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# On the spectra of algebras of analytic functions

Daniel Carando, Domingo García, Manuel Maestre, and Pablo Sevilla-Peris

ABSTRACT. In this paper we survey the most relevant recent developments on the research of the spectra of algebras of analytic functions. We concentrate mainly in three algebras, the Banach algebra  $H^\infty(B)$  of all bounded holomorphic functions on the unit ball  $B$  of a complex Banach space  $X$ , the Banach algebra of the ball  $A_u(B)$ , and the Fréchet algebra  $H_b(X)$  of all entire functions bounded on bounded sets.

## 1. Introduction

Complex analysis permeates almost any aspect of the mathematics since the early nineteenth century when it was first developed by Cauchy. But it appears that, around 1955, S. Kakutani was the first one to study the space  $H^\infty(\mathbb{D})$  of all bounded holomorphic functions on  $\mathbb{D}$ , the open unit disk of the complex plane, as a Banach algebra. In the immediately following years a group of mathematicians (Singer, Wermer, Kakutani, Buck, Royden, Gleason, Arens and Hoffman) under a joint pseudonym (I. J. Schark [62]) and in individual papers used this functional analytic point of view to develop the study of many aspects of the theory.

If we consider the space  $H^\infty(\mathbb{D})$  as a Banach algebra, a key element is to describe the spectrum  $\mathfrak{M}(H^\infty(\mathbb{D}))$ , i.e. the set of all multiplicative linear functionals on  $H^\infty(\mathbb{D})$  which is a compact set when endowed with the weak-star topology. The biggest milestone of this early period is the Corona Theorem, given by Newman (in a weak form) and Carleson [26] in 1962, that states that the evaluations at points of  $\mathbb{D}$  form a dense subset of the spectrum of  $H^\infty(\mathbb{D})$ .

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In the very relevant paper by I. J. Scharf [62] published in 1961 we find most of the main problems that has attracted the attention of many researchers to this field since then. Namely, the relationship between the evaluations at points of the open unit disk  $\mathbb{D}$  with the whole spectrum  $\mathfrak{M}(H^\infty(\mathbb{D}))$ , the fibers, the Corona theorem, the size of a fiber, the study of the clusters sets and the image of a fiber by the Gel'fand transform of an element of  $H^\infty(\mathbb{D})$ , the embedding of analytic disks on a fiber, the existence or not of analytic structure in the spectrum, and even the Shilov boundary (which we are not going to discuss here). Later we will be more precise about some of the above items, but let us mention that the I. J. Scharf's paper is probably more important for the questions that it raises than for the results themselves.

On the other hand from the beginning of the eighties of the last century researchers working in Banach algebras as Aron, Cole, Davie, Gamelin and Johnson, considered that kind of questions for Banach and Fréchet algebras of analytic functions defined on open sets in an infinite dimensional Banach space. The concept of holomorphic function on a Banach space can be traced back to Hilbert in 1907 but had been greatly developed in the seventies by the Nachbin's school. For us a holomorphic (also analytic) function on an open subset of a complex Banach space is a complex-Fréchet differentiable function at each point of that open set.

In the last years the coauthors of this survey, many times in collaboration with the aforementioned researchers have made some progress to the field under review. To make a short description of the results presented here we need some notation. If  $X$  is a (infinite dimensional) complex Banach space,  $B$  (or  $B_X$ ) will denote its open unit ball,  $B^{**}$  will be the open unit ball of the bidual  $X^{**}$ . The closed unit ball of  $X^{**}$  obviously coincides with  $\overline{B^{**}}$ , but also with  $\overline{B}^{w(X^{**}, X^*)}$ , the weak-star closure of  $B$  in  $X^{**}$ . A mapping  $P : X \rightarrow \mathbb{C}$  is called an  $m$ -homogeneous continuous polynomial if there exists an  $m$ -linear continuous form  $M : X^m \rightarrow \mathbb{C}$  such that  $P(x) = M(x, \dots, x)$  for every  $x \in X$  and  $\|P\| = \sup\{|P(x)| : \|x\| \leq 1\} < \infty$ . In that case  $\|P\|$  is called the (supremum) norm of  $P$ . A function  $f$  is holomorphic if and only if for each  $x$  it has a Taylor expansion  $\sum_{m=0}^{\infty} P_{m,x}$  on some ball centered on  $x$ , where each  $P_{m,x}$  is an  $m$ -homogeneous continuous polynomial.

The most important Banach algebras that we are going to deal with are  $H^\infty(B)$ , the space of all bounded analytic functions on  $B$ , the open unit ball of  $X$ , and the algebra of the ball  $A_u(B)$ , the Banach algebra of all uniformly continuous holomorphic functions on  $B$ . Uniformly continuous mappings on a bounded convex set are bounded, so  $A_u(B)$  is a closed subalgebra of  $H^\infty(B)$ . The Banach algebra  $A_u(B)$  can also be seen as the algebra of all functions that are holomorphic on the open unit ball  $B$  and uniformly continuous on the closed unit ball  $\bar{B}$ . It can also be described as the completion of the space of all continuous polynomials on  $X$ . Both  $H^\infty(B)$  and  $A_u(B)$  are endowed with the supremum norm on  $B$ .

Given a Banach algebra  $A$ , the set  $\mathfrak{M}(A)$  (called the spectrum of  $A$ ) is the family of all continuous linear functionals on  $A$  that are also multiplicative (actually the continuity is a redundant condition). Each  $a \in A$  defines a mapping, called the *Gel'fand transform*,  $\hat{a} : \mathfrak{M}(A) \rightarrow \mathbb{C}$  by  $\hat{a}(\varphi) = \varphi(a)$ .

In Section 2 we review the state of the art about the Corona Theorem. In Section 3, we present the main known results on a weak version of the Corona Theorem, that is called the Cluster Value Theorem both for  $H^\infty(B)$  and  $A_u(B)$ . It is important to point out that the spectrum of  $A_u(B)$  is reduced to the evaluations on the closed unit ball  $\bar{B}$  whenever  $X$  is finite dimensional, but many surprising rich structures arise when  $X$  is infinite dimensional. In Section 4 we describe the relevant known results about how big is the spectrum of the algebras  $H^\infty(B)$  and  $A_u(B)$ .

In Section 5 the Fréchet algebra of analytic functions  $H_b(U)$  is introduced. Again some notation is pertinent. If  $U$  is an open subset of a complex Banach space  $X$ , then a subset  $E$  of  $U$  is called  $U$ -bounded if it is bounded and has positive distance to the boundary of  $U$ . The Fréchet algebra of all holomorphic functions on  $U$  that are bounded on  $U$ -bounded sets endowed with the topology of the uniform convergence on the  $U$ -bounded subsets of  $U$  is denoted by  $H_b(U)$ . If  $A$  is a Banach algebra its spectrum  $\mathfrak{M}(A)$  is always a compact set endowed with the restriction of the weak-star topology, but that is not the case for  $\mathfrak{M}(H_b(U))$ . Aron, Cole and Gamelin in a seminal paper in 1991 [5], made a deep study of that spectrum. The main result in Section 5 is the fact that if the Banach space is symmetrically regular

(see inside this section for the definition) then a Riemann analytic structure can be given to  $\mathfrak{M}(H_b(U))$ .

Section 6 is devoted to the study of algebras of weighted analytic functions, of which  $H_b(U)$  can be considered a particular case. There we survey results about the existence of analytic structure on the spectrum and several properties associated to that situation.

Finally in Section 7 several applications are shown of the use of the spectra of different Fréchet algebras of analytic functions to obtain Banach-Stone type theorems for couple of Fréchet algebras that are algebra isomorphic. Also some very pathological phenomena are described.

## 2. The Corona Theorem for $H^\infty(B)$

Given a set  $U$  we denote by  $\mathcal{B}(U)$  the Banach algebra of all bounded mappings  $f : U \rightarrow \mathbb{C}$ , endowed with the supremum norm  $\|f\|_\infty = \sup\{|f(x)| : x \in U\}$ . Consider a closed subalgebra  $A$  of  $\mathcal{B}(U)$ . For each  $x \in U$ , we can define the evaluation at  $x$  as  $\delta_x(f) = f(x)$ , for  $f \in A$ , which is a homomorphism on  $\mathcal{B}(U)$ . We will denote

$$\Delta(A) = \{\delta_x : x \in U\}$$

Following Newman (1959) [57], the set  $\mathfrak{M}(A) \setminus \overline{\Delta(A)}^{w^*}$  is usually referred to as *the Corona*, where the closure is taken with respect to the weak-star topology  $w(A^*, A)$  on  $\mathfrak{M}(A)$ . We are going to say that the Corona Theorem holds for  $A$  when  $\Delta(A)$  is weak-star dense in  $\mathfrak{M}(A)$ . Equivalently, if the Corona set is the empty set. If  $A$  is a commutative Banach algebra with unity, then an element of  $A$  is invertible if and only if there does not exist a homomorphism on  $A$  that vanishes on that element (see e.g. [61, 11.5 Theorem, p. 257]). This fact, together with the description of the weak-star topology  $w(A^*, A)$ , leads to the following equivalence.

**THEOREM 2.1.** *Let  $A$  be a closed subalgebra of  $\mathcal{B}(U)$ . The set  $\Delta(A)$  is weak-star dense in  $\mathfrak{M}(A)$  if and only whenever  $f_1, \dots, f_n \in A$  satisfy*

$$|f_1(x)| + \dots + |f_n(x)| \geq \delta > 0$$

for some  $\delta > 0$  and every  $x \in \Omega$ , there exist  $g_1, \dots, g_n \in A$  with

$$\sum_{j=1}^n f_j g_j = 1.$$

A proof of this equivalence along the lines indicated above when  $A$  is the space  $H^\infty(\mathbb{D})$  of bounded holomorphic function on the open unit disk  $\mathbb{D}$  in the complex plane can be found in [50, Theorem, p. 163].

Even though we formulated the Corona problem in full generality, in this paper we are going to restrict ourselves to the case in which  $U$  is a bounded open subset of a complex Banach space  $X$ , and  $A$  is a Banach algebra of bounded holomorphic functions on  $U$ , endowed with the supremum norm. We will always assume that  $A$  contains both the constant functions and the dual  $X^*$  of  $X$ . Actually the main two cases we want to discuss here are  $H^\infty(B)$  and  $A_u(B)$ . Another Banach algebra that will be of interest is  $A_a(B)$ , the subalgebra of  $A_u(B)$  of all approximable holomorphic functions. In others words  $A_a(B)$  is the closure in  $A_u(B)$  of the algebra generated by the constant functions and  $X^*$ . If  $X$  is finite dimensional then  $A_a(B) = A_u(B)$ , but in general they are different: for example  $A_a(\ell_2)$  is a proper subalgebra of  $A_u(\ell_2)$ .

Take a Banach algebra  $A$ , with  $A_a(B) \subset A \subset H^\infty(B)$  (endowed with the supremum norm). Since  $X^*$  is always included in  $A_a(B)$ , we can define  $\pi : \mathfrak{M}(A) \rightarrow X^{**}$  by

$$(2.1) \quad \pi(\varphi) = \varphi|_{X^*},$$

i.e.,  $\pi(\varphi)$  is the restriction of the homomorphism  $\varphi$  to  $X^*$ . Since the norm of any non-zero homomorphism is 1,  $\pi(\mathfrak{M}(A))$  is a subset of  $\overline{B^{**}}$ , the closed unit ball of  $X^{**}$ . For  $z \in \overline{B^{**}}$ , the fiber of the spectrum of  $A$  at the point  $z$  is defined as

$$\mathfrak{M}_z(A) = \{\varphi \in A : \pi(\varphi) = z\} = \pi^{-1}(z).$$

The fibers of the spectrum of the Banach algebras at each point of  $\overline{B^{**}}$  that we consider are non-empty, i.e  $\pi(\mathfrak{M}(A)) = \overline{B^{**}}$ . This is a consequence of the fact that  $\pi$  is  $w(A^*, A)$ - $w(X^{**}, X^*)$  continuous, hence  $\pi(\mathfrak{M}(A)) \subset \overline{B^{**}}$  is a  $w(X^{**}, X^*)$ -compact containing in  $B$ , and  $\overline{B^{**}} = \overline{B}^{w(X^{**}, X^*)}$ . According to Aron and Berner [3] every function  $f$  in  $H_b(X)$  extends in a natural way to a holomorphic function  $\tilde{f}$  (called the *Aron-Berner extension*) on  $X^{**}$  and this extension gives an (algebraic

and topological) isomorphism of  $H_b(X)$  and a closed subalgebra of  $H_b(X^{**})$ . The following result by Davie and Gamelin (1989) on the Aron-Berner extension gives a way of building homomorphisms that belong to the fibers at each point of  $B^{**}$ .

**THEOREM 2.2.** [28, Theorem 5] *If  $f \in H^\infty(B)$  has series Taylor expansion at zero  $\sum_{m=0}^\infty P_m$  and each  $\tilde{P}_m : X^{**} \rightarrow \mathbb{C}$  is the Aron-Berner extension of the polynomial  $P_m$  to the bidual, then the series  $\sum_{m=0}^\infty \tilde{P}_m$  converges on  $B^{**}$  to a function  $\tilde{f} \in H^\infty(B^{**})$ , satisfying*

$$\|\tilde{f}\|_{B^{**}} = \|f\|_B.$$

We also have  $\widetilde{fg} = \tilde{f}\tilde{g}$  for  $f, g \in H^\infty(B)$ .

An immediate consequence is that the mapping  $\tilde{\delta} : A \rightarrow \mathbb{C}$  defined by

$$(2.2) \quad \tilde{\delta}_z(f) := \tilde{f}(z),$$

is a (continuous) homomorphism on  $A$ , and  $\tilde{\delta}_z \in \mathfrak{M}_z(A)$  for each  $z \in B^{**}$ . We will address later questions about the size of the fibers, but let us say here that in 1991 Aron, Cole and Gamelin in [5, 11.1 Theorem] proved that on an infinite dimensional space  $X$ , the fiber  $\mathfrak{M}_z(H^\infty(B))$  is infinite for every  $z$  in  $\overline{B}^{**}$ .

In the one variable problem, the Banach space under consideration is  $\mathbb{C}$ . The linear function  $z \mapsto z$  is then a basis of the dual of  $\mathbb{C}$ . Hence the mapping  $\pi : \mathfrak{M}(H^\infty(\mathbb{D})) \rightarrow \mathbb{C}^{**}$  given in (2.1) can be also considered as  $\pi : \mathfrak{M}(H^\infty(\mathbb{D})) \rightarrow \overline{\mathbb{D}}$ , with

$$(2.3) \quad \pi(\varphi) = \hat{z}(\varphi) = \varphi(z \mapsto z),$$

where  $\hat{z}$  is the Gel'fand transform of the mapping  $z \mapsto z$ . Scharck proved the following (easy result).

**THEOREM 2.3.** [62, Theorem 2.1] *The projection  $\pi$  defined by (2.3) is a continuous mapping of  $\mathfrak{M}(H^\infty(\mathbb{D}))$  onto the closed unit disk  $\overline{\mathbb{D}}$ . If  $\Delta = \pi^{-1}(\mathbb{D})$ , then  $\pi$  maps the open set  $\Delta$  onto the open unit disk  $\mathbb{D}$ .*

It was also shown that  $\pi$  is one to one on  $\Delta$ , and as a consequence  $\pi^{-1}(a) = \delta_a$  for every  $a \in \mathbb{D}$ . Scharck posed several questions in the article, two or them are the following: is  $\mathfrak{M}(H^\infty(\mathbb{D})) \setminus \Delta$  connected? Is each fiber  $\mathfrak{M}_a(H^\infty(\mathbb{D}))$  connected?

