

SOLVING SINGULAR CONVOLUTION EQUATIONS USING THE
INVERSE FAST FOURIER TRANSFORM*

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(Received August 30, 2010)

Abstract. The inverse Fast Fourier Transform is a common procedure to solve a convolution equation provided the transfer function has no zeros on the unit circle. In our paper we generalize this method to the case of a singular convolution equation and prove that if the transfer function is a trigonometric polynomial with simple zeros on the unit circle, then this method can be extended.

Keywords: singular convolution equations, fast Fourier transform, tempered distribution, polynomial transfer functions, simple zeros

MSC 2010: 42A85

INTRODUCTION

Consider a convolution equation

$$a * g = f$$

with a and f finite sequences (g can be an infinite two-sided sequence). Our task is to solve for the sequence g given a and f . Consider the Fourier Transform \mathcal{F} yielding the following analogue in the Fourier domain

$$\mathcal{F}(a)\mathcal{F}(g) = \mathcal{F}(f).$$

It is well known (Wiener's Lemma) that if $\mathcal{F}(a)$ has no zeros on the unit circle then

*The second author was supported in part by Proyecto MTM2008-03211, Ministerio de Ciencia e Innovación, by a grant BEST 2010-134 of the Generalitat Valenciana, and by a grant from the Universidad Politécnica de Valencia, PAID 2009, Spain. The fourth author was supported by a grant AVOZ 101 905 03 and IAA 100190901 (Czech Republic).

the solution g is unique (see, e.g., [5]). The inverse Fast Fourier Transform (FFT^{-1}) is a common procedure to solve for g given a and f . It is based on the following. We mesh up the unit circle, sample $\mathcal{F}(f)/\mathcal{F}(a)$ on this mesh, and then take the inverse Fast Fourier Transform. The finer the mesh is, the closer we are to the solution g . This method mimics the Riemann Sum Approximation for the continuous inverse Fourier Transform for each coefficient $g(n)$. A classical result on Riemann Sum Approximation for the Riemann integral of a continuous function stipulates that this pointwise convergence to the solution g is at least $o(1/n)$ (see, e.g., [8]).

In engineering applications this procedure is many times successfully applied also in the case where $\mathcal{F}(a)$ has zeros on the unit circle. However, this method would converge to a solution g which no longer is unique. In this paper we define an extension of this method to the case of a singular convolution equation and prove some convergence results under certain but not too restrictive conditions. We will refer to this procedure as the *FFT Inversion Method for (possibly) singular convolutions equations*.

Let us sketch the needed background (see, e.g., [1], [2], and [6]). Let \mathcal{S} denote the vector space of all rapidly decreasing (complex-valued) C^∞ -functions defined on \mathbb{R} , i.e., the set of functions $\varphi: \mathbb{R} \rightarrow \mathbb{C}$ such that $P \cdot D^n \varphi$ is a bounded function for every polynomial P and every $n \in \mathbb{N} \cup \{0\}$ (here D^n denotes the n th derivative operator). This space is endowed with a locally convex topology defined by the sequence of seminorms $\{q_n\}_{n \in \mathbb{N}}$ given by

$$q_n(\varphi) := \max_{|k| \leq n} \max_{x \in \mathbb{R}} (1 + |x|^2)^n |D^k \varphi(x)|, \quad \varphi \in \mathcal{S}.$$

Then $(\mathcal{S}, \{q_n\}_{n \in \mathbb{N} \cup \{0\}})$ becomes a Fréchet locally convex space. Its dual, \mathcal{S}' , is the space of *tempered distributions*, and we endow it with its w^* -topology, i.e., the topology on \mathcal{S}' of the pointwise convergence on the elements in \mathcal{S} .

