Balancing Consistency and Expert Judgment in AHP

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Abstract

The various mechanisms that represent the know-how of decision-makers are exposed to a common weakness, namely, a lack of consistency. To overcome this weakness within AHP (analytic hierarchy process), we propose a framework that enables balancing consistency and expert judgment. We specifically focus on a linearization process for streamlining the trade-off between expert reliability and synthetic consistency. An algorithm is developed that can be readily integrated in a suitable DSS (decision support system). This algorithm follows an iterative feedback process that achieves an acceptable level of consistency while complying to some degree with expert preferences. Finally, an application of the framework to a water management decision-making problem is presented.

Key words: decision support, AHP, consistency, positive matrices

1 Introduction

One of the best established and most modern models of decision-making is AHP (analytic hierarchy process) [6,10,11]. In AHP, the input format for decision-makers to express their preferences derives from pair-wise comparisons among various elements. Comparisons can be determined by using, for instance [4], a scale of integers 1-9 to represent opinions ranging from 'equal importance' to 'extreme importance' [8] (intermediate decimal values are sometimes useful). Homogeneous and reciprocal judgment yields an $n \times n$ matrix A with $a_{ii} = 1$ and $a_{ij} = 1/a_{ji}$, $i, j = 1, \dots, n$. This last property is called *reciprocity* and A is said to be a *reciprocal matrix*. The aim is to assign to each of n elements, E_i, priority values w_i, $i = 1, \dots, n$, that reflect the emitted

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judgments. If judgments are consistent, the relations between the judgments a_{ij} and the values w_i turn out to be $a_{ij} = w_i/w_j$, $i, j = 1, \ldots, n$, and it is said that A is a *consistent matrix*. This is equivalent to $a_{ij}a_{jk} = a_{ik}$ for $i, j, k = 1, \ldots, n$ [1]. As stated by [9,10], the leading eigenvalue and the principal (Perron) eigenvector of a comparison matrix provides information to deal with complex decisions, the normalized Perron eigenvector giving the sought priority vector. In the general case, however, A is not consistent. The hypothesis that the estimates of these values are small perturbations of the 'right' values guarantees a small perturbation of the eigenvalues (see, e.g., [12]). Now, the problem to solve is the eigenvalue problem $A\mathbf{w} = \lambda_{\max}\mathbf{w}$, where $\lambda_{\max}$ is the unique largest eigenvalue of A that gives the Perron eigenvector as an estimate of the *priority vector*.

As a measurement of inconsistency, Saaty [8] proposed using the consistency index $CI = (\lambda_{\max} - n)/(n - 1)$ and the consistency ratio $CR = CI/RI$, where RI is the so-called average consistency index [8]. If $CR < 0.1$, the estimate is accepted; otherwise, a new comparison matrix is solicited until $CR < 0.1$. To overcome inconsistency in AHP while still taking into account expert know-how, the authors propose a model to balance the latter with the former. Our model incorporates an extended version of the linearization procedure described in [2], and integrates it along with AHP to produce optimal comparison matrices.

2 Linearization process and an extension to judgment modification

We first introduce some notation and mathematical tools [2]. Hereinafter, $M_{n,m}$ will denote the set of $n \times m$ real matrices. If $A \in M_{n,m}$, $[A]_{i,j}$ denotes the i, j entry of A. $M^+_{n,m} \subset M_{n,m}$ is the subset of matrices with positive entries. We assume that vectors of $\mathbb{R}^n$ are columns, and denote $\mathbf{1}_n = [1 \cdots 1]^{\mathrm{T}} \in \mathbb{R}^n$. Let us recall that the Hadamard product, $\odot$, in $M_{n,m}$ is defined as the componentwise product. The following two mappings are one inverse of the other:

$$L : M^+_{n,m} \to M_{n,m}, \qquad [L(A)]_{i,j} = \log([A]_{i,j});$$

$$E : M_{n,m} \to M^+_{n,m}, \qquad [E(A)]_{i,j} = \mathrm{e}^{[A]_{i,j}}.$$

Clearly, $L(X \odot Y) = L(X) + L(Y)$, and $E(X) + E(Y) = E(X \odot Y)$ for all $X, Y \in M^+_{n,m}$. Because of its simplicity, we use the Frobenius norm, $\|A\|_F = [\mathrm{tr}(A^{\mathrm{T}}A)]^{1/2}$, $\mathrm{tr}(X)$ and X^{T} being the trace and the transpose of matrix X, respectively. Also, in $M^+_{n,m}$ we define the distance d given by $\mathrm{d}(A, B) = \|L(A) - L(B)\|_F$. Finally, we define $\phi_n : \mathbb{R}^n \to M_{n,n}$ given by $[\phi_n(\mathbf{x})]_{i,j} = x_i - x_j$ and $\mathscr{L}_n = \{L(A) : A \in M^+_{n,n}, A \text{ is consistent}\}$.

Theorem 1 [2] *$\mathscr{L}_n = \mathrm{Im}\,\phi_n$ is a linear subspace of $M_{n,n}$ of dimension $n - 1$.*

We will now use orthogonal projections, to solve approximation problems. Let $p_n : M_{n,n} \to \mathscr{L}_n$ be such a projection, and let us assume that $\mathrm{I\!R}^n$ is endowed with the standard inner product, inducing the Euclidean norm, and $M_{n,n}$ is endowed with the following inner product: $\langle A, B\rangle = \mathrm{tr}(A^{\mathrm{T}}B)$.

Theorem 2 [2] *Let $\{\mathbf{y}_1, \dots, \mathbf{y}_{n-1}\}$ be an orthogonal basis of the orthogonal complement to* $\mathrm{span}\{\mathbf{1}_n\}$. *Then $\{\phi_n(\mathbf{y}_1), \dots, \phi_n(\mathbf{y}_{n-1})\}$ is an orthogonal basis of $\mathscr{L}_n$ and $\|\phi_n(\mathbf{y}_i)\|_F^2 = 2n\|\mathbf{y}_i\|_2^2$ for all $i = 1, \dots, n-1$.*

Hence, the orthogonal projection of $L(A)$ onto $\mathscr{L}_n$ is given by a Fourier expansion [7].

Theorem 3 [2] *Let A and $\{\mathbf{y}_1, \dots, \mathbf{y}_{n-1}\}$ an orthogonal basis of the orthogonal complement to* $\mathrm{span}\{\mathbf{1}_n\}$. *The orthogonal projection of $L(A)$ onto $\mathscr{L}_n$ is the matrix*

$$\frac{1}{2n}\sum_{i=1}^{n-1}\frac{\mathrm{tr}\left(L(A)^{\mathrm{T}}\phi_n(\mathbf{y}_i)\right)}{\|\mathbf{y}_i\|_F^2}\phi_n(\mathbf{y}_i).$$

Remark 4 *Observe that $\phi_n(\mathbf{v}) = \mathbf{v}\mathbf{1}_n^{\mathrm{T}} - \mathbf{1}_n\mathbf{v}^{\mathrm{T}}$ for any $\mathbf{v} \in \mathrm{I\!R}^n$.*

We develop now some results that enable easy calculation of the new consistent comparison matrix if one (or more) judgments are modified. As a corollary, we give a fast algorithm to find the closest consistent matrix to a given reciprocal matrix.

2.1 Consistency retrieval after modifying one pair-wise comparison

Let us suppose that a reciprocal matrix A is obtained from some expert judgment and the consistent matrix $Y_A = E[p_n(L(A))]$ closest to A is calculated. If the judgment comparing criteria r and s is changed (where $r \neq s$ and $1 \leq r, s \leq n$), we obtain another reciprocal matrix B. In other words, $[B]_{r,s} = \alpha[A]_{r,s}$ and $[B]_{s,r} = \alpha^{-1}[A]_{s,r}$ for some $\alpha > 0$ and $[B]_{i,j} = [A]_{i,j}$ in the remaining entries.

The problem we address is how to find the consistent matrix $Y_B = E[p_n(L(B))]$ closest to B by performing fewer operations than by means of Theorem 3.