These two questions were answered in the positive in the same issue by K. Hoffman [49, Theorems 1 and 2]. But Shark posed a far tougher question (which had already been asked by S. Kakutani in 1957): is the open disk  $\Delta$  dense in  $\mathfrak{M}(H^\infty(\mathbb{D}))$ ? In other words is true that the Corona is the empty set for  $H^\infty(\mathbb{D})$ ? This question, was answered by L. Carleson in 1962, by greatly improving an independent result by Newman on interpolation sequences [57, Theorem 1] or [50, p. 197]. The statement of Carleson's result is the following:

**THEOREM 2.4** (The Corona Theorem). [26, Theorem 5] *Let  $f_1, \dots, f_j$  be given functions in  $H^\infty(\mathbb{D})$  such that*

$$|f_1(z)| + |f_2(z)| + \dots + |f_j(z)| \geq \delta > 0 ,$$

*for some  $\delta > 0$  and every  $z \in \mathbb{D}$ . Then the ideal generated by  $f_1, f_2, \dots, f_j$  coincides with  $H^\infty(\mathbb{D})$ .*

The proof is based on deep work developed by L. Carleson on some special measures, the nowadays so called Carleson measures. The interested reader can find different proofs of the Corona Theorem. For example, in 1980 T. Gamelin in [39] gave an expository account of the unpublished Wolff's proof.

By the Riemann Mapping Theorem, the Corona Theorem holds for any simply connected proper domain  $U$  of  $\mathbb{C}$ . But as far as today it is unknown if the Corona Theorem is true for every domain in  $\mathbb{C}$ .

**QUESTION 1.** Does the Corona Theorem hold in  $H^\infty(U)$ , for any domain  $U$  of  $\mathbb{C}$ ?

There are many positive partial answers. For example Stout [64] gave a proof of the Corona Theorem for finitely connected domains in  $\mathbb{C}$ . M. Behrens [9] was the first to find a class of infinitely connected domains in the complex plane for which the Corona Theorem holds. Later in 1985, J. B. Garnett and P. W. Jones [47] proved the Corona Theorem for Denjoy domains and that was extended in 1987 by C. N. Moore [55] to domains  $U = \mathbb{C} \setminus K$  where  $K$  is a compact subset of a  $C^{1+\alpha}$  Jordan curve.

Nevertheless, for general Riemann surfaces the Corona Theorem fails. That was proved by Cole and a proof can be found in [37, 4.2 Theorem, p. 49]. Nowadays,

there is a great interest in the question about the kind of Riemann surfaces for which the Corona Theorem holds. For example, in 2008 Brudnyi [20, Theorem 1.1.1] proved that the Corona Theorem is true in  $H^\infty(Y)$  where  $Y$  is a Carathéodory hyperbolic Riemann surface of finite type.

In several variables all the known results about the Corona Theorem are essentially negative. For a very simple counterexample, take a Hartog figure in  $\mathbb{C}^2$  of the form  $R = \{(z_1, z_2) : 0 < a_j < |z_j| < b_j, j = 1, 2\}$ . It is known that every  $f \in H^\infty(R)$  extends uniquely to a function  $\tilde{f} \in H^\infty(\mathbb{D}^2)$ , and moreover  $\|\tilde{f}\|_\infty = \|f\|_\infty$ . Hence the evaluations at elements of the bidisk  $\mathbb{D}^2$  are elements of  $\mathfrak{M}(H^\infty(R))$ . On the other hand, if  $\delta_{z_\alpha}$  is a net in  $\Delta(R)$  that weak-star converges to  $\delta_z$ , with  $z \in \mathbb{C}^2$ , it follows that  $z_\alpha$  is convergent to  $z$  in  $\mathbb{C}^2$ . As a consequence, the evaluations at elements of  $R$  cannot be dense in  $\mathfrak{M}(H^\infty(R))$  and the Corona Theorem fails for  $H^\infty(R)$ . Even if we restrict ourselves to domains of holomorphy in  $\mathbb{C}^n$  (or, equivalently, to pseudoconvex domains) then the Corona Theorem fails in general: N. Sibony [63] found a pseudoconvex domain  $U$  in  $\mathbb{C}^2$  for which there is no Corona Theorem. On the other hand, it remains an open question whether the Corona Theorem holds or fails for the most straightforward domains in  $\mathbb{C}^n$ : the polydisk and the Euclidean ball.

QUESTION 2. Does the Corona Theorem hold in  $H^\infty(\mathbb{D}^n)$ , for any  $n \geq 2$ ?

We denote by  $B_{\ell_2^n}$  the open unit ball of the Euclidean space  $\ell_2^n = (\mathbb{C}^n, \|\cdot\|_2)$ , that is, the  $n$ -dimensional Euclidean ball.

QUESTION 3. Does the Corona Theorem hold in  $H^\infty(B_{\ell_2^n})$ , for any  $n \geq 2$ ?

More in general,

QUESTION 4. Does the Corona Theorem hold in  $H^\infty(U)$ , for some open subset  $U$  of  $\mathbb{C}^n$ ,  $n \geq 2$ , such that  $H^\infty(U)$  is not reduced to the constants functions?

### 3. A weak Corona theorem: The Cluster Value Theorem for $H^\infty(B)$ and $A_u(B)$

Clearly there is little hope to get advances in the short term on the Corona Theorem on  $H^\infty(B)$  when  $B$  is the open unit ball of a Banach space of dimension

strictly greater than one. But I. J. Schark in the same paper proved the following result.

**THEOREM 3.1.** [62, Theorem 4.1 and Corollary to Theorem 4.3] *Let  $f \in H^\infty(\mathbb{D})$  and  $a \in \mathbb{C}$ ,  $|a| \leq 1$ . The range of the Gel'fand transform  $\hat{f}$  of  $f$  on the fiber  $\mathfrak{M}_a(H^\infty(\mathbb{D}))$  consists of those complex numbers  $\xi$  for which there is a sequence  $(\lambda_n)$  in  $\mathbb{D}$  with*

- (1)  $\lambda_n \rightarrow a$ ,
- (2)  $f(\lambda_n) \rightarrow \xi$ .

In other words,

$$\{\varphi(f): \pi(\varphi) = a, \varphi \in \mathfrak{M}(H^\infty(\mathbb{D}))\} = \{\xi \in \mathbb{C}: \exists(\lambda_n) \subset \mathbb{D}, \lambda_n \rightarrow a, f(\lambda_n) \rightarrow \xi\},$$

for all  $a \in \overline{\mathbb{D}}$  and  $f \in H^\infty(\mathbb{D})$ . The set in the right hand side is called the cluster set of  $f$  at  $a$ , and consists of all limits of values of  $f$  along sequences in  $\mathbb{D}$  converging to  $a$ . Motivated by that result the cluster sets for bounded holomorphic functions on the unit ball of a complex Banach space are defined.

**DEFINITION 3.2.** Let  $X$  be a complex Banach space and let  $B$  its open unit ball. For fixed  $f \in H^\infty(B)$  and  $x \in \overline{B^{**}}$ , the *cluster set*  $Cl_B(f, x)$  is the set of all complex numbers  $\lambda$  for which there exists a net  $(x_\alpha)$  in  $B$  converging weak-star to  $x$ , such that  $f(x_\alpha)$  converges to  $\lambda$ .

There are two key properties of the cluster sets. The first one is the following.

**PROPOSITION 3.3.** [4, Lemma 2.1] *Let  $f \in H^\infty(B)$ . Each cluster set  $Cl_B(f, x)$ ,  $x \in \overline{B^{**}}$ , is a compact connected set. Furthermore, if  $x \in B$ , then  $f(x) \in Cl_B(f, x)$ .*

By the mentioned J. I. Schark's results, if we take an interior point  $a \in \mathbb{D}$  and  $f \in H^\infty(\mathbb{D})$ , we have  $Cl_{\mathbb{D}}(f, a) = \{f(a)\}$ . When dealing with infinite dimensional Banach spaces, the cluster set can be 'as big as possible', as the next example given in [4] shows. Take  $X$  an infinite dimensional Hilbert space and let  $\{\lambda_n\} \subset \mathbb{D}$  be a sequence which is dense in the closed unit disk  $\overline{\mathbb{D}}$ . Define  $f(x) = \sum \lambda_n(x_n)^2$ , where the  $x_n$ 's are the coordinates of  $x$  with respect to some orthonormal subset  $\{e_n\}$  of  $X$ . It is clear that  $f$  is a two-homogeneous entire function and, in particular,

analytic on the open unit ball  $B$  of  $X$ . Also,  $|f| \leq 1$  on  $B$  and  $Cl_B(f, 0)$  coincides with the closed unit disk  $\overline{\mathbb{D}}$ .

The second key property of cluster sets is given by the next proposition, which is a consequence of the compactness of the spectrum.

**PROPOSITION 3.4.** [4, Lemma 2.2] *Let  $A$  be a Banach algebra of bounded analytic functions on  $B$  with the supremum norm such that  $A_u(B) \subset A \subset H^\infty(B)$ . If  $f \in A$  and  $x \in \overline{B^{**}}$ , then  $Cl_B(f, x) \subseteq \hat{f}(\mathfrak{M}_x(A))$ .*

Note that this proposition, together with the example above of the ‘big cluster set’, shows that the fiber  $\mathfrak{M}_0(A_u(B))$  is infinite for any infinite dimensional Hilbert space (this is a particular case of [5, 11.1 Theorem]).

In [4] the following concept is introduced. A *Cluster Value Theorem at  $x \in \overline{B^{**}}$*  for the algebra  $A$  is a theorem that asserts that

$$(3.1) \quad Cl_B(f, x) = \hat{f}(\mathfrak{M}_x(A)), \quad \text{for all } f \in A.$$

We will say that the *Cluster Value Theorem* holds for the algebra  $A$  if it holds for every  $x \in \overline{B^{**}}$ . An interesting consequence of Proposition 3.3 is that whenever the Cluster Value Theorem holds at  $x \in \overline{B^{**}}$  for  $A$ , the fiber  $\mathfrak{M}_x(A)$  is connected (see [4, Remark after Lemma 2.2]).

Note that with this terminology, I. J. Schark’s Theorem 3.1 states that the Cluster Value Theorem is true for  $H^\infty(\mathbb{D})$ . T. W. Gamelin in 1970 gave a general positive answer to the Cluster Value Theorem on open subsets of  $\mathbb{C}$ .

**THEOREM 3.5.** [36, Theorem 2.5] *Let  $U$  be an open subset of the Riemann sphere  $\mathbb{C} \cup \{\infty\}$  such that  $H^\infty(U)$  contains a nonconstant function. Then the cluster set of  $f \in H^\infty(U)$  and  $a \in \partial U$  (the boundary of  $U$ ) coincides with the range (of the Gel’fand transform) of  $f$  on  $\mathfrak{M}_a(H^\infty(U))$ .*

In 1973, T. W. Gamelin proved the Cluster Value Theorem for the polydisk  $\mathbb{D}^n$ .

**THEOREM 3.6.** [38, Theorem 7.5] *Let  $U = U_1 \times \dots \times U_n$ , an open set in  $\mathbb{C}^n$ , where each  $U_j$  is a bounded open set in  $\mathbb{C}$ . Then for every  $f \in H^\infty(U)$  and every  $x \in \overline{U}$*

$$\hat{f}(\mathfrak{M}_x(H^\infty(U))) = Cl_U(f, x).$$

In 1979 G. McDonald [53] obtained the Cluster Value Theorem for strongly strictly pseudoconvex domains in  $\mathbb{C}^n$ . In his paper he says that Garnett had obtained the same result for  $B_{\ell_2^n}$ , the Euclidean ball of  $\mathbb{C}^n$ .

THEOREM 3.7. [53, Theorem 2] *For every  $f \in H^\infty(B_{\ell_2^n})$  and every  $x \in \overline{B_{\ell_2^n}}$*

$$\hat{f}(\mathfrak{M}_x(H^\infty(B_{\ell_2^n}))) = Cl_{B_{\ell_2^n}}(f, x).$$

It is worth mentioning that both results rely on the solution to the  $\bar{\partial}$ -equation with extra conditions. Some natural questions remain open.

QUESTION 5. Characterize the domains of holomorphy (equivalently, the pseudoconvex domains)  $U$  in  $\mathbb{C}^n$  such that the Cluster Value Theorem is true for  $H^\infty(U)$ .

Since the Reinhardt domains in  $\mathbb{C}^n$  that are logarithmically convex are the domain of existence of powers series in  $\mathbb{C}^n$ , the next open question is a natural one.

QUESTION 6. Does the Cluster Value Theorem hold for  $H^\infty(R)$  when  $R$  is a Reinhardt bounded logarithmically convex domain in  $\mathbb{C}^n$ ?

T. W. Gamelin's result (Theorem 3.6) can be seen as a positive answer to a weak version of Question 2 and, in general, a Cluster Value Theorem can be considered a weak Corona Theorem. Let us explain this claim. As we have already pointed out, the Corona Theorem holds for an algebra  $A$  with  $A_u(B) \subset A \subset H^\infty(B)$ , if and only if whenever  $f_1, \dots, f_n \in A$  satisfy  $|f_1| + \dots + |f_n| \geq \delta > 0$  on  $B$ , for some  $\delta > 0$ , there exist  $g_1, \dots, g_n \in H$  such that  $f_1g_1 + \dots + f_ng_n = 1$ . Also, if the Corona Theorem holds, then evidently the Cluster Value Theorem holds at all points  $x \in \overline{B^{**}}$ . The following results from [4] shows how close is that a Cluster Value Theorem to the condition above.

PROPOSITION 3.8. [4, Lemma 2.3] *The Cluster Value Theorem (3.1) holds at every  $x \in \overline{B^{**}}$  if and only if whenever  $f_1, \dots, f_{n-1} \in A_a(B)$  and  $f_n \in A$  satisfy  $|f_1| + \dots + |f_n| \geq \delta > 0$  on  $B$ , for some  $\delta > 0$ , there exist  $g_1, \dots, g_n \in H$  such that  $f_1g_1 + \dots + f_ng_n = 1$ .*

As we said before, since nothing positive is known for the Corona Theorem for  $\mathbb{C}^n$   $n \geq 2$ , in the setting of infinite dimensional Banach spaces this problem is

up to day impossible to attack. However, there are interesting positive results for the Cluster Value Theorem in the infinite dimensional setting. The study of this problem for  $H^\infty(B)$ , for  $B$  the open unit ball of an infinite dimensional Banach space, has been undertaken very recently in [4] by R. M. Aron, D. Carando, T. W. Gamelin, S. Lassalle and M. Maestre. There, the main result is that the Cluster Value Theorem is true for the Banach space  $c_0$  of null sequences. It should be noted that the unit balls of  $c_0$  and  $\ell_\infty$  (which are involved in next Theorem) are the infinite dimensional analogues of the  $n$ -dimensional polydisk.

**THEOREM 3.9.** [4, Theorem 5.1] *If  $X$  is the Banach space  $c_0$  of null sequences, then the Cluster Value Theorem holds for  $H^\infty(B)$ , i.e.*

$$Cl_B(f, x) = \widehat{f}(\mathfrak{M}_x(H^\infty(B))), \quad f \in H^\infty(B), \quad x \in \bar{B}_{\ell_\infty}.$$

Now we turn our attention to  $A_u(B)$ . If  $X$  is finite dimensional, then the spectrum of  $A_u(B)$  is absolutely uninteresting, since it is reduced to evaluations at points of  $\bar{B}$ . The reason is that  $A_u(B) = A_a(B)$  (see e.g. [61, 11.7 Theorem, p. 279]). But when  $X$  is an infinite dimensional Banach space the situation is very different and, as we have seen, the size of the fibers can be quite big. Below we will further discuss this claim and show some examples of this situation.

The main recent progress about the Cluster Value Theorem for  $A_u(B)$  is the following result obtained by R. M. Aron, D. Carando, T. W. Gamelin, S. Lassalle and M. Maestre.

