\right)\mathbf{x} = \mathbf{0}.$$

Since $p(0,0,0) = 0$, there exist three polynomials p_1, p_2, p_3 in three noncommuting variables such that

$$p(z_1, z_2, z_3) = p_1(z_1, z_2, z_3)z_1 + p_2(z_1, z_1z_2, z_3)z_1z_2 + p_3(z_1, z_2, z_3)z_3,$$

By carrying out as in the proof of item (i), we can prove (ii).

Item (iii) can be proved in a similar way as in the proofs of items (i) and (ii). □

Remark 2.4. Let $\mathbf{T}_1 \in \mathcal{M}_n$ be group invertible and $\mathbf{A} \in \mathcal{M}_n$. The conditions $\mathbf{A}\mathbf{T}_1\mathbf{T}_1^{\#} = \mathbf{A}$ and $\mathbf{A}\mathbf{T}_1\mathbf{T}_1^{\#} = \mathbf{T}_1\mathbf{T}_1^{\#}\mathbf{A}$ appearing in Theorem 2.4 are independent. In fact, we can write $\mathbf{T}_1 = \mathbf{S}(\mathbf{K} \oplus \mathbf{0})\mathbf{S}^{-1}$ for some nonsingular matrices $\mathbf{S} \in \mathcal{M}_n$, $\mathbf{K} \in \mathcal{M}_r$, being $r = \mathrm{rank}(\mathbf{T}_1)$. By writing

$$\mathbf{A} = \mathbf{S}\begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Z} & \mathbf{T} \end{pmatrix}\mathbf{S}^{-1}, \qquad \mathbf{X} \in \mathcal{M}_r \tag{2.21}$$ [write_a]

and using the nonsingularity of $\mathbf{K}$, one has

$$\mathbf{A}\mathbf{T}_1\mathbf{T}_1^{\#} = \mathbf{T}_1\mathbf{T}_1^{\#}\mathbf{A} \quad \Longleftrightarrow \quad \mathbf{Y} = \mathbf{0} \text{ and } \mathbf{Z} = \mathbf{0},$$

and

$$\mathbf{A}\mathbf{T}_1\mathbf{T}_1^{\#} = \mathbf{A} \quad \Longleftrightarrow \quad \mathbf{Y} = \mathbf{0} \text{ and } \mathbf{T} = \mathbf{0}.$$

The first of the two above conditions is related to the so-called sharp ordering, introduced by Mitra [13] in 1987 (for a recent survey of matrix orderings, see [14]) is defined in the subset of $\mathcal{M}_n$ composed of group invertible matrices by

$$\mathbf{M} \overset{\#}{\leq} \mathbf{N} \quad \Longleftrightarrow \quad \mathbf{M}^{\#}\mathbf{M} = \mathbf{M}^{\#}\mathbf{N} \text{ and } \mathbf{M}\mathbf{M}^{\#} = \mathbf{N}\mathbf{M}^{\#}.$$

As is easy to see, if $\mathbf{T}_1$ is written as $\mathbf{T}_1 = \mathbf{S}(\mathbf{K} \oplus \mathbf{0})\mathbf{S}^{-1}$ and $\mathbf{A}$ is written as in (2.21), then

$$\mathbf{T}_1 \overset{\#}{\leq} \mathbf{A} \quad \Longleftrightarrow \quad \mathbf{X} = \mathbf{K}, \ \mathbf{Y} = \mathbf{0}, \text{ and } \mathbf{Z} = \mathbf{0},$$

which obviously shows that $\mathbf{T}_1 \overset{\#}{\leq} \mathbf{A}$ implies $\mathbf{A}\mathbf{T}_1\mathbf{T}_1^{\#} = \mathbf{T}_1\mathbf{T}_1^{\#}\mathbf{A}$.

It can be given some kind of the converse of Theorem 2.4 in case that $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3 \in \mathcal{M}_n$ are three mutually commuting group invertible matrices satisfying $\mathbf{T}_1\mathbf{T}_2\mathbf{T}_3 = \mathbf{T}_1^2\mathbf{T}_2 - \mathbf{T}_2^2\mathbf{T}_1 = \mathbf{T}_2^2\mathbf{T}_3 + \mathbf{T}_3^2\mathbf{T}_2 = \mathbf{T}_1^2\mathbf{T}_3 - \mathbf{T}_3^2\mathbf{T}_1$. Then

$$(\mathbf{T}_1 - \mathbf{T}_2 - \mathbf{T}_3)\mathbf{T}_1\mathbf{T}_2 = (\mathbf{T}_1 - \mathbf{T}_2 - \mathbf{T}_3)\mathbf{T}_3\mathbf{T}_1 = (\mathbf{T}_1 - \mathbf{T}_2 - \mathbf{T}_3)\mathbf{T}_2\mathbf{T}_3 = \mathbf{0},$$

and hence the invertibility of $\mathbf{T}_1 - \mathbf{T}_2 - \mathbf{T}_3$ leads to $\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_3\mathbf{T}_1 = \mathbf{T}_2\mathbf{T}_3 = \mathbf{0}$. Thus it can be written $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 - c_4(\mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_3\mathbf{T}_1 + \mathbf{T}_2\mathbf{T}_3) = c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3$, and it will be given the explicit expression of $(c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3)^{-1}$ in terms of $(\mathbf{T}_1 - \mathbf{T}_2 - \mathbf{T}_3)^{-1}$ under some conditions (similar conditions were used in a related context in [11]).

Theorem 2.5. *Let $c_1, c_2, c_3 \in \mathbb{C}^*$ and $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3 \in \mathcal{M}_n$ be three group invertible matrices such that $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$ is nonsingular. If there exists $\delta \in \mathbb{C}$ such that*

$$c_1(c_2^{-1} - \delta)\mathbf{T}_1\mathbf{T}_2\mathbf{T}_2^{\#} + c_2(c_1^{-1} - \delta)\mathbf{T}_2\mathbf{T}_1\mathbf{T}_1^{\#} = \mathbf{0}, \tag{2.22}$$

$$c_2(c_3^{-1} - \delta)\mathbf{T}_2\mathbf{T}_3\mathbf{T}_3^{\#} + c_3(c_2^{-1} - \delta)\mathbf{T}_3\mathbf{T}_2\mathbf{T}_2^{\#} = \mathbf{0}, \tag{2.23}$$

and

$$c_3(c_1^{-1} - \delta)\mathbf{T}_3\mathbf{T}_1\mathbf{T}_1^{\#} + c_1(c_3^{-1} - \delta)\mathbf{T}_1\mathbf{T}_3\mathbf{T}_3^{\#} = \mathbf{0}, \tag{2.24}$$

then $(c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3)^{-1}$ is nonsingular and

$$\begin{aligned}(c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3)^{-1} \\ = \left[(c_1^{-1} - \delta)\mathbf{T}_1\mathbf{T}_1^{\#} + (c_2^{-1} - \delta)\mathbf{T}_2\mathbf{T}_2^{\#} + (c_3^{-1} - \delta)\mathbf{T}_3\mathbf{T}_3^{\#} + \delta\mathbf{I}_n\right](\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3)^{-1}\end{aligned}$$

Proof. Let $\alpha = c_1^{-1} - \delta$, $\beta = c_2^{-1} - \delta$, and $\gamma = c_3^{-1} - \delta$. The proof of this theorem is immediately seen from the following equality:

$$(c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3)(\alpha\mathbf{T}_1\mathbf{T}_1^{\#} + \beta\mathbf{T}_2\mathbf{T}_2^{\#} + \gamma\mathbf{T}_3\mathbf{T}_3^{\#} + \delta\mathbf{I}_n) = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3.$$

□

The above Theorem 2.5 permits establish many corollaries. As an exemplary list we can state two some of them in the foregoing paragraphs:

Let $c_1, c_2 \in \mathbb{C}^*$ and $\mathbf{T}_1, \mathbf{T}_2 \in \mathcal{M}_n$ be two group invertible matrices such that $\mathbf{T}_1 + \mathbf{T}_2$ is nonsingular and $\mathbf{T}_1\mathbf{T}_2\mathbf{T}_2^{\#} = \lambda\mathbf{T}_2\mathbf{T}_1\mathbf{T}_1^{\#}$ for some $\lambda \in \mathbb{C}$. By setting $\mathbf{T}_3 = \mathbf{0}$, obviously (2.23) and (2.24) hold. If exists $\delta \in \mathbb{C}$ such that (2.22) holds then

$$\begin{vmatrix} c_1(c_2^{-1} - \delta) & c_2(c_1^{-1} - \delta) \\ -1 & \lambda \end{vmatrix} = 0. \tag{2.25}$$

By expanding (2.25), one has $\lambda c_1 c_2^{-1} - c_2 c_1^{-1} = (\lambda c_1 - c_2)\delta$. Thus, if $\lambda c_1 - c_2 \neq 0$, then we can apply Theorem 2.5 to assure that $c_1\mathbf{T}_1 + c_2\mathbf{T}_2$ is nonsingular and to find $(c_1\mathbf{T}_1 + c_2\mathbf{T}_2)^{-1}$. If $c_2 = \lambda c_1$, then $c_1\mathbf{T} + c_2\mathbf{T}_2$ is nonsingular if and only if $\mathbf{T} + \lambda\mathbf{T}_2$ is nonsingular. Now for arbitrary $x, y, z \in \mathbb{C}$ and taking into account that $\mathbf{T}_1\mathbf{T}_2\mathbf{T}_2^{\#} = \lambda\mathbf{T}_2\mathbf{T}_1\mathbf{T}_1^{\#}$, it follows

$$(\mathbf{T}_1 + \lambda\mathbf{T}_2)(x\mathbf{T}_1\mathbf{T}_1^{\#} + y\mathbf{T}_2\mathbf{T}_2^{\#} + z\mathbf{I}_n) = (x+z)\mathbf{T}_1 + \lambda(y+z)\mathbf{T}_2 + \lambda(y+x)\mathbf{T}_2\mathbf{T}_1^{\#}\mathbf{T}_1^{\#}.$$

By solving the following linear system (observe that $\lambda \neq 0$, since otherwise $c_2 = \lambda c_1 = 0$)

$$x + z = 1, \qquad y + z = \lambda^{-1}, \qquad x + y = 0,$$

one has that $(\mathbf{T}_1 + \lambda\mathbf{T}_2)(\frac{1-\lambda^{-1}}{2}\mathbf{T}_1\mathbf{T}_1^{\#} + \frac{\lambda^{-1}-1}{2}\mathbf{T}_2\mathbf{T}_2^{\#} + \frac{1+\lambda^{-1}}{2}\mathbf{I}_n) = \mathbf{T}_1 + \mathbf{T}_2$, which permits to find $(\mathbf{T}_1 + \lambda\mathbf{T}_2)^{-1}$ in terms of $(\mathbf{T}_1 + \mathbf{T}_2)^{-1}$.

Let $c_1, c_2, c_3 \in \mathbb{C}^*$ and $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3 \in \mathcal{M}_n$ be three group invertible matrices such that $\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3$ is nonsingular. Assume that $\mathbf{T}_2\mathbf{T}_1 = \mathbf{T}_2\mathbf{T}_3 = \mathbf{0}$. By setting $\delta = c_2^{-1}$, then (2.22) and (2.23) hold. Hence if $c_3(c_1^{-1} - c_2^{-1})\mathbf{T}_3\mathbf{T}_1\mathbf{T}_1^{\#} + c_1(c_3^{-1} - c_2^{-1})\mathbf{T}_1\mathbf{T}_3\mathbf{T}_3^{\#} = \mathbf{0}$ (a simpler but weaker condition is $\mathbf{T}_1\mathbf{T}_3 = \mathbf{T}_3\mathbf{T}_1 = \mathbf{0}$) then $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3$ is nonsingular and $(c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3)^{-1}$ can be expressed by using the formula of Theorem 2.5.

Remark 2.5. In Theorem 2.5, it is not necessary to set the conditions (2.22)–(2.24) in case when $c_1 = c_2 = c_3$.

Theorem 2.6. *Let $c_1, c_2, c_3, r_1, r_2, r_3 \in \mathbb{C}$ and $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3 \in \mathcal{M}_n$ such that $\mathbf{T}_1\mathbf{T}_3 = \mathbf{T}_3\mathbf{T}_1$. If $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 + (r_1c_1 + r_2c_2)\mathbf{T}_1\mathbf{T}_2 + (r_1c_1 + r_3c_3)\mathbf{T}_3\mathbf{T}_1 + (r_2c_2 + r_3c_3)\mathbf{T}_3\mathbf{T}_2$ is nonsingular, then*

$$\mathcal{N}\left[\mathbf{T}_1(\mathbf{I}_n + r_1\mathbf{T}_2 + r_1\mathbf{T}_3)\right] \cap \mathcal{N}\left[(\mathbf{I}_n + r_2\mathbf{T}_1 + r_2\mathbf{T}_3)\mathbf{T}_2\right] \cap \mathcal{N}\left[\mathbf{T}_3(\mathbf{I}_n + r_3\mathbf{T}_1 + r_3\mathbf{T}_2)\right] = \{\mathbf{0}\} \tag{2.26}$$

and

$$\mathcal{R}\left[\mathbf{T}_1(\mathbf{I}_n + r_1\mathbf{T}_2 + r_1\mathbf{T}_3)\right] + \mathcal{R}\left[(\mathbf{I}_n + r_2\mathbf{T}_1 + r_2\mathbf{T}_3)\mathbf{T}_2\right] + \mathcal{R}\left[\mathbf{T}_3(\mathbf{I}_n + r_3\mathbf{T}_1 + r_3\mathbf{T}_2)\right] = \mathbb{C}^n. \tag{2.27}$$

Proof. Let α_1, α_2, and α_3 denote $r_1c_1 + r_2c_2$, $r_1c_1 + r_3c_3$, and $r_2c_2 + r_3c_3$, respectively. Moreover, let us take

$$\mathbf{x} \in \mathcal{N}\left[\mathbf{T}_1(\mathbf{I}_n + r_1\mathbf{T}_2 + r_1\mathbf{T}_3)\right] \cap \mathcal{N}\left[(\mathbf{I}_n + r_2\mathbf{T}_1 + r_2\mathbf{T}_3)\mathbf{T}_2\right] \cap \mathcal{N}\left[\mathbf{T}_3(\mathbf{I}_n + r_3\mathbf{T}_1 + r_3\mathbf{T}_2)\right].$$

Then, $\mathbf{T}_1(\mathbf{I}_n + r_1\mathbf{T}_2 + r_1\mathbf{T}_3)\mathbf{x} = (\mathbf{I}_n + r_2\mathbf{T}_1 + r_2\mathbf{T}_3)\mathbf{T}_2\mathbf{x} = \mathbf{T}_3(\mathbf{I}_n + r_3\mathbf{T}_1 + r_3\mathbf{T}_2)\mathbf{x} = \mathbf{0}$. Postmultiplying $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 + \alpha_1\mathbf{T}_1\mathbf{T}_2 + \alpha_2\mathbf{T}_3\mathbf{T}_1 + \alpha_3\mathbf{T}_3\mathbf{T}_2$ by $\mathbf{x}$, it is obtained

$$\begin{aligned}&(c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 + \alpha_1\mathbf{T}_1\mathbf{T}_2 + \alpha_2\mathbf{T}_3\mathbf{T}_1 + \alpha_3\mathbf{T}_3\mathbf{T}_2)\,\mathbf{x}\\ &= c_1\mathbf{T}_1\left(\mathbf{I}_n + r_1\mathbf{T}_2 + r_1\mathbf{T}_3\right)\mathbf{x} + c_2\left(\mathbf{I}_n + r_2\mathbf{T}_1 + r_2\mathbf{T}_3\right)\mathbf{T}_2\mathbf{x} + c_3\mathbf{T}_3\left(\mathbf{I}_n + r_3\mathbf{T}_1 + r_3\mathbf{T}_2\right)\mathbf{x}\\ &= \mathbf{0},\end{aligned}$$

which leads to $\mathbf{x} = \mathbf{0}$. So, the proof of (2.26) is complete.