The space \mathcal{S} can be identified with a subspace of $\mathcal{C}_{\text{per}}^{(\infty)}(\mathbb{R})$, the space of all infinitely differentiable (complex-valued) 2π -periodic functions on \mathbb{R} endowed with the locally convex topology inherited from the canonical topology on $\mathcal{C}^{(\infty)}(\mathbb{R})$ induced by a fundamental system of compacta. Precisely, to $\varphi \in \mathcal{S}$ we associate the mapping $\tilde{\varphi}: [0, 2\pi] \rightarrow \mathbb{C}$ given by

$$(1) \quad \tilde{\varphi}(t) := \begin{cases} \varphi(\sigma(t)) & \text{if } t \in (0, 2\pi), \\ 0 & \text{otherwise,} \end{cases}$$

where $\sigma: (0, 2\pi) \rightarrow \mathbb{R}$ is defined as

$$\sigma(t) = \frac{1}{2\pi - t} - \frac{1}{t} \quad \text{for all } t \in (0, 2\pi).$$

The mapping $\varphi \mapsto \tilde{\varphi}$ is a linear homeomorphism from the space \mathcal{S} into $C_{\text{per}}^{(\infty)}(\mathbb{R})$. As it is well known, the space $C_{\text{per}}^{(\infty)}(\mathbb{R})$ and the space s of rapidly decreasing sequences are linearly homeomorphic, when s carries the topology defined by the system of seminorms $\{p_n\}_{n \in \mathbb{N}}$, where $p_n(a) := \sum_{k=1}^{\infty} n^k |a_k|$, $a = (a_k) \in s$ (see, e.g., [4, Theorem 2.10.H.8]). Recall that a sequence $\{a_n\}_{n \in \mathbb{Z}}$ is said to be *rapidly decreasing* if for all $r > 0$ there exists $C = C(r) > 0$ such that $|a_n| \leq C|n|^{-r}$, $n \in \mathbb{Z} \setminus \{0\}$. The sequence $\{b_n\}_{n \in \mathbb{Z}}$ is said to be *slowly growing* if there exists $r > 0$ and $C > 0$ such that $|b_n| \leq C|n|^r$ for all $n \in \mathbb{Z}$. The space of all the slowly growing sequences will be denoted by \mathcal{M} .

As it is well known, polynomially bounded functions define tempered distributions via integration, see e.g., [5]. Precisely, the following result holds.

Proposition 1. *Let g be a measurable function on \mathbb{R} such that for some $m \in \mathbb{N}$, the function $(1 + x^2)^m g(x)$ is bounded on \mathbb{R} . Then the map $F_g: \mathcal{S} \rightarrow \mathbb{C}$ defined as*

$$(2) \quad F_g(f) = \int_{\mathbb{R}} g(x)f(x) \, dx, \quad f \in \mathcal{S},$$

is a tempered distribution.

If there is no risk of misunderstanding, distributions F_g defined via (2) will be denoted simply by g . Observe that every $\varphi \in \mathcal{S}$ can be considered, according to Proposition 1, as an element in \mathcal{S}' .

A tempered distribution F is said to be (2π) -periodic if $\langle F, \varphi \rangle = \langle F, \psi \rangle$ whenever $\varphi \in \mathcal{S}$ and $\psi(x) = \varphi(x + 2\pi)$ for every $x \in \mathbb{R}$. In view of Proposition 1, the function e_n defined as $e_n(x) = e^{inx}$ for $x \in \mathbb{R}$ defines a tempered distribution (denoted again by e_n) via (2), and, obviously, e_n is a periodic distribution. More generally, given a slowly growing sequence $a = \{a_n\}_{n \in \mathbb{Z}}$, the sequence $\left\{ \sum_{n=-N}^N a_n e_n \right\}_{N \in \mathbb{N}}$ is w^* -convergent. Its w^* -limit will be denoted, accordingly, by $\sum_{n=-\infty}^{\infty} a_n e_n$, and it is again a periodic distribution, that will be denoted by $\mathcal{F}(a)$. If \mathcal{P} denotes the subspace of \mathcal{S}' of all periodic distributions, then \mathcal{F} maps \mathcal{M} into \mathcal{P} . It is known, see [6] for example, that

$$\mathcal{P} = \left\{ \sum_{n=-\infty}^{\infty} a_n e_n : \{a_n\}_{n \in \mathbb{Z}} \text{ a slowly growing sequence} \right\},$$

so in fact \mathcal{F} maps \mathcal{M} onto \mathcal{P} (see, e.g., [7]).