**PROPOSITION 3.10.** [4, Lemma 3.5] *Let  $X$  be a Banach space. Suppose each weak neighborhood of 0 in  $B$  contains the unit ball of a subspace of finite codimension with a norm-one projection. Then the Cluster Value Theorem holds for  $A_u(B)$  at  $x = 0$ .*

As a consequence, the Cluster Value Theorem is valid at 0 for Banach spaces with a 1-unconditional shrinking basis. We recall that a basis  $(e_j)$  in a Banach space  $X$ , is called 1-unconditional if

$$\left\| \sum_{j=1}^n a_j x_j e_j \right\| \leq \left\| \sum_{j=1}^n x_j e_j \right\|,$$

for every  $x_j \in \mathbb{C}$ , every  $a_j \in \bar{\mathbb{D}}$  ( $j = 1, \dots, n$ ) and every  $n \in \mathbb{N}$ . A basis  $(e_j)$  in a Banach space  $X$  is called shrinking if the associated linear forms  $(e_j^*)$  form a basis of  $X^*$ .

**THEOREM 3.11.** [4, Lemma 3.1] *If  $X$  is a Banach space with a shrinking 1-unconditional basis, then the Cluster Value Theorem holds for  $A_u(B)$  at  $x = 0$ ,*

$$Cl_B(f, 0) = \widehat{f}(\mathfrak{M}_0(A_u(B))), \quad f \in A_u(B).$$

The solution is complete for the case of a Hilbert space.

**THEOREM 3.12.** [4, Theorem 4.1] *If  $X$  is a Hilbert space, then the Cluster Value Theorem holds for  $A_u(B)$  at every  $x \in \bar{B}$ ,*

$$Cl_B(f, x) = \widehat{f}(\mathfrak{M}_x(A_u(B))), \quad f \in A_u(B), x \in \bar{B}.$$

Since in the reference above the proof is given only for the case of separable Hilbert spaces, let us show how to prove the theorem as a direct consequence of Proposition 3.10 along the same lines of the proof of the separable case. Let  $(e_i)_{i \in I}$  be a maximal orthonormal system in  $X$ . Then, by the Bessel inequality, for each finite subset  $J \subset I$ , the operator

$$P_J : x = \sum a_k e_k \rightarrow \sum_{k \notin J} a_k e_k$$

is a norm-one projection. The sets

$$U_{\varepsilon, J} = \{a = \sum a_k e_k \in B : |a_k| < \varepsilon, 1 \leq k \in J\}$$

form a basis of weak neighborhoods of 0 in  $B$ . Any weak neighborhood  $W$  of 0 in  $B$  contains  $U_{\varepsilon, J}$  for some  $\varepsilon$  and  $J$ , which in turn contains the unit ball of  $\text{Ker} P_J$ . This is a finite codimensional subspace with a norm-one projection, so we can use Theorem 3.10 to obtain the Cluster Value Theorem for  $A_u(B)$  at  $x = 0$ . The Cluster Value Theorem for any  $x \in \bar{B}$  follows from a classical result of Renaud [59] about existence of automorphisms of the open unit ball of Hilbert spaces, and the next theorem.

**THEOREM 3.13.** [4, lemma 4.4] *An automorphism  $\phi$  of the open unit ball  $B$  of a Hilbert space  $X$  induces an automorphism  $C_\phi : f \rightarrow f \circ \phi$  of the uniform algebra*

$A_u(B)$ . Further,  $\phi$  extends to a homeomorphism  $\tilde{\phi}$  of the spectrum  $\mathfrak{M}(A_u(B))$ , which maps the fiber  $\mathfrak{M}_x(A_u(B))$  homeomorphically onto the fiber  $\mathfrak{M}_{\phi(x)}(A_u(B))$ .

The difference between  $\ell_p$   $1 < p < \infty$ ,  $p \neq 2$ , and  $\ell_2$  is that for  $p \neq 2$  we do not have enough automorphisms of the ball. The same happens for general Banach spaces satisfying the hypotheses of Theorem 3.11. So we have the following natural question.

QUESTION 7. Does the Cluster Value Theorem hold for  $A_u(B)$  when  $B$  is the open unit ball of a complex Banach space with a shrinking 1-unconditional basis? (i.e., does it hold for  $x \in B$ ,  $x \neq 0$ ?)

QUESTION 8. For  $1 < p < \infty$ ,  $p \neq 2$ , does the Cluster Value Theorem hold for  $A_u(B_{\ell_p})$ ?

#### 4. The size of the fibers and cluster sets for some algebras of bounded analytic functions

In this section, we address the subject of the size of the fibers. We deal first with the case  $A_u(B)$ . In Section 2 we saw that evaluation at points in  $B^{**}$  are continuous homomorphisms on any algebra between  $A_u(B)$  and  $H^\infty(B)$ . As a consequence, we have

$$\{\tilde{\delta}_z : z \in \overline{B^{**}}\} \subset \mathfrak{M}(A_u(B)),$$

where  $\tilde{\delta}_z$  was defined in (2.2). Here points in the unit sphere of the bidual  $X^{**}$  are included due to the fact that if  $f \in A_u(B)$ , then its Aron-Berner extension  $\tilde{f}$  is uniformly continuous on  $B^{**}$ . Hence, if  $X$  is not a reflexive space the spectrum  $\mathfrak{M}(A_u(B))$  contains points that are not evaluations on  $\overline{B}$ . But even in the reflexive case far more can be said.

In [5, Example in p. 58] it is shown that the fiber at 0 of the spectra of  $H_b(\ell_p)$  (see Section 5 for definitions) contain a copy of  $\beta(\mathbb{N}) \setminus \mathbb{N}$ , where  $\beta(\mathbb{N})$  stands for the Stone-Ćech compactification of the positive integers  $\mathbb{N}$ . This copy of  $\beta(\mathbb{N}) \setminus \mathbb{N}$  is built up with adherent points of the sequence  $(\delta_{e_j})_j$  of evaluations at the elements  $e_j$  of the canonical basis of  $\ell_p$ . In order to show that these adherent points form such a set they make use of homogeneous polynomials of degree  $m \geq p$  of the form

$P(x) = \sum_{j=1}^{\infty} \alpha_j x_j^m$ , where  $(\alpha_j) \in \ell_{\infty}$ . It is clear that these polynomials belong to  $A_u(\ell_p)$ , and that functions in  $A_u(\ell_p)$  can be evaluated in the  $e_j$ 's. Thus, we can follow their example to show that  $\mathfrak{M}_0(A_u(B_{\ell_p}))$  contains  $\beta(\mathbb{N}) \setminus \mathbb{N}$ . Actually, the same result is true for  $\mathfrak{M}_x(A_u(B_{\ell_p}))$  for every point  $x \in B_{\ell_p}$ .

Recall that a point  $x \in \overline{B^{**}}$  is a *peak point* for  $A_a(B)$  if there exists  $g \in A_a(B)$  such that  $g(x) = 1$ , and  $|g(y)| < 1$  for  $y \in \overline{B^{**}}$ ,  $y \neq x$ . In this case, the function  $g$  is said to *peak at  $x$*  (see [34]). The spaces  $\ell_p$ ,  $1 < p < \infty$ , are all smooth. Hence, every  $x$  in the unit sphere of  $\ell_p$  is a peak point, the corresponding  $g$  being a continuous linear form attaining its norm at  $x$ . Then, as consequence of the following Theorem, we have  $\mathfrak{M}_x(A_u(B_{\ell_p})) = \{\delta_x\}$ .

**THEOREM 4.1.** [4, Corollary 2.5] *Suppose  $x \in \bar{B}$  is a peak point for  $A_a(B)$  and let  $A$  be a uniform algebra with  $A_a(B) \subset A \subset H^{\infty}(B)$ . If for each  $f \in A$ , the limit of  $f(y)$  as  $y \in B$  tends to  $x$  in norm exists, then the fiber  $\mathfrak{M}_x(A)$  reduces to one point,  $\mathfrak{M}_x(A) = \{x\}$ .*

This result is an improvement of one due to J. Farmer [33, Lemma 4.4]. What happens with the fibers of  $\mathfrak{M}(A_u(B_{\ell_p}))$  for  $1 < p < \infty$  is rather surprising: they are infinite for points in the interior of the ball, and singletons for points at the boundary. Recall that for a finite dimensional  $X$  we have  $\mathfrak{M}(A_u(B)) = \{\delta_x : x \in \bar{B}\}$ . On the other hand, the fiber of  $\mathfrak{M}(H^{\infty}(B))$  at points of  $B$  are reduced to singletons (the evaluation at that point), and the rich structure of this spectrum appears only in the boundary, just the opposite situation from the case of  $\mathfrak{M}(A_u(\ell_p))$ .

In the case of  $H^{\infty}(B)$ , for  $B$  the unit ball of an infinite dimensional Banach space  $X$ , we have already pointed out that fibers are always infinite sets. Now we precisely state this result, due to Aron, Cole and Gamelin.

**THEOREM 4.2.** [5, 11.1 Theorem] *Suppose that  $X$  is an infinite dimensional Banach space. Then the fiber  $\mathfrak{M}_z(H^{\infty}(B))$  contains a copy of  $\beta(\mathbb{N}) \setminus \mathbb{N}$ , for every  $z \in \overline{B^{**}}$ .*

Up to now, we have only addressed the question of the cardinality of the fibers. Let us see now that fibers have also topological structures that also give information on their sizes. Given any Banach algebra  $A$  of bounded analytic functions on  $B$

containing  $X^*$ , a mapping  $\psi : \mathbb{D} \rightarrow \mathfrak{M}(A)$  is said to be analytic if  $\hat{f} \circ \psi$  is analytic on  $\mathbb{D}$  for every  $f \in A$ . We say that there is *an analytic embedding of a disk in a fiber* (or that the *disk injects analytically into a fiber* of the spectrum) if we can construct an analytic mapping  $\psi$  from  $\mathbb{D}$  into the spectrum  $\mathfrak{M}(A)$  which is a homeomorphism onto its image, and actually maps  $\mathbb{D}$  into some fiber  $\mathfrak{M}_x(A)$ . We can also consider an open subset  $U$  of a Banach space and have an *analytic embedding of the open set  $U$*  in some fiber. I. J. Schark in [62, 5. Embedding a disc in a fiber] proved that a disk can be analytically embedded in  $\mathfrak{M}_1(H^\infty(\mathbb{D}))$ .

The main results on embedding of analytic disks in fibers on the unit ball of an infinite dimensional Banach space are due to B. J. Cole, T. W. Gamelin and W. B. Johnson in [27]. As in the classical case of  $\mathbb{D}$  they use Blaschke products of interpolating sequences in the open unit ball  $B^{**}$  of the bidual of  $X$ .

**THEOREM 4.3.** [27, 5.1. Theorem] *Let  $X$  be an infinite dimensional Banach space. Suppose that  $(z_k)$  is a sequence in  $B^{**}$  which converges weak-star to 0, such that the distance from  $z_k$  to the linear span of  $z_1, \dots, z_{k-1}$  tends to 1 as  $k \rightarrow \infty$ . Then, passing to a subsequence, we can find a sequence of analytic disks  $\lambda \rightarrow z_k(\lambda)$  ( $\lambda \in \mathbb{D}$ ,  $k \geq 1$ ) in  $B^{**}$  with  $z_k(0) = z_k$ , such that for each  $\lambda \in \mathbb{D}$ ,  $(z_k(\lambda))$  is an interpolating sequence for  $H^\infty(B)$ . Furthermore, the correspondence  $(k, \lambda) \rightarrow z_k(\lambda)$  extends to an embedding*

$$\Psi : \beta(\mathbb{N}) \times \mathbb{D} \rightarrow \mathfrak{M}(H^\infty(B))$$

such that

$$\Psi((\beta(\mathbb{N}) \setminus \mathbb{N}) \times \mathbb{D}) \subset \mathfrak{M}_0(H^\infty(B))$$

and  $\hat{f} \circ \Psi$  is analytic on each slice  $\{p\} \times \mathbb{D}$  for all  $f \in H^\infty(B)$  and  $p \in \beta(\mathbb{N})$ .

The same authors point out that if the unit ball of some infinite dimensional Banach space injects analytically in  $\mathfrak{M}_0(H^\infty(B))$ , then so does the infinite dimensional polydisk  $\mathbb{D}^{\mathbb{N}}$  (i.e., the open unit ball of  $\ell_\infty$ ), as  $\ell_\infty$  can be mapped injectively into any infinite dimensional Banach space. In the other direction, since any separable Banach space maps injectively into  $\ell_\infty$ , whenever  $\mathbb{D}^{\mathbb{N}}$  can be analytically injected into  $\mathfrak{M}_0(H^\infty(B))$ , so can the unit ball of any separable Banach space. Then, they

proceed to show that for some classes of Banach spaces (those containing the  $\ell_p$ 's and  $L_p[0, 1]$ 's for  $1 < p < \infty$ ), these analytic injections can be done.

**THEOREM 4.4.** [27, 6.1.Theorem] *Suppose that  $X$  has a normalized basis  $(e_j)$  that is shrinking, with associated functionals  $(e_j^*)$  satisfying that there exists a positive integer  $N \geq 1$  such that*

$$\sum_{j=1}^{\infty} |e_j^*(x)|^N < \infty$$

*for all  $x = \sum_{j=1}^{\infty} e_j^*(x)x_j$  in  $X$ . Then there is an analytic injection of the countable infinite dimensional polydisk  $\mathbb{D}^{\mathbb{N}}$  into the fiber  $\mathfrak{M}_0(H^\infty(B))$ .*

Cole, Gamelin and Johnson show in [27, 6.3 Theorem, 6.4 Theorem, 6.5 Theorem and 7.2 Theorem] that for superreflexive Banach spaces the open unit ball of big spaces can be analytically embedded in  $\mathfrak{M}(H^\infty(B))$ . The most typical example of superreflexive Banach space is  $\ell_2$ . We only state one of these results. For this, recall that the *Gleason metric* in  $\mathfrak{M}(H^\infty(B))$  is defined as

$$\rho(\varphi, \psi) = \sup\{|\hat{f}(\varphi) - \hat{f}(\psi)| : f \in H^\infty(B), \|f\| \leq 1\}$$

(see [34, chapter VI]).

**THEOREM 4.5.** [27, 6.3. Theorem] *If  $X$  is a superreflexive Banach space, then the unit ball of a nonseparable Hilbert space injects into the fiber  $\mathfrak{M}_0(H^\infty(B))$  via an analytic map which is uniformly bicontinuous from the metric of the unit ball of the Hilbert space to the Gleason metric of its image in  $\mathfrak{M}(H^\infty(B))$ .*

Finally, for  $c_0$  they are able to analytically inject the infinite dimensional polydisk  $\mathbb{D}^{\mathbb{N}}$  into the fiber at 0.

**THEOREM 4.6.** [27, 6.6. Theorem] *There is an analytic injection of the infinite dimensional polydisk  $\mathbb{D}^{\mathbb{N}}$  into the fiber  $\mathfrak{M}_0(H^\infty(B_{c_0}))$  which is an isometry from the Gleason metric of  $\mathbb{D}^{\mathbb{N}}$  (as the open unit ball of  $\ell_\infty$ ) to the Gleason metric of  $\mathfrak{M}(H^\infty(B_{c_0}))$ .*

This study was continued by J. Farmer in 1998. He studied the spectrum of the space  $H_w^\infty(B)$ , a Banach algebra which is defined as follows. Given  $B$  the open unit ball of a complex Banach space  $X$ , a function  $f \in H^\infty(B)$  belongs to  $H_w^\infty(B)$ ,

if its Taylor series expansion at zero  $\sum_{m=0}^{\infty} P_m$  satisfies that each  $m$ -homogeneous polynomial  $P_m$  is weakly (uniformly) continuous when restricted to the unit ball  $B$ . Obviously,  $X^*$  (and then  $A_a(B)$ ) is included in  $H_w^\infty(B)$ . Farmer concentrated in uniformly convex Banach spaces giving several results, the following being of particular interest for us.