The relationship between matrices A and B is

$$L(B) = L(A) + \log\alpha(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}}). \quad (1)$$

Since the orthogonal projection p_n is linear,

$$p_n(L(B)) = p_n(L(A)) + \log\alpha \cdot p_n(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}}). \quad (2)$$

By Theorem 3 we have

$$p_n(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}}) = \frac{1}{2n}\sum_{i=1}^{n-1}\frac{\operatorname{tr}\left((\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}})^{\mathrm{T}}\phi_n(\mathbf{y}_i)\right)}{\|\mathbf{y}_i\|_2^2}\phi_n(\mathbf{y}_i). \tag{3}$$

Let us simplify $\operatorname{tr}\left((\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}})^{\mathrm{T}}\phi_n(\mathbf{y}_i)\right)$. By using Remark 4, it is obtained

$$(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}})^{\mathrm{T}}\phi_n(\mathbf{y}_i) = (\mathbf{e}_r^{\mathrm{T}}\mathbf{y}_i)\mathbf{e}_s\mathbf{1}_n^{\mathrm{T}} - \mathbf{e}_s\mathbf{y}_i^{\mathrm{T}} - (\mathbf{e}_s^{\mathrm{T}}\mathbf{y}_i)\mathbf{e}_r\mathbf{1}_n^{\mathrm{T}} + \mathbf{e}_r\mathbf{y}_i^{\mathrm{T}}.$$

Now, since the trace is a linear mapping we have

$$\operatorname{tr}\left((\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}})^{\mathrm{T}}\phi_n(\mathbf{y}_i)\right) = 2\mathbf{e}_r^{\mathrm{T}}\mathbf{y}_i - 2\mathbf{e}_s^{\mathrm{T}}\mathbf{y}_i = 2(\mathbf{e}_r^{\mathrm{T}} - \mathbf{e}_s^{\mathrm{T}})\mathbf{y}_i.$$

Now, from (3) and Note 4, we have

$$p_n(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}}) = \frac{1}{n}\sum_{i=1}^{n-1}\frac{(\mathbf{e}_r^{\mathrm{T}} - \mathbf{e}_s^{\mathrm{T}})\mathbf{y}_i}{\|\mathbf{y}_i\|_2^2}(\mathbf{y}_i\mathbf{1}_n^{\mathrm{T}} - \mathbf{1}_n\mathbf{y}_i^{\mathrm{T}}). \tag{4}$$

Since $\{\mathbf{1}_n, \mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ is an orthogonal basis of $\mathbb{R}^n$ and $\|\mathbf{1}_n\|_2^2 = n$, we have

$$\mathbf{e}_r - \mathbf{e}_s = \frac{(\mathbf{e}_r^{\mathrm{T}} - \mathbf{e}_s^{\mathrm{T}})\mathbf{1}_n}{n}\mathbf{1}_n + \sum_{i=1}^{n-1}\frac{(\mathbf{e}_r^{\mathrm{T}} - \mathbf{e}_s^{\mathrm{T}})\mathbf{y}_i}{\|\mathbf{y}_i\|_2^2}\mathbf{y}_i. \tag{5}$$

And since $\mathbf{e}_r^{\mathrm{T}}\mathbf{1}_n = \mathbf{e}_s^{\mathrm{T}}\mathbf{1}_n = 1$, Eq. 5 reduces to $\mathbf{e}_r - \mathbf{e}_s = \sum_{i=1}^{n-1}\frac{(\mathbf{e}_r^{\mathrm{T}} - \mathbf{e}_s^{\mathrm{T}})\mathbf{y}_i}{\|\mathbf{y}_i\|_2^2}\mathbf{y}_i$. Hence (4) leads to

$$p_n(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}}) = \frac{1}{n}\left[(\mathbf{e}_r - \mathbf{e}_s)\mathbf{1}_n^{\mathrm{T}} - \mathbf{1}_n(\mathbf{e}_r - \mathbf{e}_s)^{\mathrm{T}}\right]. \tag{6}$$

Now, it can be easily proven that

$$\begin{aligned}&(\mathbf{e}_r - \mathbf{e}_s)\mathbf{1}_n^{\mathrm{T}} - \mathbf{1}_n(\mathbf{e}_r - \mathbf{e}_s)^{\mathrm{T}}\\ &\quad = 2(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}}) + \sum_{j\notin\{r,s\}}\mathbf{e}_r\mathbf{e}_j^{\mathrm{T}} - \mathbf{e}_j\mathbf{e}_r^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_j^{\mathrm{T}} + \mathbf{e}_j\mathbf{e}_s^{\mathrm{T}}.\end{aligned} \tag{7}$$