We define the *convolution* $\{c_n\}_{n \in \mathbb{Z}} = c = a * b$ of an element $a = \{a_n\}_{n \in \mathbb{Z}} \in s$ and an element $b = \{b_n\}_{n \in \mathbb{Z}} \in \mathcal{M}$ by the formula

$$(3) \quad c_n = \sum_{j=-\infty}^{\infty} a_{n-j} b_j, \quad n \in \mathbb{Z}.$$

The following result has a simple proof, that will be omitted.

Proposition 2. *Let $a \in s$ and $b \in \mathcal{M}$. Then $a * b$ is well defined and belongs to s . Moreover, $\mathcal{F}(a * b) = \mathcal{F}(a)\mathcal{F}(b)$.*

Remark. The conclusion of Proposition 2 needs some comments. It relates the distribution associated to a certain sequence to the product of two distributions. It is a problem to define properly in general the product of two distributions (see, e.g., [3]). However, if $F \in \mathcal{S}'$ and $G \in \mathcal{S}$, the multiplication $F \cdot G$ is well defined (cf. *op. cit.*). If $a \in s$, then $\mathcal{F}(a)$ is a true function, and it belongs to \mathcal{S} . In our case (see below) we are interested in elements $a \in c_{00}$, i.e., sequences with finite support. In this case, convolution of a with a (slowly growing) sequence g , and multiplication of $\mathcal{F}(a)$ with elements in \mathcal{S}' are, then, well defined.

THE MAIN RESULT

Consider, as above, the convolution equation

$$a * g = f,$$

with the assumption that a and f are finite sequences (g need not be finite). Assume now that $\mathcal{F}(a)$ (a 2π -periodic function on \mathbb{R} that can be considered as a function on the unit circle T) has zeros on T (we refer to this case by saying that the convolution equation $a * g = f$ is *singular*). Proposition 2 shows that the Fourier Transform \mathcal{F} translates the convolution equation into

$$(4) \quad \mathcal{F}(a)\mathcal{F}(g) = \mathcal{F}(f).$$

Formally, $\mathcal{F}(g) = \mathcal{F}(f)/\mathcal{F}(a)$ (if it exists, it would be certainly a periodic tempered distribution, i.e., an element in \mathcal{P}), and the solution g will be obtained just by applying \mathcal{F}^{-1} to the former equality (the (*generalized*) *FFT method*). Obviously, this approach needs to be justified, since $\mathcal{F}(a)$, as a function, is not invertible. The inverse function $\mathcal{F}^{-1}: \mathcal{P} \rightarrow \mathcal{M}$ has a simple description, due to the orthogonality of

the base $\{e_n\}_{n \in \mathbb{Z}}$ in $[0, 2\pi]$, where $e_n(x) := e^{inx}$ for $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Precisely, if $F(x) := \sum_{n=-\infty}^{\infty} a_n e^{inx}$ for $x \in \mathbb{R}$, then

$$(5) \quad a_n = \frac{1}{2\pi} \int_0^{2\pi} F(x) e_{-n}(x) dx \quad \text{for } n \in \mathbb{Z}.$$

Note that this is not, strictly speaking, the action of the distribution F on the function e_n , since, certainly, no e_n belongs to \mathcal{S} .

The key observation to solve our problem is that whenever $\mathcal{F}(a)$ has only simple roots on the unit circle T , equation (5) still has a meaning when F is replaced by $\mathcal{F}(f)/\mathcal{F}(a)$, at least in the Cauchy Principal Value sense, and that the sequence (g_n) so obtained is a solution (among possibly an infinite number of solutions) of our convolution equation. This is the content (and the proof) of the following result.