**THEOREM 4.7.** [33, Theorem 5.1 and Corollary 5.2] *Consider  $1 < p < \infty$ ,  $x \in \overline{B}_{\ell_p}$  and  $\mathcal{U}$  a free ultrafilter on the positive integers. Then there is a copy of  $B_{\Pi_{\mathcal{U}}\ell_p} \times (\beta(\mathbb{N}) \setminus \mathbb{N})$  in the fiber  $\mathfrak{M}_x(H_w^\infty(B_{\ell_p}))$ , embedded via a uniform homeomorphism with uniformly continuous inverse, that is analytic for each point of  $\beta(\mathbb{N}) \setminus \mathbb{N}$ , where  $\Pi_{\mathcal{U}}\ell_p$  is the ultrapower of copies of  $\ell_p$  along  $\mathcal{U}$ .*

## 5. The spectrum of $H_b(U)$

To obtain information about the algebra  $H^\infty(B)$  and its spectrum  $\mathfrak{M}(H^\infty(B))$ , Aron, Cole and Gamelin found it useful to study the algebra  $H_b(X)$  and its spectrum  $\mathfrak{M}(H_b(X))$ . We recall that  $H_b(X)$  is the algebra of complex valued entire functions on  $X$  which are bounded on bounded sets, which is a Fréchet algebra when endowed with the topology of uniform convergence on bounded sets. Let us give an informal account of the relationship between  $\mathfrak{M}(H^\infty(B))$  and  $\mathfrak{M}(H_b(X))$ . For this, we turn back to the one dimensional case. We note that if  $H(\mathbb{C})$  is the algebra of all holomorphic functions on the complex plane (with the topology of uniform convergence on compact sets), then its spectrum  $\mathfrak{M}(H(\mathbb{C}))$  easily identifies with  $\mathbb{C}$ . The continuous embedding  $H(\mathbb{C}) \hookrightarrow H^\infty(D)$  given by the restriction  $f \mapsto f|_D$  induces a projection

$$\begin{aligned} \rho : \mathfrak{M}(H^\infty(\mathbb{D})) &\rightarrow \mathfrak{M}(H(\mathbb{C})) \\ \varphi &\mapsto \varphi|_{H(\mathbb{C})}. \end{aligned}$$

With the identification  $\mathfrak{M}(H(\mathbb{C})) \simeq \mathbb{C}$ , we can see that  $\rho$  is nothing but the projection  $\pi$  already defined in (2.3). In the infinite dimensional setting, entire holomorphic functions are not necessarily bounded on the unit ball. However, we do have a continuous embedding  $H_b(X) \hookrightarrow H^\infty(B)$  given by the restriction map. As

a consequence, we have a projection

$$(5.1) \quad \begin{aligned} \rho : \mathfrak{M}(H^\infty(B)) &\rightarrow \mathfrak{M}(H_b(X)) \\ \varphi &\mapsto \varphi|_{H_b(X)}. \end{aligned}$$

In general,  $\rho$  does not coincide with the projection  $\pi$  defined in (2.1), since  $\mathfrak{M}(H_b(X))$  is usually much larger than  $X^{**}$ . One might say that the one-dimensional case  $\pi$  has two possible extensions to the infinite dimensional case: the projections  $\pi$  from (2.1) and  $\rho$  from (5.1). A good knowledge of  $\mathfrak{M}(H_b(X))$  will then provide information on the structure of  $\mathfrak{M}(H^\infty(B))$  via the mapping  $\rho$ . To be more precise, for  $\varphi \in \mathfrak{M}(H_b(X))$  we define its radius  $R(\varphi)$  as

$$R(\varphi) = \inf\{r > 0 : \varphi(f) \leq \|f\|_{rB}\},$$

where  $\|f\|_{rB}$  is the supremum of  $|f|$  over  $rB$ . Analogously, we can define the radius of  $\psi \in \mathfrak{M}(H^\infty(B))$ , in which case we always have  $R(\psi) \leq 1$ . The following result of Aron, Cole and Gamelin relates both spectra:

**THEOREM 5.1.** [5, 10.1 Theorem] *The projection  $\rho$  maps  $\mathfrak{M}(H^\infty(B))$  onto the set  $\{\varphi \in \mathfrak{M}(H_b(X)) : R(\varphi) \leq 1\}$ .*

*Moreover,  $\rho$  is one to one between  $\{\psi \in \mathfrak{M}(H^\infty(B)) : R(\psi) < 1\}$  and  $\{\varphi \in \mathfrak{M}(H_b(X)) : R(\varphi) < 1\}$ .*

Note that the set  $\{\varphi \in \mathfrak{M}(H_b(X)) : R(\varphi) \leq 1\}$  plays here the same role as the closed unit disk  $\overline{\mathbb{D}}$  in Theorem 2.3.

In order to understand the spectrum of  $H_b(X)$ , symmetric regularity turned out to be a fundamental concept. Let us first recall the notion of regularity. Let  $A : X \times X \rightarrow \mathbb{C}$  be a continuous bilinear function. Fix  $x \in X$  and for  $w \in X^{**}$  let  $(y_\beta)$  be a net in  $X$  weak-star convergent to  $w$ . Since  $A(x, -) \in X^*$  then there exists  $\lim_\beta A(x, y_\beta) := \bar{A}(x, w)$ . Now, fix  $w \in X^{**}$  and for  $z \in X^{**}$  let  $(x_\alpha)$  be a net in  $X$  weak-star convergent to  $z$ . Since  $\bar{A}(-, w) \in X^*$  then there exists

$$\tilde{A}(z, w) := \lim_\alpha \bar{A}(x_\alpha, w) = \lim_\alpha \lim_\beta A(x_\alpha, y_\beta).$$

**DEFINITION 5.2.** A Banach space  $X$  is regular if for all continuous bilinear function  $A : X \times X \rightarrow \mathbb{C}$  it follows that

$$\tilde{A}(z, w) = \lim_\beta \lim_\alpha A(x_\alpha, y_\beta).$$

What is imposed in this definition is that the extension of a bilinear function  $A(x, y)$  to  $X^{**} \times X^{**}$  obtained by extending  $A$  weak-star continuously first with respect to  $x$  and then with respect to  $y$  coincides with the extension of  $A(x, y)$  to  $X^{**} \times X^{**}$  obtained by extending weak-star continuously first with respect to  $y$  and then with respect to  $x$ .

This concept was introduced by Arens [2] in 1951. He defined in a natural way two products on the bidual  $X^{**}$  of a Banach algebra  $X$ , each being an extension of the product of  $X$ . These two products are known as the Aren's products. The coincidence of these gives the concept of regularity. One of the most important reasons to study regularity is that this allows to pass the commutativity from  $X$  to the bidual  $X^{**}$ .

The Banach space  $X$  will be said to be symmetrically regular if the above extensions coincide for every continuous symmetric bilinear function  $A$ . Every reflexive Banach space is trivially regular. The space of all null sequences  $c_0$  is also regular (Arens [2]) and the Banach space of all absolutely summable sequences  $\ell_1$  is not even symmetrically regular (Rennison [60], see also [32, Exercise 6.50]). The bilinear form associated to the operator  $T : \ell_1 \rightarrow \ell_\infty$  given by

$$T((x_i)_i) = \left( (-1)^{i+1} \sum_{j \leq i} x_j + \sum_{j > i} (-1)^{j+1} x_j \right)_i$$

is symmetric and does not satisfy the condition in the definition of symmetric regularity. On the other hand an important family of nonreflexive Banach spaces which are regular is formed by the spaces of continuous functions  $C(X)$ , for  $X$  compact (Gulick [48]).

In 1991, Aron, Cole and Gamelin [5] proved the following characterization of symmetrically regular spaces.

**THEOREM 5.3.** *For a Banach space  $X$  the following are equivalent:*

- (1)  $X$  is symmetrically regular.
- (2) Every continuous symmetric linear operator from  $X$  to  $X^*$  is weakly compact (an operator  $T : X \rightarrow X^*$  is symmetric if  $\langle Tx, y \rangle = \langle x, Ty \rangle$  for all  $x, y \in X$ ).

- (3) For all symmetric continuous bilinear forms  $A$ , the extension  $\tilde{A}$  is separately weak-star continuous.
- (4) For all symmetric continuous  $n$ -linear forms  $A$ , the extension  $\tilde{A}$  is separately weak-star continuous.

Let us recall that by [3] every function in  $H_b(X)$  extends to a holomorphic function on the bidual. Thus, to each point  $z$  in  $X^{**}$  we can associate a homomorphism  $\tilde{\delta}_z$ , by  $\tilde{\delta}_z(f) = \tilde{f}(z)$ , where  $\tilde{f}$  is the Aron-Berner extension of  $f$ . Then we have

$$(5.2) \quad X^{**} \hookrightarrow \mathfrak{M}(H_b(X)).$$

Of course, we can continue this Aron-Berner procedure to obtain extensions of the original function to any even dual of  $X$ . In particular, for each element  $\xi$  of the fourth dual  $X^{iv}$  of  $X$  we can define an element  $\tilde{\tilde{\delta}}_\xi \in \mathfrak{M}(H_b(X))$  by  $\tilde{\tilde{\delta}}_\xi(f) = \tilde{f}(\xi)$  for  $f \in H_b(X)$ . A natural question then is: by doing so, do we actually obtain a new element of  $\mathfrak{M}(H_b(X))$  (i.e., an element that we cannot define with an element of the bidual)? Symmetric regularity plays a crucial role in determining whether or not elements of the fourth dual produce new homomorphisms.

**THEOREM 5.4.** [7, Theorem 1.3] *To every point  $\xi \in X^{iv}$  corresponds a point  $z \in X^{**}$  such that  $\tilde{\tilde{\delta}}_\xi = \tilde{\delta}_z$  if and only if  $X$  is symmetrically regular.*

So, if  $X$  is not symmetrically regular there are more homomorphisms in  $\mathfrak{M}(H_b(X))$  than those obtained by evaluating at points of  $X^{**}$ .

Let us mention that the inclusion in (5.2) can be strict even in the case that  $X$  is reflexive (take for example  $X = \ell_2$  [7, Proposition 1.5]). Actually, examples where the equality  $\mathfrak{M}(H_b(X)) = X^{**}$  holds are scarce, two of them being the following.

**EXAMPLE 5.5.** [6]  $\mathfrak{M}(H_b(c_0)) = \ell_\infty$  and  $\mathfrak{M}(H_b(T^*)) = T^*$ , where  $T^*$  is the original Tsirelson space.

Aron, Cole and Gamelin [6] proved that if  $X^*$  has the approximation property then  $\mathfrak{M}(H_b(X))$  coincides with  $X^{**}$  (as a point set) if and only if the space  $P_f(X)$  of polynomials of finite type is dense in  $H_b(X)$ . Motivated by this result, the analogous problem of determining the spectrum of  $H_b(X; Y)$  (the space of vector

valued holomorphic mappings  $f : X \rightarrow Y$  that are bounded on the bounded sets of  $X$ ) when  $Y$  is an arbitrary Banach algebra with identity was considered in [42].

**THEOREM 5.6.** [42, Theorem 5.2] *If  $X$  is a Banach space,  $Y$  is a uniform algebra with identity and  $X^*$  has the approximation property then  $\mathfrak{M}(H_b(X; Y)) = X^{**} \times \mathfrak{M}(Y)$  if and only if the continuous  $n$ -homogeneous polynomials from  $X$  into  $Y$  are weakly continuous on bounded sets for all  $n \in \mathbb{N}$ .*

So far the spectrum of  $H_b(X)$  is mainly studied from an algebraic point of view, but finding a natural analytic structure on it is also interesting. This was done in [7] for symmetrically regular spaces, answering a question of Aron, Cole and Gamelin. With this analytic structure, the Gel'fand extension of any function  $f \in H_b(X)$  turns out to be holomorphic on  $\mathfrak{M}(H_b(X))$ . Thus, the connected component of  $\mathfrak{M}(H_b(X))$  containing  $X$  (via evaluation) may be regarded as the envelope of  $H_b$ -holomorphy of  $X$ . Since this connected component is an analytic copy of  $X^{**}$ , we could say that  $X^{**}$  is the mentioned envelope [32, Chapter 6]. There are, however, some arguments against this way to understand the envelope. For example, uniqueness of extensions is a property enjoyed by the usual definitions of envelopes and, as commented in [24], extensions to the bidual are never unique for nonreflexive spaces. Indeed, since  $X$  is a proper closed subspace of  $X^{**}$ , it is contained in the kernel of some non-zero functional  $\eta \in X^{***}$ . As a consequence, if  $\tilde{f}$  is the Aron-Berner extension of  $f$ , the function  $\tilde{f} + p \circ \eta$  is an extension of  $f$  to  $X^{**}$  for any one-variable polynomial, and each polynomial gives a different extension. We will turn back to this question later.

As in the finite-dimensional case we have the following picture

$$\begin{array}{ccc}
 X \subset & \xrightarrow{\delta} & \mathfrak{M}(H_b(X)) \\
 & \searrow & \downarrow \pi \\
 & & X^{**}
 \end{array}$$

where, as usual,  $\delta$  is the point evaluation mapping and  $\pi$  is defined by  $\pi(\phi) = \phi|_{X^*} \in X^{**}$ ,  $\phi \in \mathfrak{M}(H_b(X))$ . For each  $z \in X^{**}$  define  $\tau_z : X \rightarrow X^{**}$  by  $\tau_z(x) = x + z$

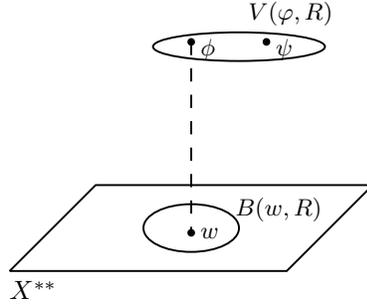
for all  $x \in X$ . This mapping induces a type of adjoint  $\tau_z^* : H_b(X) \rightarrow H_b(X)$  by  $\tau_z^*(f) = \tilde{f} \circ \tau_z|_X$ , where  $\tilde{f}$  denotes the Aron-Berner extension of  $f$ . For every  $z \in X^{**}$ , the mapping  $\tau_z^*$  is a continuous homomorphism, so given  $\phi \in \mathfrak{M}(H_b(X))$  we have  $\phi \circ \tau_z^* \in \mathfrak{M}(H_b(X))$ . This allows us to define:

DEFINITION 5.7. For  $R > 0$ ,  $\phi \in \mathfrak{M}(H_b(X))$

$$V(\phi, R) := \{\phi \circ \tau_z^* : \|z\| < R, z \in X^{**}\}.$$

It is not difficult to see that  $\pi(\phi \circ \tau_z^*) = \pi(\phi) + z$ , for all  $z \in X^{**}$  and all  $\phi \in \mathfrak{M}(H_b(X))$ . So, the question is now: Is  $\{V(\phi, R), R > 0\}$  a neighborhood basis of  $\phi$ ?

The answer is positive when  $X$  is symmetrically regular. Fix  $\psi \in V(\phi, R)$  and put  $w = \pi(\phi)$ . We have



where  $B(w, R)$  is the open ball of radius  $R$  centered at  $w$ . Is there  $S > 0$  so that  $V(\psi, S) \subset V(\phi, R)$ ? We have that  $\psi = \phi \circ \tau_z^*$  for some  $z \in X^{**}$ ,  $\|z\| = r < R$ . Now, taking  $v \in X^{**}$  such that  $\|v\| < R - r$  we would like to have  $\psi \circ \tau_v^* \in V(\phi, R)$ . Since  $\pi(\psi \circ \tau_v^*) = \pi(\psi) + v = w + z + v$ , if  $\psi \circ \tau_v^*$  is in  $V(\phi, R)$  then  $\psi \circ \tau_v^* = \phi \circ \tau_{z+v}^*$ . Thus, we would have  $\phi \circ \tau_z^* \circ \tau_v^* = \phi \circ \tau_{z+v}^*$  and this equality holds if and only if  $X$  is symmetrically regular.