Since $c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 + \alpha_1\mathbf{T}_1\mathbf{T}_2 + \alpha_2\mathbf{T}_3\mathbf{T}_1 + \alpha_3\mathbf{T}_3\mathbf{T}_2$ is nonsingular, then $\bar{c}_1\mathbf{T}_1^* + \bar{c}_2\mathbf{T}_2^* + \bar{c}_3\mathbf{T}_3^* + \bar{\alpha}_1\mathbf{T}_2^*\mathbf{T}_1^* + \bar{\alpha}_2\mathbf{T}_1^*\mathbf{T}_3^* + \bar{\alpha}_3\mathbf{T}_2^*\mathbf{T}_3^*$ is nonsingular. On the other hand, it can be written

$$\mathcal{N}\left[\left(\mathbf{I}_n + \bar{r}_3\mathbf{T}_1^* + \bar{r}_3\mathbf{T}_2^*\right)\mathbf{T}_3^*\right] \cap \mathcal{N}\left[\mathbf{T}_2^*\left(\mathbf{I}_n + \bar{r}_2\mathbf{T}_1^* + \bar{r}_2\mathbf{T}_3^*\right)\right] \cap \mathcal{N}\left[\left(\mathbf{I}_n + \bar{r}_1\mathbf{T}_2^* + \bar{r}_1\mathbf{T}_3^*\right)\mathbf{T}_1^*\right] = \{\mathbf{0}\}.$$

In view of this equation and [3, pages 74 and 188], it is clearly seen that (2.27) is true. So, the proof is complete. □

In the following theorem, an expression of the inverse of

$$c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 - c_4(\mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_3\mathbf{T}_1 + \mathbf{T}_2\mathbf{T}_3),$$

where $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3 \in \mathcal{M}_n$ are tripotent matrices, $c_1, c_2, c_3 \in \mathbb{C}^*$, and $c_4 \in \mathbb{C}$ is given under some conditions using [10, Theorem 2.5]. It is noteworthy that there is a simple mistake with a minus sign in the formula (2.11) in [10, Theorem 2.5 (ii)]. The corrected form of this formula is

$$\begin{aligned}&\left[(c_1+c_2)^2 - c_3^2\right](c_1\mathbf{T}_1 + c_2\mathbf{T}_2 - c_3\mathbf{T}_1\mathbf{T}_2)^{-1}\\ &\quad = (c_1+c_2)\mathbf{T}_2 + c_3\mathbf{T}_2\mathbf{T}_1 + c_2^{-1}(c_1^2 + c_1c_2 - c_3^2)(\mathbf{T}_2 - \mathbf{T}_2\mathbf{T}_1^2).\end{aligned}$$

Of course, this expression is used in the foregoing theorem.

Thefive

Theorem 2.7. *Let c_1, c_2, $c_3 \in \mathbb{C}^*$, $c_4 \in \mathbb{C}$, $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3 \in \mathcal{M}_n$ be nonzero tripotent matrices such that $\mathbf{T}_1^2\mathbf{T}_2 - \mathbf{T}_2^2\mathbf{T}_1 = \mathbf{T}_2^2\mathbf{T}_3 + \mathbf{T}_3^2\mathbf{T}_2 = \mathbf{T}_1^2\mathbf{T}_3 - \mathbf{T}_3^2\mathbf{T}_1 = \mathbf{0}$ and let us say, for the sake of simplicity, $\alpha = (c_1+c_3)^2 - c_4^2$, $\beta = (c_1+c_2)^2 - c_4^2$, $\gamma = (c_2-c_3)^2 - c_4^2$,*

$$\mathbf{T}_- = c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 - c_4(\mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_3\mathbf{T}_1 + \mathbf{T}_2\mathbf{T}_3),$$

and

$$\mathbf{T}_+ = c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 + c_4(\mathbf{T}_1\mathbf{T}_2 + \mathbf{T}_3\mathbf{T}_1 + \mathbf{T}_2\mathbf{T}_3).$$

(i) *Let $\mathbf{T}_1$ be nonsingular and $\alpha \neq 0$. If $\beta = 0$, then $\mathbf{T}_-$ or $\mathbf{T}_+$ is singular. If $\beta \neq 0$, then $\mathbf{T}_-$ is nonsingular and*

$$\begin{aligned}&\alpha\beta\mathbf{T}_-^{-1}\\ &= \alpha\left[(c_1+c_2)\mathbf{T}_1\mathbf{T}_2^2 + c_4\mathbf{T}_1\mathbf{T}_2 + \frac{c_4}{c_1}(c_1+c_2)(\mathbf{T}_2^2 - \mathbf{T}_1\mathbf{T}_2) + \frac{c_4^2}{c_1}(\mathbf{T}_2 - \mathbf{T}_1\mathbf{T}_2^2)\right]\\ &\quad + \beta\left[c_4\mathbf{T}_1\mathbf{T}_3 + \frac{\alpha}{c_1}(\mathbf{T}_1 - \mathbf{T}_1\mathbf{T}_2^2 - \mathbf{T}_1\mathbf{T}_3^2) + (c_1+c_3)\mathbf{T}_1\mathbf{T}_3^2\right].\end{aligned} \tag{2.28}$$

eqfiveone

(ii) *Let $\mathbf{T}_2$ be nonsingular and $\beta \neq 0$. If $\gamma = 0$, then $\mathbf{T}_-$ or $\mathbf{T}_+$ is singular. If $\gamma \neq 0$, then $\mathbf{T}_-$ is nonsingular and*

$$\begin{aligned}&\beta\gamma\mathbf{T}_-^{-1}\\ &= \beta\left[(c_2-c_3)\mathbf{T}_2\mathbf{T}_3^2 + c_4\mathbf{T}_2\mathbf{T}_3 + \frac{c_4}{c_2}(c_2-c_3)(\mathbf{T}_3^2 + \mathbf{T}_2\mathbf{T}_3) - \frac{c_4^2}{c_2}(\mathbf{T}_3 + \mathbf{T}_2\mathbf{T}_3^2)\right]\\ &\quad + \gamma\left[c_4\mathbf{T}_2\mathbf{T}_1 + \frac{\beta}{c_2}(\mathbf{T}_2 - \mathbf{T}_2\mathbf{T}_3^2 - \mathbf{T}_2\mathbf{T}_1^2) + (c_1+c_2)\mathbf{T}_2\mathbf{T}_1^2\right].\end{aligned} \tag{2.29}$$

eqfivetwo

(iii) *Let $\mathbf{T}_3$ be nonsingular and $\alpha \neq 0$. If $\gamma = 0$, then $\mathbf{T}_-$ or $\mathbf{T}_+$ is singular. If $\gamma \neq 0$, then $\mathbf{T}_-$ is nonsingular and*

$$\begin{aligned}\alpha\gamma\mathbf{T}_-^{-1} = {}& \alpha\left[(c_3-c_2)\mathbf{T}_3\mathbf{T}_2^2 + c_4\mathbf{T}_3\mathbf{T}_2\right]\\ &+ \frac{\gamma}{c_3}\left[\alpha(\mathbf{T}_3 - \mathbf{T}_3\mathbf{T}_2^2) + c_4(c_1+c_3)\mathbf{T}_1^2 - c_1c_4\mathbf{T}_3\mathbf{T}_1 - c_1(c_1+c_3)\mathbf{T}_3\mathbf{T}_1^2 + c_4^2\mathbf{T}_1\right].\end{aligned} \tag{2.30}$$

eqfivethree

Proof. First, let us prove the following claim:

Claim: Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathcal{M}_n$ be nonzero tripotent matrices such that $\mathbf{X}$ is nonsingular and

$$\mathbf{Y} = \mathbf{Y}^2\mathbf{X}, \qquad \mathbf{Y}^2\mathbf{Z} + \mathbf{Z}^2\mathbf{Y} = \mathbf{0}, \qquad \mathbf{Z} = \mathbf{Z}^2\mathbf{X}. \tag{2.31}$$

claim_a

Then $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ can be represented as follows:

$$\mathbf{X} = \mathbf{S}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{D} & \mathbf{E}\end{pmatrix}\mathbf{S}^{-1}, \quad \mathbf{Y} = \mathbf{S}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{S}^{-1}, \quad \mathbf{Z} = \mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{pmatrix}\mathbf{S}^{-1}, \tag{2.32}$$

represent_xyz

where $\mathbf{S} \in \mathcal{M}_n$ is nonsingular, $\mathbf{A} \in \mathcal{M}_r$, $\mathbf{K} \in \mathcal{M}_{n-r}$, and

$$\mathbf{K}\mathbf{D} = \mathbf{0}, \qquad \mathbf{K}^2\mathbf{E} = \mathbf{K}, \qquad \mathbf{A}^2 = \mathbf{I}_r, \qquad \mathbf{E}^2 = \mathbf{I}_{n-r}, \qquad \mathbf{D}\mathbf{A} = -\mathbf{E}\mathbf{D}. \tag{2.33}$$

claim_b

Proof of the claim. Since $\mathbf{Y}$ is tripotent, there exists a nonsingular $\mathbf{S} \in M_n$ such that $\mathbf{Y} = \mathbf{S}(\mathbf{A} \oplus \mathbf{0})\mathbf{S}^{-1}$, where $\mathbf{A} \in \mathcal{M}_r$ and $r = \text{rank}(\mathbf{A})$. Since $\mathbf{A}$ is nonsingular and $\mathbf{Y}^3 = \mathbf{Y}$, we have $\mathbf{A}^2 = \mathbf{I}_r$. Let us write

$$\mathbf{X} = \mathbf{S}\begin{pmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{D} & \mathbf{E} \end{pmatrix}\mathbf{S}^{-1}, \quad \mathbf{Z} = \mathbf{S}\begin{pmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{H} & \mathbf{K} \end{pmatrix}\mathbf{S}^{-1}, \qquad \mathbf{B},\ \mathbf{F} \in \mathcal{M}_r.$$

From the first equality of (2.31) it follows that

$$\mathbf{B} = \mathbf{A}, \qquad \mathbf{C} = \mathbf{0}. \tag{2.34}$$

eqmatrices_a

The middle equality of (2.31) together with $\mathbf{A}^2 = \mathbf{I}_r$ lead to

$$\mathbf{F}^2\mathbf{A} + \mathbf{F} = \mathbf{0}, \qquad \mathbf{G} = \mathbf{0}, \qquad \mathbf{HF} + \mathbf{KH} = \mathbf{0}. \tag{2.35}$$

eqmatrices_b

The last equality of (2.31) in conjunction with (2.34), $\mathbf{G} = \mathbf{0}$, and $\mathbf{HF} + \mathbf{KH} = \mathbf{0}$ yield

$$\mathbf{F} = \mathbf{F}^2\mathbf{A}, \qquad \mathbf{H} = \mathbf{K}^2\mathbf{D}, \qquad \mathbf{K} = \mathbf{K}^2\mathbf{E}. \tag{2.36}$$

eqmatrices_c

The first equalities of (2.35) and (2.36) imply $\mathbf{F} = \mathbf{0}$. Premultiplying by $\mathbf{Z}$ the second equality of (2.31) and using the tripotency of $\mathbf{T}_3$ lead to $\mathbf{ZY}^2\mathbf{Z} + \mathbf{ZY} = \mathbf{0}$, and this latter equality yields $\mathbf{HA} = \mathbf{0}$, and having in mind the nonsingularity of $\mathbf{A}$ we can deduce $\mathbf{H} = \mathbf{0}$. Thus, the representations given in (2.32) are proven.

Furthermore, the tripotency of $\mathbf{Z}$ and $\mathbf{G} = \mathbf{0}$ imply $\mathbf{K}^3 = \mathbf{K}$, and thus, from the second equality of (2.36) it follows that $\mathbf{KD} = \mathbf{0}$. Thus we have proved the first equality of (2.33). The second equality of (2.33) was deduced in (2.36), while the remaining equalities of (2.33) follow from $\mathbf{X}^2 = \mathbf{I}_n$. □

(i) Let us assume that $\mathbf{T}_1$ is nonsingular and $\alpha \neq 0$. The condition $\mathbf{T}_1^2\mathbf{T}_2 - \mathbf{T}_2^2\mathbf{T}_1 = \mathbf{T}_2^2\mathbf{T}_3 + \mathbf{T}_3^2\mathbf{T}_2 = \mathbf{T}_1^2\mathbf{T}_3 - \mathbf{T}_3^2\mathbf{T}_1 = \mathbf{0}$ turns into

$$\mathbf{T}_2 = \mathbf{T}_2^2\mathbf{T}_1, \qquad \mathbf{T}_2^2\mathbf{T}_3 + \mathbf{T}_3^2\mathbf{T}_2 = \mathbf{0}, \qquad \mathbf{T}_3 = \mathbf{T}_3^2\mathbf{T}_1$$

since $\mathbf{T}_1^2 = \mathbf{I}_n$. By applying the claim, we can write

$$\mathbf{T}_1 = \mathbf{S}\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{D} & \mathbf{E} \end{pmatrix}\mathbf{S}^{-1}, \quad \mathbf{T}_2 = \mathbf{S}\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{S}^{-1}, \quad \mathbf{T}_3 = \mathbf{S}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{pmatrix}\mathbf{S}^{-1}, \tag{2.37}$$

eqT1T2T32

and in addition, the relations (2.33) hold. Observe that $\mathbf{K}$ must be a nonzero tripotent matrix since $\mathbf{T}_3$ is nonzero and tripotent. On the other hand, using (2.37), it can be written

$$\mathbf{T}_- = \mathbf{S}\begin{pmatrix} (c_1 + c_2)\mathbf{A} - c_4\mathbf{I}_r & \mathbf{0} \\ c_1\mathbf{D} - c_4\mathbf{DA} & c_3\mathbf{K} + c_1\mathbf{E} - c_4\mathbf{KE} \end{pmatrix}\mathbf{S}^{-1}. \tag{2.38}$$

eqnewcom

According to [10, Theorem 2.5 (ii)], the matrix $c_3\mathbf{K} + c_1\mathbf{E} - c_4\mathbf{KE}$ is nonsingular and

$$(c_3\mathbf{K} + c_1\mathbf{E} - c_4\mathbf{KE})^{-1} = \alpha^{-1}\left[(c_1 + c_3)\mathbf{E} + c_4\mathbf{EK} + c_1^{-1}(c_3^2 + c_3c_1 - c_4^2)(\mathbf{E} - \mathbf{EK}^2)\right],$$

which having in mind $\alpha = (c_1 + c_3)^2 - c_4^2$, becomes to

$$(c_3\mathbf{K} + c_1\mathbf{E} - c_4\mathbf{KE})^{-1} = \alpha^{-1}\left[c_4\mathbf{EK} + \alpha c_1^{-1}(\mathbf{E} - \mathbf{EK}^2) + (c_1 + c_3)\mathbf{EK}^2\right]. \tag{2.39}$$

eqtwoinv

From (2.38) it is obtained that $\mathbf{T}_-^{-1}$ is nonsingular if and only if $(c_1 + c_2)\,\mathbf{A} - c_4\mathbf{I}_r$ is nonsingular (recall that the first row in the block matrix appearing in (2.38) must be present, since otherwise, $\mathbf{T}_2 = \mathbf{0}$). The following equality is evident:

$$[(c_1 + c_2)\mathbf{A} - c_4\mathbf{I}_r]\,[(c_1 + c_2)\mathbf{A} + c_4\mathbf{I}_r] = \beta\mathbf{I}_r, \tag{2.40}$$

eqbetaone

If $\beta = 0$, then (2.40) implies that $(c_1 + c_2)\mathbf{A} - c_4\mathbf{I}_r$ or $(c_1 + c_2)\mathbf{A} + c_4\mathbf{I}_r$ is singular. Hence $\mathbf{T}_-$ or $\mathbf{T}_+$ is singular by (2.38).