Theorem 3. *Consider a singular convolution equation $a * g = f$, with finite sequences a and f , and assume that the transfer function $\mathcal{F}(a)$ has simple zeros on the unit circle. Then this convolution equation is solvable by the (generalized) FFT method and moreover the (non-unique) solution g is a bounded (two-sided) sequence.*

P r o o f. It is not difficult to see that, without loss of generality, we can reduce the problem to the case that $\mathcal{F}(a) = 1 - e^{ix}$ (i.e., when $a = (\dots, 0, 0, 1, -1, 0, 0, \dots)$, where 1 is in position 0 and -1 in position 1). This is done by decomposing $\mathcal{F}(f)/\mathcal{F}(a)$ by using partial fractions and then scaling. We shall prove that, for every $n \in \mathbb{Z}$, the following integral exists in the sense of its Cauchy Principal Value:

$$(6) \quad g_n := (\text{VP}) \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathcal{F}(f)(x)}{1 - e^{ix}} e^{-inx} dx.$$

This is simple: Just observe that, for $m \in \mathbb{Z}$,

$$(7) \quad \frac{e^{imx}}{1 - e^{ix}} = \frac{1}{2} \left[\left(\cos mx + \frac{\sin mx \sin x}{1 - \cos x} \right) + i \left(\sin mx - \frac{\cos mx \sin x}{1 - \cos x} \right) \right].$$

The real part of (7) is integrable in $[0, 2\pi]$ for all $m \in \mathbb{Z}$, and the imaginary part is antisymmetric with respect to π (this follows easier by checking that the complex conjugate of $e^{im(\pi+x)}(1 - e^{i(\pi+x)})^{-1}$ is $e^{im(\pi-x)}(1 - e^{i(\pi-x)})^{-1}$). This gives the convergence of the integral in (6).

Let us check that the sequence $(g_j)_{j \in \mathbb{Z}}$ obtained in (6) satisfies the requirements: It is enough, as has been mentioned above, to take $f = (f_n)_{n \in \mathbb{Z}} \in c_{00}$ and $a = (a_n)_{n \in \mathbb{Z}}$,

where $a_0 = 1$, $a_1 = -1$, and $a_n = 0$ otherwise. Then, for $n \in \mathbb{Z}$,

$$\begin{aligned} \sum_{j=-\infty}^{\infty} a_{n-j}g(j) &= a_0g_n + a_1g_{n-1} = g_n - g_{n-1} \\ &= (\text{VP}) \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathcal{F}(f)(x)}{1 - e^{ix}} e^{-inx} dx - (\text{VP}) \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathcal{F}(f)(x)}{1 - e^{ix}} e^{-i(n-1)x} dx \\ &= (\text{VP}) \frac{1}{2\pi} \int_0^{2\pi} \frac{\mathcal{F}(f)(x)}{1 - e^{ix}} e^{-inx} (1 - e^{ix}) dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}(f)(x) e^{-inx} dx = f_n. \end{aligned}$$

The sequence $(g_n)_{n \in \mathbb{Z}}$ is bounded. Indeed, recall that the Dirichlet kernel D_n , for $n \in \mathbb{N}$, is the function

$$D_n(x) = \frac{\sin(n + \frac{1}{2})x}{\sin(\frac{1}{2}x)} \quad \text{for } x \in (0, 2\pi).$$

Note that

$$\frac{\sin mx \sin x}{1 - \cos x} = \frac{\sin mx}{\sin x} (1 + \cos x) \quad \text{for } x \in (0, 2\pi).$$

This shows, together with the fact D_n satisfies $\int_0^{2\pi} D_n(x) dx = 2\pi$ for all $n \in \mathbb{N}$, that the real part in (7) remains bounded when $m \rightarrow \infty$. \square

Once the convergence of the integral (6) has been proved for all $n \in \mathbb{N}$, the computation is done by using the meshing procedure sketched in the Introduction. For this we rely on the FFT techniques, as usual.