THEOREM 5.8. [7, Theorem 2.2] *If  $X$  is a symmetrically regular Banach space then the family  $\{V(\phi, R) : \phi \in \mathfrak{M}(H_b(X)), R > 0\}$  is a basic neighborhood system for a Hausdorff topology on  $\mathfrak{M}(H_b(X))$ .*

COROLLARY 5.9. [7, Corollary 2.4] *If  $X$  is a symmetrically regular Banach space then  $\pi$  is a local homeomorphism over  $X^{**}$  and  $\mathfrak{M}(H_b(X))$  has an analytic structure over  $X^{**}$ .*

With this analytic structure,  $\mathfrak{M}(H_b(X))$  can be seen as a disjoint union of analytic copies of  $X^{**}$ , which are the connected components of  $\mathfrak{M}(H_b(X))$ .

PROPOSITION 5.10. [32, Proposition 6.30] *If  $X$  is a symmetrically regular Banach space and  $f \in H_b(X)$  then  $f^{**}(\phi) := \phi(f)$  is a holomorphic function on  $\mathfrak{M}(H_b(X))$  which extends the Aron-Berner extension  $\tilde{f}$  and such that the following diagram commutes*

$$\begin{array}{ccc}
 X^{**} & \xrightarrow{\tilde{\delta}} & \mathfrak{M}(H_b(X)) \\
 \uparrow J_X & \searrow \tilde{f} & \downarrow f^{**} \\
 X & \xrightarrow{f} & \mathbb{C}
 \end{array}$$

where  $\tilde{\delta}(z)(f) = \tilde{f}(z)$ , for all  $z \in X^{**}$ , and  $J_X$  is the natural embedding.

Looking at the vector valued case, Zalduendo [65] in 1990 presented the space  $G_{XY} = \mathcal{L}(\mathcal{L}(X; Y); Y)$ , endowed with the usual operator norm, as the canonical candidate to extend holomorphically the elements of  $H_b(X; Y)$ . When  $Y = \mathbb{C}$  we have  $G_{XY} = X^{**}$  and the Zalduendo extension coincides with that of Aron-Berner. Whenever  $Y$  is a Banach algebra, the space  $H_b(X; Y)$  is a Fréchet algebra and so we can study its spectrum  $\mathfrak{M}(H_b(X; Y); Y)$  as we did in the scalar-valued case. In general, the sets  $G_{XY}$  and  $\mathfrak{M}(H_b(X; Y); Y)$  do not coincide [41, Proposition 3.3]. In the following result [41, Corollary 4.4] the analytic structure of  $\mathfrak{M}(H_b(X; Y); Y)$  is obtained.

THEOREM 5.11. *If  $X$  is a symmetrically regular Banach space and  $Y$  is a Banach algebra such that every continuous linear mapping from  $X$  into  $Y$  is weakly compact, then  $\mathfrak{M}(H_b(X; Y); Y)$  has an analytic structure as a manifold over  $X^{**}$ . Moreover if  $Y$  is a commutative Banach algebra which is not isometrically isomorphic to  $\mathbb{C}$ , then  $\mathfrak{M}(H_b(X; Y); Y)$  is non-connected.*

So far, we have dealt with entire functions of bounded type. Let us see what happens when we consider open sets as domains. The philosophy to define a basis

of neighborhoods of  $\phi \in \mathfrak{M}(H_b(U))$  is essentially the same: to go ‘down’ to  $X^{**}$  with  $\pi$ , take some ball centered on  $\pi(\phi)$  and then to go ‘up’ again to define a neighborhood of  $\phi$ . However, the problem gets technically more complicated. We sketch now roughly the construction of the neighborhoods, given in [7, Section 2]. Given  $\phi \in \mathfrak{M}(H_b(U))$  there exists  $U_r = \{x \in X : \|x\| \leq r \text{ and } d(x, X \setminus U) > \frac{1}{r}\}$  such that  $|\phi(f)| \leq \sup_{x \in U_r} |f(x)|$  for all  $f \in H_b(U)$ . For a function  $f$  we consider the Taylor expansion at a point  $x$ , denoted  $\sum_{n=0}^{\infty} P_{n,x}$ . If  $\tilde{P}_{n,x}$  denotes the extension of each polynomial to  $X^{**}$ , for each fixed  $z \in X^{**}$  we can consider the mapping  $x \in U \rightsquigarrow \tilde{P}_{n,x}(z)$ . It is shown that this mapping is in  $H_b(U)$  and then if  $\|z\| < \frac{1}{r}$  the expression

$$(5.3) \quad \phi^z(f) = \sum_{n=0}^{\infty} \phi(x \rightsquigarrow \tilde{P}_{n,x}(z))$$

defines a mapping  $\phi^z : H_b(U) \rightarrow \mathbb{C}$ . In fact,  $\phi^z \in \mathfrak{M}(H_b(U))$  and  $\pi(\phi^z) = \pi(\phi) + z$ .

Note that if  $U = X$  then  $\phi^z = \phi \circ \tau_z^*$ .

For  $m \in \mathbb{N}$  with  $m > r$  we consider the set

$$(5.4) \quad V_{\phi,m} = \{\phi^z : z \in X^{**}, \|z\| \leq \frac{1}{r}\}.$$

**THEOREM 5.12.** [7, Theorem 2] *If  $X$  is a symmetrically regular Banach space and  $U$  is an open subset of  $X$ , then the family  $\{V_{\phi,m} : \phi \in \mathfrak{M}(H_b(U)) \text{ and } m \text{ chosen as before}\}$  is a basic neighborhood system for a Hausdorff topology on  $\mathfrak{M}(H_b(U))$ .*

The key point of the proof is to show that  $(\phi^z)^w = \phi^{z+w}$ . This goes through calculating the Taylor series expansion of the function  $x \rightsquigarrow \tilde{P}_{n,x}(z)$  and using the symmetric regularity of  $X$ .

We end this section with some comments on the envelope of  $H_b$ -holomorphy of an open subset  $U \subset X$ . Loosely speaking, the  $H_b$ -envelope of  $U$  is the largest domain containing  $U$  to which every holomorphic function of bounded type on  $U$  has a unique holomorphic extension. A description of this envelope is given in [24, Theorem 1.2] for general Riemann domains. Let us describe it in the case we deal with an open set  $U$ . First, it is observed in [24, Lemma 1.1] that the subset  $\pi^{-1}(X)$  of  $\mathfrak{M}(H_b(U))$  can be given an analytic structure as a Riemann domain over  $X$ , regardless of the symmetric regularity of  $X$ . To see this, note that if we only

consider  $z \in X$  in (5.3), there is no need to use Aron-Berner extensions. Also, we can define sets analogous to those in (5.4) in the following way:

$$V_{\phi,m} = \{\phi^z : z \in X, \|z\| \leq \frac{1}{r}\}.$$

The sets thus constructed turn out to be a basic neighborhood system for a Hausdorff topology, and this makes  $\pi^{-1}(X)$  a Riemann domain spread over  $X$ , with  $\pi|_{\pi^{-1}(X)}$  the local homeomorphism. Now we present the characterization of the  $H_b$ -envelope of holomorphy, which is a restatement of [24, Theorem 1.2].

**THEOREM 5.13.** *If  $U$  is an open subset of  $X$ , the  $H_b$ -envelope of holomorphy is the connected component of  $\pi^{-1}(X)$  containing  $\delta(U)$ .*

Whenever  $U$  is balanced, there is a simple description of its  $H_b$ -envelope. For  $E$  a  $U$ -bounded set, its polynomially convex hull is defined by

$$\widehat{E}_P = \{x \in X : |P(x)| \leq \sup_{y \in E} |P(y)| \text{ for every polynomial } P \text{ on } X\}.$$

Now we can define the polynomially convex hull of  $U$  as

$$(5.5) \quad \widehat{U}_P := \bigcup_E \widehat{E}_P,$$

where the union is taken over all  $U$ -bounded sets  $E$ . It is clear that the union could also be taken over a fundamental sequence of  $U$ -bounded sets.

**THEOREM 5.14.** [24, Theorem 2.2] *Let  $U$  be an open balanced subset of a Banach space  $X$ . Then  $\widehat{U}_P$  is the  $H_b$ -envelope of  $U$ . Moreover, any  $f \in H_b(U)$  extends to a holomorphic function  $\tilde{f}$  on  $\widehat{U}_P$  which is bounded on  $\widehat{E}_P$  for every  $U$ -bounded set  $E$ .*

## 6. Weighted algebras of holomorphic functions

Given an open set  $U \subset X$ , we consider a countable family  $V = (v_n)_n$  of continuous functions  $v_n : U \rightarrow [0, \infty[$  (called weights). Following [11, 12, 13, 14, 16, 15, 17, 18, 25, 43, 44, 45] we define the space

$$HV(U) = \{f : U \rightarrow \mathbb{C} : \text{holom. } \|f\|_v = \sup_{x \in U} v(x)|f(x)| < \infty \text{ for all } v \in V\}.$$

It is worth mentioning that, since each  $\|\cdot\|_v$  is a seminorm and the family  $V$  is countable, we are dealing with Fréchet spaces and (when that is the case) Fréchet

algebras.

Given a weight  $v$ , the associated weight  $\tilde{v}$  was defined in [12] by

$$\tilde{v}(x) = \frac{1}{\sup\{|f(x)| : f \text{ holomorphic}, \|f\|_v \leq 1\}}.$$

It is well known that  $v \leq \tilde{v}$  [12, Proposition 1.2] and that, if  $U$  is absolutely convex, then  $\|f\|_v = \|f\|_{\tilde{v}}$  for every  $f$  [12, Observation 1.12].

**DEFINITION 6.1.** [43] We will say that a family of weights satisfies Condition I if for every  $U$ -bounded set  $E$  there exists some  $v \in V$  such that  $\inf_{x \in E} v(x) > 0$ .

If Condition I holds, then  $HV(U)$  is continuously included in  $H_b(U)$ .

**6.1. On  $HV(U)$  as an algebra and its spectrum.** In [25, Proposition 1] a characterization is given of when is  $HV(U)$  an algebra.

**PROPOSITION 6.2.** *Let  $U$  be an open and balanced subset of  $X$  and  $V$  be a family of radial, bounded weights satisfying Condition I. Then  $HV(U)$  is an algebra if and only if for every  $v$  there exist  $w \in V$  and  $C > 0$  so that*

$$(6.1) \quad v(x) \leq C\tilde{w}(x)^2 \text{ for all } x \in U.$$

The problem of establishing if a weighted space of functions is an algebra was considered by L. Oubbi in [58] for weighted spaces of continuous functions. In that setting,  $CV(X)$  is an algebra if and only if for every  $v \in V$  there are  $C > 0$  and  $w \in V$  so that, for every  $x \in X$

$$(6.2) \quad v(x) \leq Cw(x)^2.$$

Let us note that for holomorphic functions, since  $w \leq \tilde{w}$ , if (6.2) holds then  $HV(U)$  is an algebra. On the other hand, if the family  $V$  consists of weights satisfying that there is a constant  $C$  so that  $\tilde{v} \leq Cv$  (such weights are called *essential*), then  $HV(U)$  is an algebra if and only if (6.2) holds.

Usually, dealing with or computing the associated weight is difficult. But for the one-dimensional case, associated weights have been widely studied, and several conditions for  $v$  to be essential are known. For example, [12, Corollary 1.6] states that if  $v(z) = f(|z|)$ , where  $f$  is a holomorphic function whose Taylor expansion at

0 has nonnegative coefficients, then  $v = \tilde{v}$ . Also, by [12, Proposition 3.1], if  $\frac{1}{v}$  is increasing and logarithmically convex, then  $v$  is essential. These conditions can in some cases be carried on to the infinite dimensional setting, namely in the following situation: if  $v$  is defined on  $X$  by letting  $v(x) = \varphi(\|x\|)$  where  $\varphi : [0, \infty[ \rightarrow ]0, \infty[$  is a decreasing, continuous function such that  $\lim_{t \rightarrow \infty} t^k \varphi(t) = 0$  for every  $k \in \mathbb{N}$ . Then the following result [25, Proposition 2] holds for both finite and infinite dimensional Banach spaces  $X$ .

**PROPOSITION 6.3.** *Let  $X$  be a Banach space and  $v$  a weight defined by  $v(x) = \varphi(\|x\|)$  for  $x \in X$  ( $\varphi$  as above). Then  $\tilde{v}(x) = \tilde{\varphi}(\|x\|)$  for all  $x \in X$ , where  $\tilde{\varphi}$  is one-dimensional weight associated to  $\varphi(|\cdot|)$  (the radially extension to  $\mathbb{C}$  of  $\varphi$ , i.e.  $\varphi(z) = \varphi(|z|)$  for  $z \in \mathbb{C}$ ).*

As a consequence,  $v$  is essential or equal to its associated weight if and only if so is  $\varphi$  (as a weight on  $\mathbb{C}$ ).

In order to give the spectrum of  $HV(U)$  an analytic structure, we need to introduce some conditions on the weights.

**DEFINITION 6.4.** We will say that a family of weights  $V$  has *good local control* if it satisfies Condition I,  $X^*$  is contained in  $HV(U)$  and for each  $v \in V$  there exist  $s > 0$ ,  $w \in V$  and  $C > 0$  so that

$$(6.3) \quad \text{supp } v + \overline{B}_X(0, s) \subseteq U,$$

$$(6.4) \quad v(x) \leq Cw(x+y) \text{ for all } x \in \text{supp } v \text{ and all } y \in X \text{ with } \|y\| \leq s.$$

Here  $\overline{B}_X(0, s)$  stands for the closed ball of  $X$  centered at 0 and radius  $s$ .

If  $U = X$  then (6.3) is trivially satisfied. On the other hand, if we define weights

$$(6.5) \quad v_n(x) = \varphi(\|x\|)^{1/n}, \quad n \in \mathbb{N}$$

then condition (6.4) translates into restrictions on the decreasing rate of  $\varphi$ .

**PROPOSITION 6.5.** *The family of weights defined in (6.5) has good local control if and only if there exist  $\alpha \geq 1$  and  $s > 0$  such that*

$$(6.6) \quad \sup_{t \in \mathbb{R}} \frac{\varphi(t)^\alpha}{\varphi(t+s)} < \infty.$$

Condition (6.6) is clearly satisfied if  $\varphi$  is such that

$$(6.7) \quad \varphi(s)\varphi(t) \leq C\varphi(s+t)$$

for some constant  $C > 0$  and all  $t, s$ .

If  $U \subset X$  is a bounded, open set and  $V$  is a family of bounded weights, it is easy to check that if  $V$  satisfies (6.3) then  $HV(U) = H_b(U)$ . The condition that the weights be bounded is an extra hypothesis, but it is actually fulfilled by all usual examples. Thus, we will always consider unbounded sets.

The following examples of families having (or not) good local control can be found in [23]

EXAMPLE 6.6.

- (1) In [43, Example 14] a family of weights  $V$  is defined so that  $HV(U) = H_b(U)$ . Obviously, this family  $V$  has good local control.
- (2) The function  $\varphi(t) = e^{-t}$  obviously satisfies (6.7).
- (3) The function  $\varphi(t) = e^{-e^t}$  satisfies (6.6) but does not satisfy (6.7).
- (4) Let  $(a_n)_n$  be a sequence such that  $a_n \geq 0$  for all  $n$ ,  $a_0 > 0$  and  $\frac{a_n}{a_{n-1}} \leq \frac{1}{n}$  (or, equivalently,  $\frac{a_n}{a_k} \leq \frac{k!}{n!}$  for all  $k \leq n$ ). Then the function defined by

$$\varphi(t) = \left( \sum_{n=0}^{\infty} a_n t^n \right)^{-1}.$$

satisfies (6.6).