If $\beta \neq 0$, from (2.40) the matrix $(c_1+c_2)\mathbf{A} - c_4\mathbf{I}_r$ is nonsingular and

$$[(c_1+c_2)\mathbf{A} - c_4\mathbf{I}_r]^{-1} = \beta^{-1}\left[(c_1+c_2)\mathbf{A} + c_4\mathbf{I}_r\right]. \tag{2.41}$$

eqbetatwo

Using [18, Problem 19 (c), p.42], the inverse of matrix in (2.38) is obtained as

$$\mathbf{T}_-^{-1} = \mathbf{S}\begin{pmatrix} [(c_1+c_2)\mathbf{A} - c_4\mathbf{I}_r]^{-1} & \mathbf{0} \\ \mathbf{M} & [c_3\mathbf{K} + c_1\mathbf{E} - c_4\mathbf{KE}]^{-1} \end{pmatrix}\mathbf{S}^{-1}, \tag{2.42}$$

invnewcom

where

$$\mathbf{M} = -[c_3\mathbf{K} + c_1\mathbf{E} - c_4\mathbf{KE}]^{-1}(c_1\mathbf{D} - c_4\mathbf{DA})\left[(c_1+c_2)\mathbf{A} - c_4\mathbf{I}_r\right]^{-1}. \tag{2.43}$$

invnewcom_bis

Observe that by (2.33), and (2.39), one has

$$[c_3\mathbf{K} + c_1\mathbf{E} - c_4\mathbf{KE}]^{-1}(c_1\mathbf{D} - c_4\mathbf{DA}) = \mathbf{ED} + c_1^{-1}c_4\mathbf{D} \tag{2.44}$$

part_of_x

By using (2.33), (2.41), and (2.44), the matrix $\mathbf{M}$ defined in (2.43) can be simplified:

$$\begin{aligned} \mathbf{M} &= -\beta^{-1}\left[\mathbf{ED} + c_1^{-1}c_4\mathbf{D}\right]\left[(c_1+c_2)\mathbf{A} + c_4\mathbf{I}_r\right] \\ &= -\beta^{-1}\left[(c_1+c_2)\mathbf{EDA} + c_4\mathbf{ED} + c_1^{-1}c_4(c_1+c_2)\mathbf{DA} + c_1^{-1}c_4^2\mathbf{D}\right] \\ &= -\beta^{-1}\left[(c_1^{-1}c_4^2 - c_1 - c_2)\mathbf{D} + c_1^{-1}c_4c_2\mathbf{DA}\right]. \end{aligned} \tag{2.45}$$

Combining (2.39), (2.41), (2.42), and (2.45), it is obtained

$$\begin{aligned} \alpha\beta\mathbf{T}_-^{-1} = \mathbf{S}\Bigg\{ & \alpha\Bigg[(c_1+c_2)\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{D} & \mathbf{0}\end{pmatrix} + c_4\begin{pmatrix}\mathbf{I}_r & \mathbf{0}\\ -\mathbf{ED} & \mathbf{0}\end{pmatrix} \\ & + c_1^{-1}c_4(c_1+c_2)\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{ED} & \mathbf{0}\end{pmatrix} + c_1^{-1}c_4^2\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ -\mathbf{D} & \mathbf{0}\end{pmatrix}\Bigg] \\ & + \beta\Bigg[c_4\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{EK}\end{pmatrix} + \alpha c_1^{-1}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{E}-\mathbf{EK}^2\end{pmatrix} \\ & + (c_1+c_3)\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{EK}^2\end{pmatrix}\Bigg]\Bigg\}\mathbf{S}^{-1}. \end{aligned} \tag{2.46}$$

Then, considering the following equalities in (2.46)

$$\mathbf{T}_1\mathbf{T}_3 = \mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{EK}\end{pmatrix}\mathbf{S}^{-1}, \qquad \mathbf{T}_1\mathbf{T}_3^2 = \mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{EK}^2\end{pmatrix}\mathbf{S}^{-1},$$

$$\mathbf{T}_2^2 - \mathbf{T}_1\mathbf{T}_2 = \mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{ED} & \mathbf{0}\end{pmatrix}\mathbf{S}^{-1}, \qquad \mathbf{T}_1\mathbf{T}_2^2 = \mathbf{S}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{D} & \mathbf{0}\end{pmatrix}\mathbf{S}^{-1},$$

$$\mathbf{T}_1 - \mathbf{T}_1\mathbf{T}_2^2 - \mathbf{T}_1\mathbf{T}_3^2 = \mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{E}-\mathbf{EK}^2\end{pmatrix}\mathbf{S}^{-1},$$

and

$$\mathbf{T}_1\mathbf{T}_2 = \mathbf{S}\begin{pmatrix}\mathbf{I}_r & \mathbf{0}\\ -\mathbf{ED} & \mathbf{0}\end{pmatrix}\mathbf{S}^{-1}, \qquad \mathbf{T}_2 - \mathbf{T}_1\mathbf{T}_2^2 = \mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ -\mathbf{D} & \mathbf{0}\end{pmatrix}\mathbf{S}^{-1}$$

leads to the formula (2.28). So the proof of part (i) is complete.

(ii) Let us assume that $\mathbf{T}_2$ is nonsingular and $\beta \neq 0$. The condition $\mathbf{T}_1^2\mathbf{T}_2 - \mathbf{T}_2^2\mathbf{T}_1 = \mathbf{T}_2^2\mathbf{T}_3 + \mathbf{T}_3^2\mathbf{T}_2 = \mathbf{T}_1^2\mathbf{T}_3 - \mathbf{T}_3^2\mathbf{T}_1 = \mathbf{0}$ turns into

$$\mathbf{T}_1^2\mathbf{T}_2 = \mathbf{T}_1, \qquad \mathbf{T}_3 + \mathbf{T}_3^2\mathbf{T}_2 = \mathbf{0}, \qquad \mathbf{T}_1^2\mathbf{T}_3 = \mathbf{T}_3^2\mathbf{T}_1 \tag{2.47}$$

eqconii

since $\mathbf{T}_2^2 = \mathbf{I}_n$. We can apply the claim for $\mathbf{X} = -\mathbf{T}_2$, $\mathbf{Y} = \mathbf{T}_3$, and $\mathbf{Z} = -\mathbf{T}_1$ obtaining that $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ can be written as

$$\mathbf{T}_1 = \mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{pmatrix}\mathbf{S}^{-1}, \quad \mathbf{T}_2 = \mathbf{S}\begin{pmatrix}-\mathbf{A} & \mathbf{0}\\ \mathbf{D} & \mathbf{E}\end{pmatrix}\mathbf{S}^{-1}, \quad \mathbf{T}_3 = \mathbf{S}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{S}^{-1} \tag{2.48}$$

eqnewmat

(we rename $\mathbf{K} \leftrightarrow -\mathbf{K}$, $\mathbf{D} \leftrightarrow -\mathbf{D}$, and $\mathbf{E} \leftrightarrow -\mathbf{E}$). The blocks appearing in (2.48) satisfy the following relations derived from the corresponding ones in (2.33):

$$\mathbf{KD} = \mathbf{0}, \qquad \mathbf{K}^2\mathbf{E} = \mathbf{K}, \qquad \mathbf{A}^2 = \mathbf{I}_r, \qquad \mathbf{E}^2 = \mathbf{I}_{n-r}, \qquad \mathbf{DA} = \mathbf{ED}. \tag{2.49}$$