Consider a mesh $\{x_k\} = \{2\pi k/N\}_{k=0}^{N-1}$ of the interval $[0, 2\pi]$. Define

$$\mathbf{u}_{M,N}(k) = \begin{cases} \frac{\mathcal{F}(f)(e^{ix_k})}{(1 - e^{ix_k})} & \text{if } \left| \frac{\mathcal{F}(f)(e^{ix_k})}{(1 - e^{ix_k})} \right| < M, \\ 0 & \text{otherwise,} \end{cases}$$

and note each $\mathbf{u}_{M,N}$ induces a tempered distribution given by the following action

$$\langle \mathbf{u}_{M,N}, \varphi \rangle = \frac{1}{N} \sum_j \sum_k \mathbf{u}_{M,N}(k) \varphi(x_k + 2\pi j)$$

for all $\varphi \in S$. Observe that

$$\lim_{M,N \rightarrow \infty} \langle \mathbf{u}_{M,N}, \varphi \rangle = \langle \mathbf{u}, \varphi \rangle \quad \text{for all } \varphi \in S,$$

where \mathbf{u} is the tempered distribution defined by the action

$$\langle \mathbf{u}, \varphi \rangle = \int_{\mathbf{R}} \frac{\mathcal{F}(f)(e^{ix})}{1 - e^{ix}} \varphi(x) dx.$$

It follows that $\mathbf{u}_{M,N}$ converges to \mathbf{u} as $M, N \rightarrow \infty$ in the w^* topology. Let $g_{M,N} = \text{ifft}(\mathbf{u}_{M,N})$ where ifft stands for the inverse Fast Fourier Transform. Therefore, there exists a bounded sequence $g \in M$ so that $g_{M,N} \rightarrow g$ coordinatewise.

Example I. We are solving

$$a * g = f$$

with the unit impulse $f = (f_n)_{n \in \mathbb{Z}}$, i.e., $f_n = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$ and $f_0 = 1$, and the mask $a = (a_n)_{n \in \mathbb{Z}}$ such that $a_0 = a_1 = a_2 = 1$ and $a_n = 0$ otherwise. We thus have

$$g_n + g_{n+1} + g_{n+2} = f_n \quad \text{for all } n \in \mathbb{N}.$$

The corresponding transfer function

$$\mathcal{F}(a)(z) = 1 + z + z^2$$

has zeros $z = e^{\pm 2\pi i/3}$ on the unit circle. Indeed, we have non-trivial solutions to the homogeneous equation

$$g_n + g_{n+1} + g_{n+2} = \mathbf{0},$$

for example

$$\dots, -1, -1, 2, -1, -1, 2, -1, -1, 2, \dots$$

and so the solution is not unique.

We apply the FFT inversion method and obtain the sequence g (using MATLAB):

```
>> clear; N=2^4; for k=0:N-1 z=exp(i*2*pi*k/N);
    u(k+1)=1/(z^2+z+1); end; g=ifft(u)
Columns 1 through 7
    0.3333 + 0.0000i    0.3333 + 0.0000i    0.3333 - 0.0000i
   -0.6667 + 0.0000i    0.3333 + 0.0000i    0.3333 - 0.0000i
   -0.6667 + 0.0000i
Columns 8 through 14
    0.3333 + 0.0000i    0.3333 - 0.0000i   -0.6667 - 0.0000i
    0.3333 + 0.0000i    0.3333 - 0.0000i   -0.6667 - 0.0000i
    0.3333 + 0.0000i
Columns 15 through 16
    0.3333 - 0.0000i   -0.6667 - 0.0000i
```

Re-grouping accordingly we get

$$g = \dots, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, \underline{\mathbf{3}}, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, \dots$$

Example II. We are solving

$$a * g = f$$

with the unit impulse f , $f_n = 0$ except $f_0 = 1$. We have the filter a with $a_0 = 1$, $a_1 = -2$, $a_2 = 1$ and $a_n = 0$ otherwise. Our convolution equation becomes

$$g_n - 2g_{n+1} + g_{n+2} = f_n.$$

The corresponding polynomial transfer function

$$a(z) = 1 - 2z + z^2 = (z - 1)^2$$

has a multiple zero at $z = 1$ on the unit circle. The FFT Inversion Method does not work in this case and

$$\int_0^{2\pi} \frac{1}{(1 - e^{ix})^2} dx$$

fails to exist.

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