Examples of sequences satisfying this condition can be constructed by taking  $p_{n+1} \geq p_n > 1$  and defining  $a_n = \left(\frac{1}{n!}\right)^{p_n}$  (e.g.  $a_n = \left(\frac{1}{n!}\right)^p$  or  $a_n = \left(\frac{1}{n!}\right)^n$ ). Obviously, for  $a_n = \frac{1}{n!}$  we get  $\varphi(t) = e^{-t}$ .

- (5) The function  $\varphi(t) = e^{-e^{t^2}}$  does not satisfy (6.6) (This shows that condition (6.6) implies that the function  $\varphi$  cannot decrease ‘too fast’).
- (6) Let  $X$  be a complex Banach space and  $\varphi : [0, \infty[ \rightarrow ]0, \infty[$  be an increasing and convex continuous function. Define the weights  $v_\lambda(x) = e^{-\lambda\varphi(\|x\|/\lambda)}$  for  $0 < \lambda \in \mathbb{Q}$  and the family of weights  $V = \{v_\lambda\}$ . This family has good local control and satisfies condition (6.2).
- (7) Consider  $p : \mathbb{C} \rightarrow [0, \infty[$  with the following properties:
  - (i)  $p$  is continuous and subharmonic.
  - (ii)  $\log(1 + |z|^2) = o(p(z))$ .

(iii) There exists  $C \geq 1$  such that for all  $y \in \mathbb{C}$

$$\sup_{|z-y| \leq 1} p(z) \leq C \inf_{|z-y| \leq 1} p(z) + C,$$

and hence

(iv)  $p(x+y) \leq Cp(x) + C$  for all  $x \in \mathbb{C}$  and all  $y \in \mathbb{C}$  with  $|y| \leq 1$ .

The algebra  $A_p^0 = HV(\mathbb{C})$  defined by this family is considered in [10, 19, 54]. Actually, in [10, 19], condition (iii) is replaced by

(iii')  $p(2z) = O(p(z))$

that also implies (iv).

Then  $A_p^0$  is a Fréchet algebra and  $V$  has good local control (actually,  $V$  has what is called below *excellent local control*).

(8) In  $\mathbb{C}^2$  we consider  $U = \mathbb{C} \times \mathbb{D}$  (where  $\mathbb{D}$  is the open unit disk). Then we define functions  $\psi_n$  on  $[0, 1]$  letting  $\psi_n \equiv 1$  on  $[0, 1/n]$ ,  $\psi_n \equiv 0$  on  $[1/(n+1), 1]$  and linear on  $[1/n, 1/(n+1)]$  and we consider weights defined on  $U$  by

$$v_n(z_1, z_2) = e^{-\frac{|z_1|}{n}} \psi_n(|z_2|).$$

$HV(U)$  is a Fréchet algebra and the sequence  $V = (v_n)_n$  has good local control.

(9) Let  $X_1, X_2$  be two Banach spaces,  $X = X_1 \oplus_p X_2$  and  $U = X_1 \oplus B_{X_2}(0, R)$ , where  $B_{X_2}(0, R)$  stands for the open ball of  $X_2$  centered at 0 and radius  $R$ . We choose a strictly increasing sequence  $(b_n)_n$  such that  $b_n > 0$  for all  $n$  and  $\lim_n b_n = R$ . We consider  $\psi_n$  such that  $\psi_n \equiv 1$  on  $[0, b_n]$ ,  $\psi_n \equiv 0$  on  $[b_{n+1}, R]$  and  $\psi_n$  is linear on  $[b_n, b_{n+1}]$  and take  $\varphi$  satisfying (6.6). Then we define weights by

$$v_n(x_1, x_2) = \varphi(\|x_1\|)^{1/n} \psi_n(\|x_2\|).$$

The family  $V = (v_n)_n$  has good local control and  $HV(U)$  is a Fréchet algebra.

If  $U$  is an unbounded open subset of a symmetrically regular Banach space  $X$  and the family  $V$  has good local control then an analytical structure can be defined on  $\mathfrak{M}(HV(U))$ . The idea is similar to the one described above to define the structure on  $\mathfrak{M}(H_b(U))$ . Let us recall that the first step is to define, for given

$\phi \in \mathfrak{M}(HV(U))$  and  $z \in X^{**}$ , some  $\phi^z \in \mathfrak{M}(HV(U))$ . The key point to do this is the fact that the mapping  $x \in U \rightsquigarrow \tilde{P}_{n,x}(z)$  is again in the algebra (now  $HV(U)$ ) for every function  $f \in HV(U)$  with Taylor expansion at  $x$  given by  $f = \sum_{n=0}^{\infty} P_{n,x}$ . This follows from [23, Lemma 2.10] from which if  $V$  has good local control, then for each  $v$  there exist positive numbers  $C$  and  $s$  and  $w \in V$  (coming from (6.3) and (6.4)) such that  $\|x \rightsquigarrow \tilde{P}_{n,x}(z)\|_v \leq C\|f\|_w$  for every  $\|z\| < s$  and every  $n$ . With this, the mapping  $\phi^z : HV(U) \rightarrow \mathbb{C}$  given by  $\phi^z(f) = \sum_{n=0}^{\infty} \phi[x \rightsquigarrow \tilde{P}_{n,x}(z)]$  is an element of  $\mathfrak{M}(HV(U))$  for every  $\|z\| < s$  that moreover satisfies  $\pi(\phi^z) = \pi(\phi) + z$ . In this way, the analytical structure of  $\mathfrak{M}(HV(U))$  can be defined.

**THEOREM 6.7.** [23, Theorem 2.12] *Suppose  $U$  is an open subset of a symmetrically regular Banach space  $X$  and  $V$  is a countable family of weights which has good local control such that  $HV(U)$  is a Fréchet algebra. Then, the family*

$$V_{\phi,\varepsilon} = \{\phi^z : z \in X^{**}, \|z\| < \varepsilon\},$$

*where  $\varepsilon < s$  for some  $s > 0$  depending on  $\phi \in \mathfrak{M}(HV(U))$ , forms a basis of neighborhoods of a Hausdorff topology on  $\mathfrak{M}(HV(U))$ . Furthermore,  $\pi : \mathfrak{M}(HV(U)) \rightarrow X^{**}$  gives a structure of a Riemann analytic manifold on  $\mathfrak{M}(HV(U))$ .*

**6.2. Extensions to the bidual.** As it happens for  $H_b(X)$ , the description of the spectrum can be simplified in the case of entire functions. When we look for such a simplification for  $\mathfrak{M}(HV(X))$ , we face the following problem. The weights in the family  $V$  are defined only on  $X$ , but we need to extend the topology defined by  $V$  to the bidual, in order to get a weighted algebra of holomorphic functions on  $X^{**}$  that behaves well with respect to the Aron-Berner extension. The first step to solve the problem is to *extend* weights to the bidual. In this section the weights are assumed to be bounded and that each of them satisfies Condition I.

**DEFINITION 6.8.** [23] Given a weight  $v$  on  $X$ . We define, in the spirit of the associated weight, the associated extension

$$(6.8) \quad \hat{v}(z) = \frac{1}{\sup\{|\tilde{f}(z)| : f \in Hv(X), \|f\|_v \leq 1\}},$$

for  $z \in X^{**}$ , where  $\tilde{f}$  is the Aron-Berner extension of  $f$ . Note that  $\hat{v}(x) = \tilde{v}(x)$  whenever  $x$  belongs to  $X$ .

This definition and the next results appear in [23, 3. Extensions to the bidual and the spectrum of  $HV(X)$ ]. It is important to point out that in that paper it is implicitly used that for each weight  $v$ ,  $\{v\}$  satisfies Condition I, to make sure that inclusion  $Hv(X) \subset H_b(X)$  holds. As a consequence given  $f \in Hv(X)$  there exists its Aron-Berner extension to  $X^{**}$ . A second consequence of that  $\hat{v}(z) > 0$  for all  $z \in X^{**}$ .

**PROPOSITION 6.9.** *Let  $v$  be a weight on  $X$  and  $\hat{v}$  be its associated extension to  $X^{**}$ . For each  $f \in Hv(X)$  we have that  $\tilde{f}$  belongs to  $H\hat{v}(X^{**})$  and  $\|\tilde{f}\|_{\hat{v}} = \|f\|_v$ .*

In other words, the Aron-Berner extension is an isometry from  $Hv(X)$  into  $H\hat{v}(X^{**})$ . These extensions also preserve some good properties of the original family, as the following theorems show.

**THEOREM 6.10.** *If the family  $V$  of weights on the symmetrically regular Banach space  $X$  satisfies condition (6.4), then so does the family  $\hat{V} = \{\hat{v} : v \in V\}$ .*

**DEFINITION 6.11.** A family  $V$  of weights on  $X$  is said to satisfy condition  $(\star)$  if there exists  $s > 0$  such that, for any  $v \in V$ , we can find  $C > 0$  and  $w \in V$  for which  $v(x) \leq Cw(x+y)$  for all  $x, y \in X$ ,  $\|y\| < s$ . We will say that the family  $V$  has excellent local control when it satisfies all the conditions involved in the good local control, but changing condition (6.4) to condition  $(\star)$ .

**THEOREM 6.12.** *If the family  $V$  of weights on the symmetrically regular Banach space  $X$  satisfies condition  $(\star)$ , then so does the family  $\hat{V} = \{\hat{v} : v \in V\}$ .*

So we have:

**COROLLARY 6.13.** *Let  $V$  be a family of weights on  $X$  satisfying condition  $(\star)$  and suppose  $X$  is symmetrically regular. For each  $z \in X^{**}$ , the mapping  $\tau_z : HV(X) \rightarrow HV(X)$  given by  $\tau_z f(x) = \tilde{f}(x+z)$  is continuous.*

Now we are ready to simplify the description of  $\mathfrak{M}(HV(X))$  for  $X$  symmetrically regular and  $V$  with excellent local control. Indeed, given  $\phi \in \mathfrak{M}(HV(X))$  we can give an alternative definition of  $\phi^z$  that works for all  $z \in X^{**}$ . First, let us define  $J_\phi(z) : HV(X) \rightarrow \mathbb{C}$  by  $J_\phi(z)(f) = \phi(\tau_z f)$ . Since  $\tau_z$  is multiplicative

(because so is the Aron-Berner extension) and, as we have shown, is continuous, we conclude that  $J_\phi(z)$  belongs to  $\mathfrak{M}(HV(X))$ .

Since  $X$  is symmetrically regular, we have  $\tau_{z'+z}(g) = \tau_{z'} \circ \tau_z(g)$  for all  $z, z' \in X^{**}$  and all  $g \in H_b(X)$  [5, Theorem 8.3.(vii)] or [7, Lemma 2.1]. Therefore, since  $HV(X)$  is contained in  $H_b(X)$  we have  $J_\phi(z'+z)(f) = J_\phi(z')(\tau_z f)$  for all  $z, z' \in X^{**}$  and all  $f \in HV(X)$ . With this fact, the following lemma can be proved.

LEMMA 6.14. *If  $X$  is a symmetrically regular Banach space and  $V$  is a countable family of weights with excellent local control such that  $HV(X)$  is a Fréchet algebra, then the mapping*

$$J_\phi : X^{**} \rightarrow \mathfrak{M}(HV(X))$$

*is bicontinuous into its image (in fact,  $J_\phi$  is bianalytic), when  $\mathfrak{M}(HV(X))$  is endowed with the analytic structure provided by Theorem 6.7.*

As a consequence of Lemma 6.14 we have an analytic copy of  $X^{**}$  in the connected component of  $\mathfrak{M}(HV(X))$  containing  $\phi$ . Since this analytic copy of  $X^{**}$  is necessarily open and closed, it must coincide with the connected component. Then we have [23, Theorem 3.7]:

THEOREM 6.15. *Let  $X$  be a symmetrically regular Banach space and  $V$  a countable family of weights on  $X$  with excellent local control such that  $HV(X)$  is a Fréchet algebra. Then,  $\mathfrak{M}(HV(X))$  is a disjoint union of analytic copies of  $X^{**}$ . Each copy is given by  $\{\phi \circ \tau_z : z \in X^{**}\}$  for some  $\phi \in \mathfrak{M}(HV(X))$ , where  $\tau_z f(x) = \tilde{f}(x+z)$  for all  $x \in X$ ,  $z \in X^{**}$  and  $f \in HV(X)$ .*

## 7. Homomorphisms between algebras

We recall that the space of approximable polynomials  $P_a(mX)$  ( $m \in \mathbb{N}$ ) is the closure in the space  $P(mX)$  (of continuous  $m$ -homogeneous polynomials) of the space generated by  $m$ -homogeneous polynomials of the form  $(x^*)^m$ , where  $x^*$  belongs to  $X^*$  [32, Definition 2.1]. It is not difficult to check that if  $X$  and  $Y$  are complex Banach spaces such that their dual are topologically isomorphic ( $X^* \cong Y^*$ ), then  $P_a(mX) \cong P_a(mY)$  for all  $m$ . Based on this fact, J.C. Díaz and S. Dineen [29] raised in 1988 the following question.

QUESTION 9. [29] If  $X$  and  $Y$  are complex Banach spaces such that  $X^*$  and  $Y^*$  are topologically isomorphic, does this imply that the spaces of continuous  $m$ -homogenous polynomials  $P(^mX)$  and  $P(^mY)$  are topologically isomorphic too for all  $m$ ?

They also gave a partial positive answer.

THEOREM 7.1. [29, Proposition 4] *Let  $X$  and  $Y$  be complex Banach spaces such that  $X^*$  and  $Y^*$  are topologically isomorphic. If  $X^*$  has the Schur property and the approximation property then  $P(^mX)$  and  $P(^mY)$  are topologically isomorphic for all  $m$ .*

Several authors have recently obtained more partial positive answers to this question. S. Lassalle and I. Zalduendo in [52] proved that the question has a positive answer if  $X$  and  $Y$  are symmetrically regular Banach spaces and  $X^* \cong Y^*$ .

THEOREM 7.2. [52, Theorem 4] *Let  $X$  and  $Y$  be complex Banach spaces whose duals are isomorphic (isometric). Then,*

- (1) *the spaces of continuous  $m$ -homogeneous polynomials which are weakly continuous on the unit ball  $P_w(^mX)$  and  $P_w(^mY)$  are isomorphic (isometric) for all  $m$ .*
- (2) *If also  $X$  and  $Y$  are symmetrically regular, then  $P(^mX)$  and  $P(^mY)$  are isomorphic (isometric) for all  $m$ .*

THEOREM 7.3. [21, Theorems 1 and 2] *Let  $X$  and  $Y$  be complex Banach spaces such that  $X^* \cong Y^*$ .*

- (1) *If  $X$  is regular then  $P(^mX) \cong P(^mY)$  for all  $m$ .*
- (2) *If  $X$  and  $Y$  are stable then  $P(^mX) \cong P(^mY)$  for all  $m$ .*

As a corollary, we have

COROLLARY 7.4. [21, Corollary 2] *Let  $X$  and  $Y$  be complex Banach spaces such that  $X^* \cong Y^*$ . If  $X$  is regular then  $H_b(X) \cong H_b(Y)$  as Fréchet algebras.*

It should be noted that this Corollary also holds if we replace the regularity assumption on  $X$  by symmetric regularity of both  $X$  and  $Y$ , applying Theorem 7.2

instead of Theorem 7.3. In [21, Proposition 1] it is shown that if  $X$  is regular and  $X^* \cong Y^*$ , then  $Y$  is also regular. As a consequence, one might think that for dual isomorphic spaces, the hypotheses of symmetric regularity on both spaces is somehow weaker than the regularity of one of them.