claim_c

Matrix $\mathbf{K}$ must be nonzero tripotent since $\mathbf{T}_1$ is nonzero tripotent. Observe that from (2.49) it follows that $\mathbf{E}$ is nonsingular and $\mathbf{D} = \mathbf{EDA}$. On the other hand, using (2.48) and (2.49), it can be written

$$\mathbf{T}_- = \mathbf{S} \begin{pmatrix} (-c_2 + c_3)\mathbf{A} + c_4\mathbf{I}_r & \mathbf{0} \\ c_2\mathbf{D} - c_4\mathbf{DA} & c_1\mathbf{K} + c_2\mathbf{E} - c_4\mathbf{KE} \end{pmatrix} \mathbf{S}^{-1}. \tag{2.50}$$

eqnewcomii

According to [10, Thorem 2.5 (ii)], the matrix $c_1\mathbf{K} + c_2\mathbf{E} - c_4\mathbf{KE}$ is nonsingular and

$$(c_1\mathbf{K} + c_2\mathbf{E} - c_4\mathbf{KE})^{-1} = \beta^{-1}\left[c_4\mathbf{EK} + \beta c_2^{-1}\left(\mathbf{E} - \mathbf{EK}^2\right) + (c_1 + c_2)\,\mathbf{EK}^2\right]. \tag{2.51}$$

eqtwoinvii

From (2.50), it is obtained that $\mathbf{T}_-$ is nonsingular if and only if $(-c_2 + c_3)\,\mathbf{A} + c_4\mathbf{I}_r$ is nonsingular. The following equality is obvious:

$$[(-c_2 + c_3)\,\mathbf{A} - c_4\mathbf{I}_r]\,[(-c_2 + c_3)\,\mathbf{A} + c_4\mathbf{I}_r] = \gamma\mathbf{I}_r, \tag{2.52}$$

eqgammaone

If $\gamma = 0$, then (2.52) implies that $(-c_2 + c_3)\mathbf{A} - c_4\mathbf{I}_r$ or $(-c_2 + c_3)\mathbf{A} + c_4\mathbf{I}_r$ is singular. Hence $\mathbf{T}_-$ or $\mathbf{T}_+$ is singular, by (2.50).

Now, let $\gamma \neq 0$. From (2.52), the matrix $(-c_2 + c_3)\mathbf{A} + c_4\mathbf{I}_r$ is nonsingular and

$$[(-c_2 + c_3)\,\mathbf{A} + c_4\mathbf{I}_r]^{-1} = \gamma^{-1}\,[(-c_2 + c_3)\,\mathbf{A} - c_4\mathbf{I}_r]\,. \tag{2.53}$$

eqgammatwo

Using [18, Problem 19 (c)], the inverse of the matrix $\mathbf{T}_-$ written in (2.50) is obtained as

$$\mathbf{T}_-^{-1} = \mathbf{S} \begin{pmatrix} [(-c_2 + c_3)\mathbf{A} + c_4\mathbf{I}_r]^{-1} & \mathbf{0} \\ \mathbf{M} & [c_1\mathbf{K} + c_2\mathbf{E} - c_4\mathbf{KE}]^{-1} \end{pmatrix} \mathbf{S}^{-1}, \tag{2.54}$$

invnewcomii

where

$$\mathbf{M} = -\,[c_1\mathbf{K} + c_2\mathbf{E} - c_4\mathbf{KE}]^{-1}\,(c_2\mathbf{D} - c_4\mathbf{DA})\,[(-c_2 + c_3)\mathbf{A} + c_4\mathbf{I}_r]^{-1}\,.$$

By the first equality of (2.49) and (2.51)

$$[c_1\mathbf{K} + c_2\mathbf{E} - c_4\mathbf{KE}]^{-1}\,(c_2\mathbf{D} - c_4\mathbf{DA}) = \mathbf{ED} - c_2^{-1}c_4\mathbf{D}.$$

By doing some elementary algebra and using (2.49 and (2.53) we can simplify $\mathbf{M}$ obtaining

$$\mathbf{M} = \gamma^{-1}\left[(c_2 - c_3 - c_2^{-1}c_4^2)\mathbf{D} + c_2^{-1}c_3c_4\mathbf{DA}\right]. \tag{2.55}$$

define_m

Combining (2.51), (2.53), (2.54), and (2.55) it is obtained

$$\begin{aligned} \beta\gamma\mathbf{T}_-^{-1} \quad = \quad & \mathbf{S}\Bigg\{\beta\Bigg[(-c_2 + c_3)\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{D} & \mathbf{0} \end{pmatrix} + c_4\begin{pmatrix} -\mathbf{I}_r & \mathbf{0} \\ \mathbf{ED} & \mathbf{0} \end{pmatrix} \\ & \qquad + c_2^{-1}c_4(-c_2 + c_3)\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{ED} & \mathbf{0} \end{pmatrix} - c_2^{-1}c_4^2\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{D} & \mathbf{0} \end{pmatrix}\Bigg] \\ & \quad + \gamma\Bigg[c_4\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{EK} \end{pmatrix} + \beta c_2^{-1}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} - \mathbf{EK}^2 \end{pmatrix} \\ & \qquad + (c_1 + c_2)\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{EK}^2 \end{pmatrix}\Bigg]\Bigg\}\mathbf{S}^{-1}. \end{aligned} \tag{2.56}$$

On the other hand, the following equalities can be written:

$$\mathbf{T}_2\mathbf{T}_3 = \mathbf{S}\begin{pmatrix} -\mathbf{I}_r & \mathbf{0} \\ \mathbf{ED} & \mathbf{0} \end{pmatrix}\mathbf{S}^{-1}, \qquad \mathbf{T}_2\mathbf{T}_1^2 = \mathbf{S}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{EK}^2 \end{pmatrix}\mathbf{S}^{-1},$$

$$\mathbf{T}_3^2+\mathbf{T}_2\mathbf{T}_3=\mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{ED} & \mathbf{0}\end{pmatrix}\mathbf{S}^{-1}, \qquad \mathbf{T}_2\mathbf{T}_1=\mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{EK}\end{pmatrix}\mathbf{S}^{-1},$$

$$\mathbf{T}_2\mathbf{T}_3^2=\mathbf{S}\begin{pmatrix}-\mathbf{A} & \mathbf{0}\\ \mathbf{D} & \mathbf{0}\end{pmatrix}\mathbf{S}^{-1}, \qquad \mathbf{T}_3+\mathbf{T}_2\mathbf{T}_3^2=\mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{D} & \mathbf{0}\end{pmatrix}\mathbf{S}^{-1},$$

and

$$\mathbf{T}_2-\mathbf{T}_2\mathbf{T}_3^2-\mathbf{T}_2\mathbf{T}_1^2=\mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{E}-\mathbf{EK}^2\end{pmatrix}\mathbf{S}^{-1}.$$

Substituting these equalities in (2.56) leads to the formula (2.29) which is the desired result.