From (2), (6), and (7) we obtain

$$\begin{aligned}&p_n(L(B)) = p_n(L(A)) +\\ &\quad + \frac{\log\alpha}{n}\left(2(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}}) + \sum_{j\notin\{r,s\}}\mathbf{e}_r\mathbf{e}_j^{\mathrm{T}} - \mathbf{e}_j\mathbf{e}_r^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_j^{\mathrm{T}} + \mathbf{e}_j\mathbf{e}_s^{\mathrm{T}}\right),\end{aligned}$$

and, finally,

$$Y_B = Y_A \odot E\left(\frac{\log\alpha}{n}\left(2(\mathbf{e}_r\mathbf{e}_s^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_r^{\mathrm{T}}) + \sum_{j\notin\{r,s\}} \mathbf{e}_r\mathbf{e}_j^{\mathrm{T}} - \mathbf{e}_j\mathbf{e}_r^{\mathrm{T}} - \mathbf{e}_s\mathbf{e}_j^{\mathrm{T}} + \mathbf{e}_j\mathbf{e}_s^{\mathrm{T}}\right)\right).$$

We obtain the following result from this last expression.

Theorem 5 *Let $A \in M_{n,n}^{+}$ and let Y_A be the consistent matrix closest to A. If B is defined by (1) and Y_B is the consistent matrix closest to B, then $[Y_B]_{r,s} = \alpha^{2/n}[Y_A]_{r,s}$, $[Y_B]_{s,r} = \alpha^{-2/n}[Y_A]_{s,r}$; if $j \notin \{r,s\}$, then $[Y_B]_{r,j} = \alpha^{1/n}[Y_A]_{r,j}$, $[Y_B]_{j,r} = \alpha^{-1/n}[Y_A]_{j,r}$, $[Y_B]_{s,j} = \alpha^{-1/n}[Y_A]_{s,j}$, $[Y_B]_{j,s} = \alpha^{1/n}[Y_A]_{j,s}$; and in the remaining cases, $[Y_B]_{i,j} = [Y_A]_{i,j}$.*

Using the Hadamard product, Theorem 5 can be rewritten in a more compact form.

Theorem 6 *Let $A \in M_{n,n}^{+}$ and let Y_A be the consistent matrix closest to A. If B is defined by (1) and Y_B is the consistent matrix closest to B, then*

$$Y_B = Y_A \odot (\mathbf{x}\mathbf{y}^{\mathrm{T}}),$$

where $\mathbf{x} = [x_1 \ \cdots \ x_n]^{\mathrm{T}}$, $\mathbf{y} = [y_1 \ \cdots \ y_n]$, $x_r = y_s = \alpha^{1/n}$, $x_s = y_r = \alpha^{-1/n}$, and $x_i = y_i$ when $i \notin \{r,s\}$.

2.2 Fast computation of the consistent matrix closest to a reciprocal matrix

Although Theorem 3 enables finding the closest consistent matrix to a given reciprocal matrix, we now present a very fast calculation algorithm. Let us remark that reciprocity of the involved matrices is not necessary in Theorems 5 and 6; however, the procedure we present here, which directly implements the calculations described in Theorems 5 and 6, is valid only for reciprocal matrices. We write the algorithm using MatLab language.

The idea of the algorithm is the following: to start with the matrix in $M_{n\times n}$ all of whose components are 1, and then, step by step, to modify this starting matrix and use Theorem 6.

```
function [w,X] = aprox(A)
[n,m] = size(A);
if n == m
    w = ones(n,1);
    for i=1:n
        for j=i+1:n
            w(i)=w(i)*A(i,j)^(1/n);
```

```
            w(j)=w(j)*A(i,j)^(-1/n);
        end
    end
X = w*(w.^(-1))';
else error('Matrix must be square')
end
```

In this file, the input is a reciprocal matrix A, and the output is the closest consistent matrix to A, denoted by X, and the vector $\mathbf{w}$ such that

$$X = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \begin{bmatrix} w_1^{-1} & \cdots & w_n^{-1} \end{bmatrix}.$$

3 An integrated model for achieving consistency in AHP

In this section we develop an iterative process to streamline the trade-off between expert know-how and synthetic consistency from the results of the previous section.