It is a remarkable thing that some of these results can be deduced from a variant of the following lemma from the article by Aron, Cole and Gamelin [5].

LEMMA 7.5. *Suppose  $X$  is symmetrically regular. If  $T : X^* \rightarrow X^*$  is linear and continuous, the composition operator  $f \mapsto f \circ T^*$  on  $H_b(X^{**})$  leaves  $H_b(X)$  invariant.*

Here, we should identify  $H_b(X)$  with its image inside  $H_b(X^{**})$  under the Aron-Berner extension. The lemma states that if  $f \in H_b(X^{**})$  is the Aron-Berner extension of a certain function of  $H_b(X)$ , then  $f \circ T^*$  is also the Aron-Berner extension of some function. The proof readily works to show the following: if  $X$  and  $Y$  are symmetrically regular and  $T : X^* \rightarrow Y^*$  is linear and continuous, then the composition operator  $f \mapsto f \circ T^*$  from  $H_b(Y^{**})$  to  $H_b(X^{**})$  maps  $H_b(Y)$  in  $H_b(X)$ . As a consequence, if  $X^* \cong Y^*$  and  $T$  is the isomorphism, the composition operator associated to  $T^*$  turns out to be an isomorphism from  $H_b(Y)$  to  $H_b(X)$  (its inverse being the composition operator associated to  $(T^{-1})^* = (T^*)^{-1}$ ). Also, since  $T$  is linear, these composition operators map homogeneous polynomials to homogeneous polynomials of the same degree. Therefore, it induces isomorphisms between the spaces of  $m$ -homogeneous polynomial for every  $m$ . These isomorphisms are clearly isometric if so is  $T$ .

In [22] a kind of converse problem is studied. The purpose there is to study Banach-Stone type theorems for algebras of holomorphic functions of bounded type. In a more precise way: given two open sets  $U \subset X$  and  $V \subset Y$  and two Fréchet algebras  $\mathcal{F}(U)$  and  $\mathcal{F}(V)$  of holomorphic functions of bounded type on  $U$  and  $V$  respectively, the question is the following: *if  $\mathcal{F}(U)$  and  $\mathcal{F}(V)$  are topologically isomorphic algebras, can we conclude that  $X$  and  $Y$  (or  $X^*$  and  $Y^*$ ) (or  $X^{**}$  and  $Y^{**}$ ) are topologically isomorphic?*

A naive way to face the problem is the following. Suppose  $U = X$  and  $V = Y$ . If  $B : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$  is a topological isomorphism of algebras, then its transpose  $B^t : \mathcal{F}(Y)' \rightarrow \mathcal{F}(X)'$  is another topological isomorphism, either for the strong or the weak-star topologies.

Since  $B$  is multiplicative, if we consider its restriction to the spectrum of  $\mathcal{F}(Y)$ , we obtain a bijective map

$$\theta_B : \mathfrak{M}(\mathcal{F}(Y)) \rightarrow \mathfrak{M}(\mathcal{F}(X))$$

which is, actually, a homeomorphism (the topology in the spectrum is the restriction of the weak-star topology). For  $x \in X$  we denote by  $\delta_x$ , as always, the evaluation at  $x$ . Clearly,  $\{\delta_x : x \in X\} \subset \mathfrak{M}(\mathcal{F}(X))$  and  $\{\delta_y : y \in Y\} \subset \mathfrak{M}(\mathcal{F}(Y))$ .

One may hope that  $\theta_B$  maps evaluations in evaluations and, moreover, that

$$(7.1) \quad \theta_B : \{\delta_y : y \in Y\} \rightarrow \{\delta_x : x \in X\}$$

is a bijection. Hence we can define  $g : Y \rightarrow X$  by  $g(y) := x$  where

$$\delta_x = \theta_B(\delta_y).$$

We have

$$B(f)(y) = (\delta_y \circ B)(f) = B^t(\delta_y)(f) = \theta_B(\delta_y)(f) = \delta_x(f) = f(x) = f(g(y)),$$

for all  $f \in \mathcal{F}(X)$  and all  $y \in Y$ , so we obtain

$$B(f) = f \circ g,$$

and  $B$  is a composition operator. If we can do the same with  $B^{-1}$ , we obtain a function  $h$  from  $X$  to  $Y$  which will be the inverse of  $g$ . Once  $g$  and  $h$  are defined, one has to show that they have good properties, which are usually related to the properties of the functions in  $\mathcal{F}$ , and our Banach-Stone theorem is done. So, an important task in [22] is to clarify the relationship between topological homomorphisms (and, particularly, isomorphisms) of algebras of analytic functions and composition operators. The problem is that, in general, we do not have something like (7.1), since the mapping  $\theta_B$  does not map evaluations in evaluations.

A Banach-Stone type theorem in several complex variables was proved by Cartan in the forties: given two complete Reinhardt domains  $U$  and  $V$  in  $\mathbb{C}^n$  (i.e., two balanced and  $n$ -circled open sets  $U$  and  $V$  in  $\mathbb{C}^n$ ) the spaces of holomorphic

functions  $H(U)$  and  $H(V)$  are topologically algebra isomorphic if and only if there exists a bijective biholomorphic function  $f : U \rightarrow V$ . On the other hand, in 1960 Aizenberg and Mityagin [1] proved that for any two bounded complete Reinhardt domains  $U$  and  $V$ , the spaces  $H(U)$  and  $H(V)$  are topologically isomorphic. It is well known that the Euclidean unit ball and the unit polydisk in  $\mathbb{C}^n$  are two bounded complete Reinhardt domains that are not biholomorphic. As a consequence, to obtain the kind of results we are looking for, we must consider only topological algebra isomorphisms.

**7.1. Homomorphisms on  $H_{w^*u}(U)$  and  $H_{wu}(U)$ .** Let  $U \subset X^*$  be open. We will denote by  $H_{w^*u}(U)$  the space of holomorphic functions  $f : U \rightarrow \mathbb{C}$  that are uniformly  $w(X^*, X)$ -continuous on  $U$ -bounded sets. As  $H_{w^*u}(U)$  is a closed subalgebra of  $H_b(U)$ , it is again a Fréchet algebra endowed with the topology of the uniform convergence on  $U$ -bounded subsets.

Let  $\mathfrak{M}(H_{w^*u}(U))$  be the spectrum of  $H_{w^*u}(U)$ . For  $x^* \in U$  we have  $\delta_{x^*} \in \mathfrak{M}(H_{w^*u}(U))$ , where  $\delta_{x^*}(f) := f(x^*)$  for  $f \in H_{w^*u}(U)$ . Since  $X$  is contained in  $H_{w^*u}(U)$ , we can define, as always, a projection

$$\pi : \mathfrak{M}(H_{w^*u}(U)) \rightarrow X^*$$

given by

$$\pi(\phi) = \phi|_X.$$

We also define, for each  $n$ , the following  $U$ -bounded set.

$$(7.2) \quad U_n = \{x^* \in U : \|x^*\| \leq n \text{ and } \text{dist}(x^*, X^* \setminus U) \geq \frac{1}{n}\}.$$

The family  $(U_n)_n$  is a fundamental sequence of  $U$ -bounded sets (i.e. given a  $U$ -bounded subset  $E$  of  $U$  there exists  $n$  such that  $E \subset U_n$ ). We also define

$$(7.3) \quad \tilde{U} := \bigcup_n \overline{U_n}^{w^*},$$

which is an open subset of  $X^*$  containing  $U$ .

**PROPOSITION 7.6.** *Let  $U$  be an open subset of  $X^*$ . With the previous notation, we have*

$$\tilde{U} \subset \pi(M_{w^*u}(U)) \subset \bigcup_n \overline{\Gamma(U_n)}^{w^*},$$

where  $\Gamma$  stands for the absolutely convex hull. In particular, if  $U$  is a convex and balanced open set of  $X^*$ , then  $\pi(\mathfrak{M}(H_{w^*u}(U))) = \tilde{U}$ .

Let  $U$  be an open subset of  $X^*$  and  $\mathcal{B} = (B_n)_{n=1}^\infty$  a countable family of weak-star closed  $U$ -bounded sets satisfying  $\cup_{n=1}^\infty B_n = U$  and such that for each  $n$  there is  $\varepsilon_n > 0$  with  $B_n + \varepsilon_n B_{X^*} \subset B_{n+1}$ . We define the Fréchet algebra

$$H_{\mathcal{B}w^*u}(U) := \{f \in H(U) : f \text{ is weak-star uniformly continuous on each } B_n, n = 1, \dots\},$$

endowed with the family of seminorms  $(\|\cdot\|_{B_n})_{n=1}^\infty$ . If  $\mathcal{B}$  is a fundamental sequence of  $U$ -bounded sets, then we have  $H_{\mathcal{B}w^*u}(U) = H_{w^*u}(U)$  algebraically and topologically. This algebra looks rather artificial and gives results of apparently partial character. But it is introduced in [22] because it allows to give a full answer in the class  $H_{wu}(U)$ , which we define below. In next proposition,  $U_n$  stands for the set defined in (7.2).

PROPOSITION 7.7.

- (1) Let  $U$  be a balanced open subset of  $X^*$  and  $\mathcal{B} = (\overline{U_n}^{w^*})_{n=1}^\infty$ . Every  $f \in H_{w^*u}(U)$  extends uniquely to an  $\hat{f} \in H_{\mathcal{B}w^*u}(\tilde{U})$  and the mapping  $i : H_{w^*u}(U) \longrightarrow H_{\mathcal{B}w^*u}(\tilde{U})$ ,  $i(f) := \hat{f}$  is a topological algebra isomorphism.
- (2) If  $U$  is a convex balanced open subset of  $X^*$  and  $X$  has the approximation property then  $\mathfrak{M}(H_{w^*u}(U)) = \delta(\tilde{U})$ .

This proposition is a generalization of a result by R. M. Aron and P. Rueda for entire functions in [8]. The next theorem is a first answer to the questions for the algebras  $H_{w^*u}$ .

THEOREM 7.8. [22, Theorem 9 and Corollary 10] *Let  $X$  and  $Y$  be Banach spaces, one of them having the approximation property. Let  $U \subset X^*$  and  $V \subset Y^*$  be convex and balanced open sets. If  $H_{w^*u}(U)$  and  $H_{w^*u}(V)$  are topologically isomorphic as algebras, then  $X$  and  $Y$  are isomorphic Banach spaces. In particular,  $H_{w^*u}(X^*)$  and  $H_{w^*u}(Y^*)$  are topologically algebra isomorphic if and only if  $X$  and  $Y$  are isomorphic Banach spaces.*

In [22] the concept of boundedly-regular open set is introduced: an open set  $U \subset X^*$  is boundedly-regular if given a fundamental sequence  $(U_n)$  of  $U$ -bounded sets, the family  $(\overline{U_n}^{w^*})$  is a fundamental sequence of  $\tilde{U}$ -bounded sets. The whole dual space  $X^*$  and every convex, balanced and bounded open set are examples of boundedly-regular open sets.

The naive approach to the problem studied in this section showed that homomorphisms between our algebras of holomorphic functions were very close to composition operators. Theorem 7.9 shows that, indeed, they are composition operators, but only when we look at them as homomorphisms between algebras of holomorphic functions defined on the bigger open sets  $\tilde{U}$  and  $\tilde{V}$  defined in (7.3). To simplify the notation we will write  $g \in H_{w^*u}(\tilde{V}, \tilde{U})$  whenever  $g$  is holomorphic,  $g$  is weak-star to weak-star uniformly continuous on  $\tilde{U}$ -bounded sets and such that  $g$  maps  $\tilde{U}$ -bounded sets into  $\tilde{V}$ -bounded sets.

If  $U \subset X^*$  is open,  $V \subset Y^*$  is a balanced boundedly-regular open set and  $F : H_{w^*u}(U) \rightarrow H_{w^*u}(V)$  is a continuous multiplicative operator, then the mapping  $\hat{F} : H_{w^*u}(\tilde{U}) \rightarrow H_{w^*u}(\tilde{V})$  defined as  $\hat{F}(f) := \widehat{F(f|_U)}$  for  $f \in H_{w^*u}(\tilde{U})$  is also a homomorphism. If additionally  $U$  is balanced and boundedly-regular, then  $F$  is an algebra isomorphism if and only if  $\hat{F}$  is an algebra isomorphism.

**THEOREM 7.9.** [22, Theorem 11] *Let  $X$  and  $Y$  be Banach spaces, one of them having the approximation property. Let  $U \subset X^*$  and  $V \subset Y^*$  be convex, balanced, boundedly-regular open sets. A mapping  $F : H_{w^*u}(U) \rightarrow H_{w^*u}(V)$  is a continuous homomorphism if and only if there exists a function  $g \in H_{w^*u}(\tilde{V}, \tilde{U})$  such that the operator  $\hat{F} : H_{w^*u}(\tilde{U}) \rightarrow H_{w^*u}(\tilde{V})$  is the composition operator generated by  $g$  (i.e.,  $\hat{F}f = f \circ g$  for all  $f \in H_{w^*u}(\tilde{U})$ ).*

**COROLLARY 7.10.** [22, Corollary 12] *Let  $X$  and  $Y$  be Banach spaces, one of them having the approximation property. Let  $U \subset X^*$  and  $V \subset Y^*$  be convex, balanced, boundedly-regular open sets. There exists  $F : H_{w^*u}(U) \rightarrow H_{w^*u}(V)$ , a topological algebra isomorphism, if and only if there exists a biholomorphic function  $g \in H_{w^*u}(\tilde{V}, \tilde{U})$  with  $g^{-1} \in H_{w^*u}(\tilde{U}, \tilde{V})$ , such that the operator  $\hat{F} : H_{w^*u}(\tilde{U}) \rightarrow H_{w^*u}(\tilde{V})$  is the composition operator generated by  $g$ . In that case,  $X$  and  $Y$  are isomorphic Banach spaces.*

Now we look at the homomorphisms on the space  $H_{wu}(U)$  of holomorphic functions on an open convex and balanced set  $U$  in a Banach space  $X$  which are weakly uniformly continuous on the  $U$ -bounded sets. This problem is a particular case of the study done above for boundedly-regular open sets in  $X^{**}$  since  $H_{wu}(U)$  and  $H_{w^*u}(\overset{\circ}{U}^{w^*})$  are topologically algebra isomorphic, where  $\overset{\circ}{U}^{w^*}$  is the norm interior of the weak-star closure of  $U$  in  $X^{**}$  (see [22]). As a consequence, all the previous results have their analogous for  $H_{wu}$ .

**THEOREM 7.11.** [22, Theorem 15] *Let  $U$  be a convex and balanced open subset of  $X$ , suppose  $X^*$  has the approximation property and let  $V \subset Y$  be open, convex and balanced. A mapping  $F : H_{wu}(U) \rightarrow H_{wu}(V)$  is a continuous multiplicative operator if and only if there exists  $g \in H_{w^*u}(\overset{\circ}{V}^{w^*}, \overset{\circ}{U}^{w^*})$  such that  $Ff = \tilde{f} \circ g|_U$  for all  $f \in H_{wu}(U)$ , where  $w^*$  refers to  $w(X^{**}, X^*)$  topology and  $\tilde{f}$  is the Aron-Berner extension of  $f$  to  $\overset{\circ}{U}^{w^*}$ .*

If  $F : H_{wu}(U) \rightarrow H_{wu}(V)$  is a continuous multiplicative operator we define  $\tilde{F} : H_{w^*u}(\overset{\circ}{U}^{w^*}) \rightarrow H_{w^*u}(\overset{\circ}{V}^{w^*})$  as  $\tilde{F}(f) = \widetilde{F(f|_U)}$  for all  $f \in H_{w^*u}(\overset{\circ}{U}^{w^*})$ . The mapping  $F$  is a topological algebra isomorphism if and only if  $\tilde{F}$  is a topological algebra isomorphism.