(iii) Let us assume that $\mathbf{T}_3$ is nonsingular and $\alpha \neq 0$. The condition $\mathbf{T}_1^2\mathbf{T}_2-\mathbf{T}_2^2\mathbf{T}_1=\mathbf{T}_2^2\mathbf{T}_3+\mathbf{T}_3^2\mathbf{T}_2=\mathbf{T}_1^2\mathbf{T}_3-\mathbf{T}_3^2\mathbf{T}_1=\mathbf{0}$ turns into

$$\mathbf{T}_1^2\mathbf{T}_2=\mathbf{T}_2^2\mathbf{T}_1, \qquad \mathbf{T}_2^2\mathbf{T}_3+\mathbf{T}_2=\mathbf{0}, \qquad \mathbf{T}_1^2\mathbf{T}_3=\mathbf{T}_1$$

since $\mathbf{T}_3^2=\mathbf{I}_n$. By applying the claim for $\mathbf{X}=\mathbf{T}_3$, $\mathbf{Y}=-\mathbf{T}_2$, and $\mathbf{Z}=\mathbf{T}_1$, we can write

$$\mathbf{T}_1=\mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{pmatrix}\mathbf{S}^{-1}, \quad \mathbf{T}_2=\mathbf{S}\begin{pmatrix}-\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{S}^{-1}, \quad \mathbf{T}_3=\mathbf{S}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{D} & \mathbf{E}\end{pmatrix}\mathbf{S}^{-1}, \tag{2.57}$$

eqTis2

where $\mathbf{S}\in\mathcal{M}_n$ is nonsingular, $\mathbf{A}\in\mathcal{M}_r$, $\mathbf{K}\in\mathcal{M}_{n-r}$, and blocks $\mathbf{A},\mathbf{D},\mathbf{E},\mathbf{K}$ satisfy (2.33). Using (2.57), it can be written

$$\mathbf{T}_-=\mathbf{S}\begin{pmatrix}(-c_2+c_3)\mathbf{A}+c_4\mathbf{I}_r & \mathbf{0}\\ c_3\mathbf{D} & c_3\mathbf{E}+c_1\mathbf{K}-c_4\mathbf{EK}\end{pmatrix}\mathbf{S}^{-1}. \tag{2.58}$$

eqnewcom3

Observe that $\mathbf{K}\neq\mathbf{0}$, since otherwise $\mathbf{T}_1=\mathbf{0}$. Also, $\mathbf{E}$ is nonsingular because $\mathbf{T}_3$ is nonsingular. According to [10, Thorem 2.5 (i)], the matrix $c_3\mathbf{E}+c_1\mathbf{K}-c_4\mathbf{EK}$ is nonsingular and

$$\begin{aligned}&(c_3\mathbf{E}+c_1\mathbf{K}-c_4\mathbf{EK})^{-1}\\ &\quad=\alpha^{-1}c_3^{-1}\left[\alpha\mathbf{E}+c_4(c_3+c_1)\mathbf{K}^2-c_1c_4\mathbf{EK}-c_1(c_3+c_1)\mathbf{EK}^2+c_4^2\mathbf{K}\right].\end{aligned} \tag{2.59}$$

From (2.58), it is obtained that $\mathbf{T}_-$ is nonsingular if and only if $(-c_2+c_3)\mathbf{A}+c_4\mathbf{I}_r$ is nonsingular. It is evident that

$$\left[(-c_2+c_3)\,\mathbf{A}+c_4\mathbf{I}_r\right]\left[(-c_2+c_3)\,\mathbf{A}-c_4\mathbf{I}_r\right]=\gamma\mathbf{I}_r. \tag{2.60}$$

eqgamtwo

If $\gamma=0$, then (2.53) yields that $(-c_2+c_3)\,\mathbf{A}+c_4\mathbf{I}_r$ or $(-c_2+c_3)\,\mathbf{A}-c_4\mathbf{I}_r$ is singular. Hence $\mathbf{T}_-$ or $\mathbf{T}_+$ is singular, by (2.58).

Now, let $\gamma\neq 0$. From (2.60) the matrix $(-c_2+c_3)\,\mathbf{A}+c_4\mathbf{I}_r$ is nonsingular and

$$\left[(-c_2+c_3)\,\mathbf{A}+c_4\mathbf{I}_r\right]^{-1}=\gamma^{-1}\left[(-c_2+c_3)\,\mathbf{A}-c_4\mathbf{I}_r\right]. \tag{2.61}$$

eqgam2

Using [18, Problem 19 (c)], the inverse of matrix in (2.58) is obtained as

$$\mathbf{T}_-^{-1}=\mathbf{S}\begin{pmatrix}\left[(-c_2+c_3)\mathbf{A}+c_4\mathbf{I}_r\right]^{-1} & \mathbf{0}\\ \mathbf{M} & \left[c_3\mathbf{E}+c_1\mathbf{K}-c_4\mathbf{EK}\right]^{-1}\end{pmatrix}\mathbf{S}^{-1}, \tag{2.62}$$

invnewcomiii

where

$$\mathbf{M}=-\left[c_3\mathbf{E}+c_1\mathbf{K}-c_4\mathbf{EK}\right]^{-1}c_3\mathbf{D}\left[(-c_2+c_3)\mathbf{A}+c_4\mathbf{I}_r\right]^{-1}.$$

Since $\mathbf{K}$ and $\mathbf{D}$ satisfy (2.33), then (2.59) implies $\left[c_3\mathbf{E}+c_1\mathbf{K}-c_4\mathbf{EK}\right]^{-1}\mathbf{D}=c_3^{-1}\mathbf{ED}$. Therefore, (2.60) and (2.33) lead to

$$\mathbf{M}=-\gamma^{-1}\mathbf{ED}[(-c_2+c_3)\mathbf{A}-c_4\mathbf{I}_r]=\gamma^{-1}\left[(-c_2+c_3)\mathbf{D}-c_4\mathbf{ED}\right]. \tag{2.63}$$

eqmatrices3

Combining (2.59), (2.61), (2.62), and (2.63) it is obtained

$$\begin{aligned}\alpha\gamma\mathbf{T}_{-}^{-1} &= \mathbf{S}\Bigg\{\alpha\left[(-c_2+c_3)\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{D} & \mathbf{0}\end{pmatrix}+c_4\begin{pmatrix}-\mathbf{I}_r & \mathbf{0}\\ \mathbf{ED} & \mathbf{0}\end{pmatrix}\right]\\ &\quad+\gamma c_3^{-1}\left[\alpha\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{E}\end{pmatrix}+c_4(c_1+c_3)\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}^2\end{pmatrix}-c_1c_4\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{EK}\end{pmatrix}\right.\\ &\quad\left.-c_1(c_1+c_3)\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{EK}^2\end{pmatrix}+c_4^2\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}\end{pmatrix}\right]\Bigg\}\mathbf{S}^{-1}. \qquad (2.64)\end{aligned}$$

On the other hand, by employing (2.57) and the relations given in (2.33), the following equalities can be written

$$\mathbf{T}_3\mathbf{T}_1^2=\mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{EK}^2\end{pmatrix}\mathbf{S}^{-1},\qquad \mathbf{T}_1^2=\mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{K}^2\end{pmatrix}\mathbf{S}^{-1},$$

$$\mathbf{T}_3\mathbf{T}_2=\mathbf{S}\begin{pmatrix}-\mathbf{I}_r & \mathbf{0}\\ \mathbf{ED} & \mathbf{0}\end{pmatrix}\mathbf{S}^{-1},\qquad \mathbf{T}_3\mathbf{T}_2^2=\mathbf{S}\begin{pmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{D} & \mathbf{0}\end{pmatrix}\mathbf{S}^{-1},$$

and

$$\mathbf{T}_3-\mathbf{T}_3\mathbf{T}_2^2=\mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{E}\end{pmatrix}\mathbf{S}^{-1},\qquad \mathbf{T}_3\mathbf{T}_1=\mathbf{S}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{EK}\end{pmatrix}\mathbf{S}^{-1}.$$

Substituting these equalities in (2.64) leads to the formula (2.30) which is desired result. So the proof is complete. □

In case when $c_4 = 0$, we get the following corollary.

Corollary 2.3. *Let c_1, c_2, $c_3 \in \mathbb{C}^*$, $\mathbf{T}_1, \mathbf{T}_2$, and $\mathbf{T}_3 \in \mathcal{M}_n$ be nonzero tripotent matrices such that* $\mathbf{T}_1^2\mathbf{T}_2-\mathbf{T}_2^2\mathbf{T}_1=\mathbf{T}_2^2\mathbf{T}_3+\mathbf{T}_3^2\mathbf{T}_2=\mathbf{T}_1^2\mathbf{T}_3-\mathbf{T}_3^2\mathbf{T}_1=\mathbf{0}$.