After having stated the problem by clearly defining the top-bottom hierarchical structure of objectives, criteria and alternatives, an AHP-based, real-world, decision-making process starts by collecting expert information. The results are compiled and the entries of the criteria and alternative comparison matrices represent the knowledge of the experts. Let's take a look at any one of these comparison matrices.

Such a matrix will almost certainly be non-consistent, and with non-negligible probability it will not have an acceptable consistency ratio CR, according to Saaty's criterion. The linearization process can now be used to build a consistent matrix. The new matrix may be considered by the expert(s) to partially reflect their opinions and, perhaps, they will choose to modify some of the matrix entries. Shifting one or more entries of the matrix while preserving reciprocity will produce an inconsistent matrix, and a similar process can again be undergone in an attempt to reach a reasonable trade-off between consistency and expert know-how compliance. Obtaining a new consistent matrix from another matrix obtained by shifting one or more entries of a consistent matrix is developed in a straightforward manner by following the process of the linearization-extension described in the previous paragraph.

The following pseudo-code contains the whole process through an iterative dialog between the expert(s) and the consistency points of view:

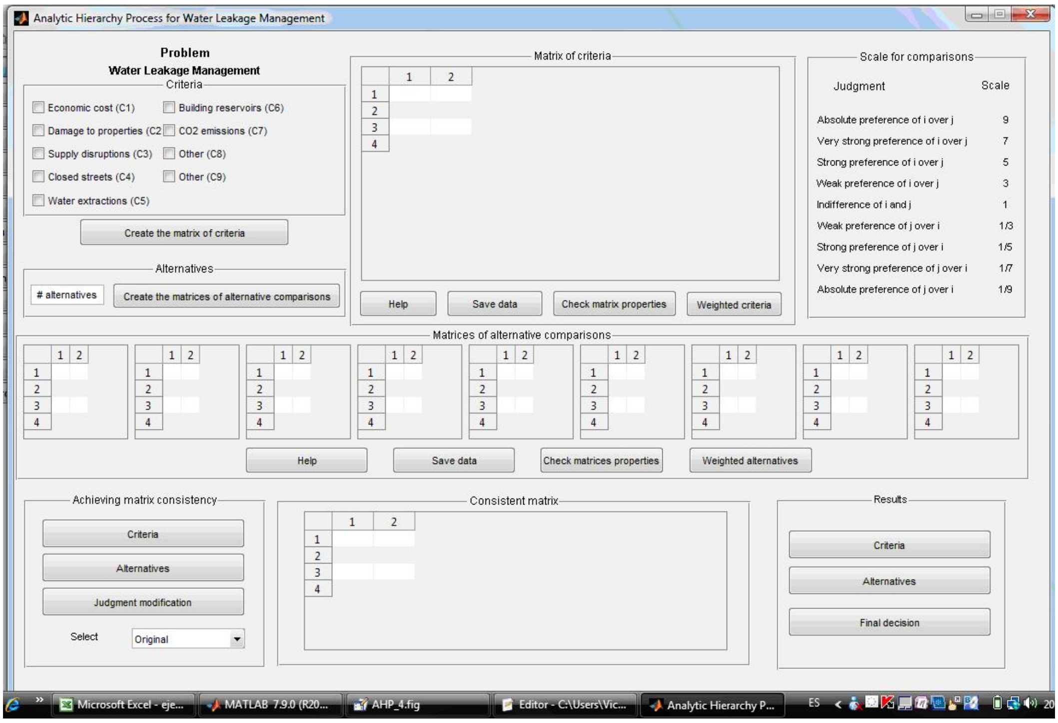

Fig. 1. MatLab GUI for an AHP decision support system

0. Matrix construction by using expert judgment.
   a. Matrix passes consistency test: accept matrix.
   b. Matrix fails consistency test: continue with the following process.
1. Construction of the closest consistent matrix using linearization.
2. Expert assessment to evaluate changes produced by linearization.
   a. Expert agreement: accept matrix.
   b. Expert decides to change: apply the following linearization-extension.
3. Automatic construction of a new consistent matrix by following the procedure described in Section 2.
4. Go to 2 until completion.