**COROLLARY 7.12.** [22, Corollary 16] *Let  $X$  and  $Y$  be Banach spaces, such that  $X^*$  or  $Y^*$  has the approximation property. Let  $U \subset X$  and  $V \subset Y$  be convex and balanced open sets. Then  $F : H_{wu}(U) \rightarrow H_{wu}(V)$  is a topological algebra isomorphism if and only if there is a biholomorphic function  $g \in H_{w^*u}(\overset{\circ}{V}^{w^*}, \overset{\circ}{U}^{w^*})$  whose inverse is in  $H_{w^*u}(\overset{\circ}{U}^{w^*}, \overset{\circ}{V}^{w^*})$  such that  $\tilde{F}(f) = f \circ g$  for all  $f \in H_{w^*u}(\overset{\circ}{U}^{w^*})$ . In this case, we have that  $X^*$  and  $Y^*$  must be isomorphic Banach spaces.*

As a consequence of the result of Lassalle and Zaldueño [52] presented in Theorem 7.2, if  $X^*$  and  $Y^*$  are isomorphic,  $H_{wu}(X)$  and  $H_{wu}(Y)$  are isomorphic algebras. Therefore we have:

**COROLLARY 7.13.** [22, Corollary 17] *Let  $X$  and  $Y$  be Banach spaces, one of their duals having the approximation property.  $H_{wu}(X)$  and  $H_{wu}(Y)$  are topologically isomorphic algebras if and only if  $X^*$  and  $Y^*$  are isomorphic Banach spaces.*

**7.2. Homomorphisms on  $H_b(U)$ .** As a first approach, one is tempted to carry out a study for  $H_b$ -algebras in the lines of that given for  $H_{wu}$ -algebras. However, this will be possible only when the size of the algebra  $H_b(U)$  is ‘small’, in the sense that all homogeneous polynomials are approximable.

Again, to simplify the notation we write  $g \in H_b(\overline{V}^{\circ w^*}, \overline{U}^{\circ w^*})$  if  $g$  is holomorphic and  $g$  maps  $\overline{V}^{\circ w^*}$ -bounded sets into  $\overline{U}^{\circ w^*}$ -bounded sets. The next three results from [22] are partial positive answers to our questions for the case of algebras of holomorphic functions of bounded type.

**THEOREM 7.14.** *Let  $U \subset X$  and  $V \subset Y$  be convex and balanced open sets and suppose that every polynomial on  $X$  is approximable. A mapping  $F : H_b(U) \rightarrow H_b(V)$  is a continuous multiplicative operator if and only if there exists  $g \in H_b(\overline{V}^{\circ w^*}, \overline{U}^{\circ w^*})$  such that  $\widetilde{F}f = \widetilde{f} \circ g$ .*

**COROLLARY 7.15.** *Let  $X$  and  $Y$  be Banach spaces. Let  $U \subset X$  and  $V \subset Y$  be convex and balanced open sets and suppose that every polynomial on  $X^{**}$  is approximable. There exists a topological algebra isomorphism  $F : H_b(U) \rightarrow H_b(V)$  if and only if there exists a biholomorphic function  $g \in H_{w^*u}(\overline{V}^{\circ w^*}, \overline{U}^{\circ w^*})$  whose inverse is in  $H_{w^*u}(\overline{U}^{\circ w^*}, \overline{V}^{\circ w^*})$  such that  $\widetilde{F}f = \widetilde{f} \circ g$  for all  $f \in H_b(U)$ . In this case  $X^*$  and  $Y^*$  must be isomorphic Banach spaces.*

**COROLLARY 7.16.** *If every polynomial on  $X^{**}$  is approximable,  $H_b(X)$  and  $H_b(Y)$  are algebra isomorphic if and only if  $X^*$  and  $Y^*$  are isomorphic Banach spaces.*

In [24], Corollary 7.15 was extended to open balanced sets (not necessarily convex). The main difference is that for general balanced sets, the set  $\overline{U}^{\circ w^*}$  must be replaced by a more suitable one. In order to describe that set we first define, for each  $U$ -bounded  $E$ ,

$$\widehat{E}_P'' = \{z \in X^{**} : |\widetilde{P}(z)| \leq \sup_{y \in E} |P(y)| \text{ for every polynomial } P \text{ on } X\},$$

where  $\widetilde{P}$  is the Aron-Berner extension of  $P$ . Now we define what can be seen as an extension of the polynomially convex hull of  $U$  (5.5) to the bidual:

$$\widehat{U}_P'' := \bigcup_E \widehat{E}_P'',$$

where the union is taken over all  $U$ -bounded sets  $E$  (or over any fundamental sequence of  $U$ -bounded sets). So Corollary 7.15 holds for balanced sets if we use  $\widehat{U}_P''$  and  $\widehat{V}_P''$  instead of  $\overline{U}^{\circ w^*}$  and  $\overline{V}^{\circ w^*}$ .

Some words about  $\widehat{U}_P''$  are in order. For a balanced set  $U$ , the polynomially convex hull  $\widehat{U}_P$  is the largest domain to which every function in  $H_b(U)$  has a unique analytic extension (Theorem 5.14). The set  $\widehat{U}_P''$  has the following property: it is the larger domain in  $X^{**}$  to which each function in  $H_b(U)$  has a unique analytic extension which coincides locally with its Aron-Berner extension. As a consequence, it could be seen as a kind of envelope of  $H_b$ -holomorphy of  $U$  modulo Aron-Berner extensions.

The original Tsirelson space  $T^*$  and the Tsirelson-James space  $T_J^*$  are quasi-reflexive spaces satisfying the conditions of Corollaries 7.15 and 7.16. These corollaries are Banach Stone type theorems for holomorphic functions of bounded type. Recently and using Corollary 7.15 a Banach-Stone theorem for germs of holomorphic functions has been obtained. A Banach space  $X$  is called a Tsirelson-James-like space if every continuous  $m$ -homogeneous polynomial on  $X$  is approximable, i.e. if the space  $P_f(mX)$  is dense in  $P(mX)$ , for all  $m \in \mathbb{N}$ .

**THEOREM 7.17.** [46] *Let  $X$  and  $Y$  be Tsirelson-James-like spaces. Let  $K \subset X$  and  $L \subset Y$  be convex and balanced compact subsets. Then the following conditions are equivalent.*

- (1) *There exists open subsets  $U \subset X^{**}$  and  $V \subset Y^{**}$  with  $K \subset U$  and  $L \subset V$  and a biholomorphic mapping  $\varphi : V \rightarrow U$  such that  $\varphi(L) = K$ .*
- (2) *The algebras of germs  $H(K)$  and  $H(L)$  are topologically isomorphic.*

Now we face the far more difficult situation in which we do not assume that every continuous polynomial on  $X^{**}$  is approximable, and we are going to restrict ourselves to the case of entire functions of bounded type. All the following results can be found in [22]. Recall that for a symmetrically regular  $X$ , the spectrum of  $\mathfrak{M}(H_b(X))$  is a disjoint union of analytic copies of  $X^{**}$ . The following theorem can be seen as a complicated version of the naive approach.

**THEOREM 7.18.** *Let  $X$  and  $Y$  be symmetrically regular Banach spaces and  $F : H_b(X) \rightarrow H_b(Y)$  be an isomorphism. Suppose that there exist open subsets  $\mathcal{V} \subset \mathfrak{M}(H_b(Y))$  and  $\mathcal{U} \subset \mathfrak{M}(H_b(X))$  such that  $\theta_F : \mathcal{V} \rightarrow \mathcal{U}$  is a homeomorphism. Then  $X^{**}$  and  $Y^{**}$  are isomorphic.*

A natural question is, then, if a topological and algebraic morphism  $F : H_b(X) \rightarrow H_b(Y)$  always induces a continuous mapping  $\theta_F$  from  $\mathfrak{M}(H_b(Y))$  to  $\mathfrak{M}(H_b(X))$ . That would allow us to take an almost-naive approach to the problem. However, this is not the case. Let us take  $g \in H_b(Y, X)$  and consider the composition homomorphism  $F_g : H_b(X) \rightarrow H_b(Y)$  given by  $F_g(f)(x) = f \circ g(x)$ . The next two theorems show that the induced mapping  $\theta_{F_g} : \mathfrak{M}(H_b(Y)) \rightarrow \mathfrak{M}(H_b(X))$  is not necessarily continuous on  $\mathfrak{M}(H_b(Y))$ , even if  $g$  is a continuous homogeneous polynomial.

**THEOREM 7.19.** *Let  $X$  be a symmetrically regular Banach space with an unconditional finite dimensional Schauder decomposition and suppose that there exists a continuous  $m$ -homogeneous polynomial which is not weakly sequentially continuous. Then there exists an  $(m+1)$ -homogeneous polynomial  $P : X \rightarrow X$  such that  $\theta_{F_P}$  is not continuous.*

**THEOREM 7.20.** *Let  $X$  be a symmetrically regular Banach space with a weakly null symmetric basis  $(e_j)_j$  and suppose there exists a homogeneous polynomial  $Q$  such that  $\lim_j Q(e_j) \neq 0$ . Then there exists a biholomorphic polynomial  $g : X \rightarrow X$  such that the composition algebra isomorphism  $F_g : H_b(X) \rightarrow H_b(X)$  given by  $F_g f = f \circ g$  induces a non-continuous  $\theta_{F_g}$ .*

We end this subsection by defining a class of operators that contain the composition ones. We will say that  $F : H_b(X) \rightarrow H_b(Y)$  is an *AB-composition* homomorphism if there exists  $g \in H_b(Y^{**}, X^{**})$  such that  $\widetilde{F}(f)(y^{**}) = \widetilde{f}(g(y^{**}))$  for all  $f \in H_b(U)$  and all  $y^{**} \in Y^{**}$ . Where again  $\widetilde{f}$  stands for the Aron-Berner extension of  $f$ . The following result characterizes the spaces for which every homomorphism is an *AB-composition* one.

**COROLLARY 7.21.** *Let  $X$  be a symmetrically regular Banach space. If  $X^*$  has the approximation property, the following are equivalent:*

- (1) *Every polynomial on  $X$  is weakly continuous on bounded sets.*
- (2) *Every homomorphism  $F : H_b(X) \rightarrow H_b(Y)$  is an AB-composition one, for any symmetrically regular Banach space  $Y$ .*
- (3) *Every homomorphism  $F : H_b(X) \rightarrow H_b(X)$  is an AB-composition one.*

It remains as an open problem:

QUESTION 10. Do two complex Banach spaces  $X$  and  $Y$  exist such that  $H_b(X)$  and  $H_b(Y)$  are isomorphic as algebras but  $X^{**}$  and  $Y^{**}$  (or  $X^*$  and  $Y^*$ ) are not topologically isomorphic?

**7.3. Holomorphic functions of exponential 0-type.** We consider now the algebra of 0-exponential type functions:

$$\text{Exp}(X) = \{f \in H(X) : \sup_x |f(x)|e^{-\|x\|/n} < \infty \text{ for all } n \in \mathbb{N}\}.$$

This is a weighted algebra defined by the family of weights  $v_n = e^{-\|x\|/n}$ , with  $n \in \mathbb{N}$ . Note that every polynomial belongs to  $\text{Exp}(X)$ . However, we will see that it is not necessary that polynomials be approximable to obtain positive results. Next lemma has no analogous for  $H_b(X)$ , and is probably the basis of the good behavior of the algebra  $\text{Exp}(X)$ .

LEMMA 7.22. *Let  $F : \text{Exp}(X) \rightarrow \text{Exp}(Y)$  be an algebra homomorphism. Then  $Fx^*$  is a degree 1 polynomial for all  $x^* \in X^*$  (i.e.  $F$  maps linear forms on  $X$  to affine forms on  $Y$ ).*

COROLLARY 7.23. *If the composition operator  $F_g : \text{Exp}(X) \rightarrow \text{Exp}(Y)$  is continuous, then  $g$  is affine.*

The following theorem should be compared to Corollary 7.16, where polynomials on  $X^{**}$  were assumed to be approximable to obtain the analogous result.

THEOREM 7.24. *If  $\text{Exp}(X) \cong \text{Exp}(Y)$  as topological algebras, then  $X^* \cong Y^*$ . If moreover both  $X$  and  $Y$  are symmetrically regular or  $X$  is regular, then  $\text{Exp}(X) \cong \text{Exp}(Y)$  if and only if  $X^* \cong Y^*$ .*

Finally, we give a characterization of the homomorphisms  $F : \text{Exp}(X) \rightarrow \text{Exp}(Y)$  which induce continuous mappings between the spectra.

THEOREM 7.25. *Let  $X$  and  $Y$  be symmetrically regular Banach spaces and  $F : \text{Exp}(X) \rightarrow \text{Exp}(Y)$  an algebra homomorphism. Then, the following are equivalent.*

- (1) *There exist  $\phi \in \mathfrak{M}(\text{Exp}(X))$  and  $T : Y^{**} \rightarrow X^{**}$  affine and  $w^*$ - $w^*$ -continuous so that  $Ff(y) = \phi(\tilde{f}(\cdot + Ty))$  for all  $y \in Y$ , where  $\tilde{f}$  is the Aron-Berner extension of  $f$ .*
- (2)  *$\theta_F$  is continuous.*
- (3)  *$\theta_F$  maps  $Y^{**}$  into a connected component.*

*In particular,  $\theta_F$  is continuous if and only if it is continuous on  $Y^{**}$*

Let us note that, although the spectra of  $H_b(X)$  and  $\text{Exp}(X)$  have similar structures (each of them consisting of analytical copies of the bidual), the behavior of the homomorphisms on  $H_b(X)$  and  $\text{Exp}(X)$  is very different. As we see in Theorem 7.25,  $\theta_F$  is continuous on  $\mathfrak{M}(\text{Exp}(Y))$  if and only if  $\theta_F$  maps  $Y^{**}$  into a connected component of  $\mathfrak{M}(\text{Exp}(X))$ . However, we have seen in Theorem 7.19 that there exists a composition homomorphism  $F : H_b(X) \rightarrow H_b(Y)$  such that  $\theta_F$  which is not continuous. Since it is a composition homomorphism,  $\theta_F$  is continuous on  $Y^{**}$  by Corollary 7.21. This means that  $Y^{**}$  is mapped into a sole connected component but some other copy of  $Y^{**}$  is split into several connected components of  $\mathfrak{M}(H_b(X))$ . The difference lies on the very different behavior of composition operators in these two spaces. The example given in Theorem 7.19 consists of a composition operator defined by a homogeneous polynomial of degree greater than one. This is continuous on  $H_b(X)$  but, as Corollary 7.23 shows, such a composition operator is not continuous from  $\text{Exp}(X)$  to  $\text{Exp}(Y)$ .

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DEPARTAMENTO DE MATEMÁTICA, FAC. C. EXACTAS Y NATURALES, UNIVERSIDAD DE BUENOS AIRES, PAB I, CIUDAD UNIVERSITARIA, 1428, BUENOS AIRES, ARGENTINA AND CONICET

*E-mail address:* `dcarando@dm.uba.ar`

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE VALENCIA, DOCTOR MOLINER 50, 46100 BURJASOT (VALENCIA), SPAIN

*E-mail address:* `domingo.garcia@uv.es`

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE VALENCIA, DOCTOR MOLINER 50, 46100 BURJASOT (VALENCIA), SPAIN

*E-mail address:* `manuel.maestre@uv.es`

INSTITUTO UNIVERSITARIO DE MATEMÁTICA PURA Y APLICADA AND DMA, ETSIAMN, UNIVERSITAT POLITÈCNICA DE VALÈNCIA, VALENCIA, SPAIN

*E-mail address:* `psevilla@mat.upv.es`