(i) *If $\mathbf{T}_1$ is nonsingular, $c_1+c_3\neq 0$, and $c_1+c_2\neq 0$, then*

$$\begin{aligned}&(c_1+c_2)(c_1+c_3)\left[c_1\mathbf{T}_1+c_2\mathbf{T}_2+c_3\mathbf{T}_3\right]^{-1}\\ &=(c_1+c_3)\mathbf{T}_1\mathbf{T}_2^2+(c_1+c_2)\left[c_1^{-1}(c_1+c_3)\left(\mathbf{T}_1-\mathbf{T}_1\mathbf{T}_2^2-\mathbf{T}_1\mathbf{T}_3^2\right)+\mathbf{T}_1\mathbf{T}_3^2\right],\end{aligned}$$

(ii) *If $\mathbf{T}_2$ is nonsingular, $c_1+c_2\neq 0$, and $c_2-c_3\neq 0$, then*

$$\begin{aligned}&(c_1+c_2)(c_2-c_3)\left[c_1\mathbf{T}_1+c_2\mathbf{T}_2+c_3\mathbf{T}_3\right]^{-1}\\ &=(c_1+c_2)\mathbf{T}_2\mathbf{T}_3^2+(c_2-c_3)\left[c_2^{-1}(c_1+c_2)\left(\mathbf{T}_2-\mathbf{T}_2\mathbf{T}_3^2-\mathbf{T}_2\mathbf{T}_1^2\right)+\mathbf{T}_2\mathbf{T}_1^2\right],\end{aligned}$$

(iii) *If $\mathbf{T}_3$ is nonsingular, $c_1+c_3\neq 0$, and $c_2-c_3\neq 0$, then*

$$\begin{aligned}&(c_1+c_3)(c_3-c_2)\left(c_1\mathbf{T}_1+c_2\mathbf{T}_2+c_3\mathbf{T}_3\right)^{-1}\\ &\quad=(c_1+c_3)\mathbf{T}_3{\mathbf{T}_2}^2+(c_3-c_2){c_3}^{-1}\left[(c_1+c_3)\left(\mathbf{T}_3-\mathbf{T}_3{\mathbf{T}_2}^2\right)-c_1\mathbf{T}_3{\mathbf{T}_1}^2\right].\end{aligned}$$

Next theorem shows that the nonsingularity of

$$c_1\mathbf{T}_1+c_2\mathbf{T}_2+c_3\mathbf{T}_3-c_4(\mathbf{T}_1\mathbf{T}_2+\mathbf{T}_3\mathbf{T}_1+\mathbf{T}_2\mathbf{T}_3)$$

is also related to the nonsingularity of a combination of $\left(\mathbf{T}_2^2+\mathbf{T}_3^2\right)\mathbf{T}_1$, $\left(\mathbf{T}_1^2+\mathbf{T}_3^2\right)\mathbf{T}_2$ and $\left(\mathbf{T}_1^2+\mathbf{T}_2^2\right)\mathbf{T}_3$ or $\mathbf{T}_1\left(\mathbf{T}_2^2+\mathbf{T}_3^2\right)$, $\mathbf{T}_2\left(\mathbf{T}_1^2+\mathbf{T}_3^2\right)$ and $\mathbf{T}_3\left(\mathbf{T}_1^2+\mathbf{T}_2^2\right)$.

Theorem 2.8. *Let c_1, c_2, $c_3 \in \mathbb{C}^*$, $c_4 \in \mathbb{C}$, and $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3 \in \mathcal{M}_n$ be tripotent matrices. The following statements are equivalent:*

(i) $c_1\left(\mathbf{T}_2^2+\mathbf{T}_3^2\right)\mathbf{T}_1+c_2\left(\mathbf{T}_1^2+\mathbf{T}_3^2\right)\mathbf{T}_2+c_3\left(\mathbf{T}_1^2+\mathbf{T}_2^2\right)\mathbf{T}_3-c_4\left(\left(\mathbf{T}_2^2+\mathbf{T}_3^2\right)\mathbf{T}_1\mathbf{T}_2\right.$ $\left.+\left(\mathbf{T}_1^2+\mathbf{T}_2^2\right)\mathbf{T}_3\mathbf{T}_1+\left(\mathbf{T}_1^2+\mathbf{T}_3^2\right)\mathbf{T}_2\mathbf{T}_3\right)$ *is nonsingular.*

(ii) $c_1\mathbf{T}_1\left(\mathbf{T}_2^2+\mathbf{T}_3^2\right)+c_2\mathbf{T}_2\left(\mathbf{T}_1^2+\mathbf{T}_3^2\right)+c_3\mathbf{T}_3\left(\mathbf{T}_1^2+\mathbf{T}_2^2\right)-c_4\left(\mathbf{T}_3\mathbf{T}_1\left(\mathbf{T}_2^2+\mathbf{T}_3^2\right)\right.$ $\left.+\mathbf{T}_2\mathbf{T}_3\left(\mathbf{T}_1^2+\mathbf{T}_2^2\right)+\mathbf{T}_1\mathbf{T}_2\left(\mathbf{T}_1^2+\mathbf{T}_3^2\right)\right)$ *is nonsingular.*

(iii) $c_1\mathbf{T}_1+c_2\mathbf{T}_2+c_3\mathbf{T}_3-c_4(\mathbf{T}_1\mathbf{T}_2+\mathbf{T}_3\mathbf{T}_1+\mathbf{T}_2\mathbf{T}_3)$ *and* $\mathbf{T}_1^2+\mathbf{T}_2^2+\mathbf{T}_3^2-\mathbf{I}_n$ *are nonsingular.*

The proof of this theorem is followed immediately from the equalities

$$\begin{aligned}&\left(\mathbf{T}_1^2+\mathbf{T}_2^2+\mathbf{T}_3^2-\mathbf{I}_n\right)[c_1\mathbf{T}_1+c_2\mathbf{T}_2+c_3\mathbf{T}_3-c_4(\mathbf{T}_1\mathbf{T}_2+\mathbf{T}_2\mathbf{T}_3+\mathbf{T}_3\mathbf{T}_1)]\\&\quad=c_1(\mathbf{T}_2^2+\mathbf{T}_3^2)\mathbf{T}_1+c_2(\mathbf{T}_3^2+\mathbf{T}_1^2)\mathbf{T}_2+c_3(\mathbf{T}_1^2+\mathbf{T}_2^2)\mathbf{T}_3\\&\qquad-c_4\left[(\mathbf{T}_2^2+\mathbf{T}_3^2)\mathbf{T}_1\mathbf{T}_2+(\mathbf{T}_3^2+\mathbf{T}_1^2)\mathbf{T}_2\mathbf{T}_3+(\mathbf{T}_1^2+\mathbf{T}_2^2)\mathbf{T}_3\mathbf{T}_1\right]\end{aligned}$$

and

$$\begin{aligned}&[c_1\mathbf{T}_1+c_2\mathbf{T}_2+c_3\mathbf{T}_3-c_4(\mathbf{T}_1\mathbf{T}_2+\mathbf{T}_2\mathbf{T}_3+\mathbf{T}_3\mathbf{T}_1)]\left(\mathbf{T}_1^2+\mathbf{T}_2^2+\mathbf{T}_3^2-\mathbf{I}_n\right)\\&\quad=c_1\mathbf{T}_1(\mathbf{T}_2^2+\mathbf{T}_3^2)+c_2\mathbf{T}_2(\mathbf{T}_3^2+\mathbf{T}_1^2)+c_3\mathbf{T}_3(\mathbf{T}_1^2+\mathbf{T}_2^2)\\&\qquad-c_4\left[\mathbf{T}_1\mathbf{T}_2(\mathbf{T}_1^2+\mathbf{T}_3^2)+\mathbf{T}_2\mathbf{T}_3(\mathbf{T}_1^2+\mathbf{T}_2^2)+\mathbf{T}_3\mathbf{T}_1(\mathbf{T}_2^2+\mathbf{T}_3^2)\right].\end{aligned}$$

Observe that setting $c_4=0$ in the last result, we get a characterization of the nonsingularity of a linear combination of three tripotent matrices without any further restriction on these matrices.

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