This process has been implemented in a tool developed in MatLab. Figure 1 shows the GUI (guided user interface) containing the problem elements. Various methods for consistency improvements may be selected - including the method presented in this paper. The matrix the user has selected is shown in the lower-center part of the GUI. The final decision is accessed by using the buttons in the lower right-hand area.

This tool has been used to develop the decision-making process described in the paragraph below.

4 Application to leakage policy in a water distribution system

We consider here one of the problems that poses a great challenge to water supply managers: the minimization of water loss. Great sums of money are devoted annually to this aim worldwide. We consider a problem with only two management alternatives for leakage control. Active leakage control (ALC) involves taking actions in distribution systems to identify and repair not reported leaks. Passive leakage control (PLC) boils down to just repairing reported or evident leaks [5].

The criteria used to decide on the alternatives are multiple, but decision-makers should be concerned with the tangible and quantitative factors, such as cost, in engineering selection problems; as well as the intangible and qualitative factors, such as environmental and social impacts [3]. We consider the following criteria:

C_1: planning development cost and its implementation;
C_2: damage to property and other service networks;
C_3: effects (cost or compensations) of supply disruptions;
C_4: inconveniences caused by closed or restricted streets;
C_5: water extractions (benefits for aquifers, wetlands, or rivers);
C_6: construction of tanks and reservoirs (environmental and recreational impacts);
C_7: CO_2 emissions (related to energy used in pumping stations).

We use here the point of view of the management department of a supply company - OOAPAS (Public Water Company) in Morelia, Michoacán (Mexico). Upon evaluation using the 9-point Saaty scale, the matrix A in Table 1 was produced.

Table 1
Matrix of criteria, A

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
C_1	1	7	9	5	7	5	3
C_2	1/7	1	5	9	5	7	5
C_3	1/9	1/5	1	7	3	7	3
C_4	1/5	1/9	1/7	1	7	5	5
C_5	1/7	1/5	1/3	1/7	1	9	7
C_6	1/5	1/7	1/7	1/5	1/9	1	5
C_7	1/3	1/5	1/3	1/5	1/7	1/5	1

This matrix is positive, homogeneous and reciprocal, yet inconsistent. Since

$\lambda_{\max} \simeq 10.56$, $CI \simeq 0.5948$, and $CR \simeq 44.06\%$, consistency of this matrix is inadmissible.

The process described in Section 3 was launched. The final consensus between consistency and expertise is given in Table 2.

Table 2
Trade-off between consistency and expert judgment

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
C_1	1	1.28	1.17	1.77	2.12	4.49	9.89
C_2	0.78	1	0.92	1.39	1.66	3.52	7.74
C_3	0.85	1.09	1	1.51	1.81	3.83	8.42
C_4	0.57	0.72	0.66	1	1.20	2.54	5.59
C_5	0.47	0.60	0.55	0.83	1	2.16	4.66
C_6	0.22	0.28	0.26	0.39	0.47	1	2.20
C_7	0.10	0.13	0.12	0.18	0.22	0.45	1

The Perron eigenvector providing the priority vector, normalized so that $\mathbf{1}_7^{\mathrm{T}}\mathbf{w} = 1$, is $\mathbf{Z} = [0.25,\ 0.20,\ 0.21,\ 0.14,\ 0.12,\ 0.06,\ 0.02]^{\mathrm{T}}$. Thus, greater importance is placed on criterion C_1 closely followed by C_3 and C_2.

5 Conclusions

In this paper, by extending a linearization process already published by the authors [2], and describing an efficient implementation for the calculations, an algorithm is developed that follows an iterative feedback process and achieves an acceptable level of consistency while complying to some degree with expert preferences.

Our ultimate objective was to devise a method and then integrate it into a suitable DSS tool. This method would help practitioners build comparison matrices that both rely on their judgment and are efficient and reliable in deriving priorities.

An application to a real decision-making problem in water management has been presented. The study enhances the relevance of the economic aspects, showing the leading role in the decision played by planning development implementation and costs. The interesting aspect regarding the application of AHP is indeed the inclusion of social and environmental costs in decision-making.

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