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Research Article

A Family of Iterative Methods with Accelerated Eighth-Order Convergence

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We propose a family of eighth-order iterative methods without memory for solving nonlinear equations. The new iterative methods are developed by using weight function method and using an approximation for the last derivative, which reduces the required number of functional evaluations per step. Their efficiency indices are all found to be 1.682. Several examples allow us to compare our algorithms with known ones and confirm the theoretical results.

1. Introduction

Finding the solution of nonlinear equations is one of the most important problems in numerical analysis. There are some texts that have become classic, as the one of Traub (see [1]) and Neta (see [2]) which include a vast collection of methods and their efficiency, or the paper by Neta and Johnson, [3]), Jarratt (see [4]), or Homeier (see [5]), among others.

As the order of an iterative method increases, so does the number of functional evaluations per step. The efficiency index (see [6]) gives a measure of the balance between those quantities, according to the formula $I = p^{1/d}$, where p is the order of convergence of the method and d is the number of functional evaluations per step. Kung and Traub conjectured in [7] that the order of convergence of any multipoint method without memory cannot exceed the bound 2^{d-1} (called the optimal order). Thus, the optimal order for a method with 3 functional evaluations per step would be 4. King's method [8], Chun's schemes (see [9, 10]), Chun et al. [11], Maheshwari's procedure (see [12]), and Jarratt's method [4] are some of optimal fourth-order methods, because they only perform three functional evaluations per step.

More recently, optimal eighth-order iterative methods have been investigated by many researchers (see, e.g., [13] where the authors show optimal methods of order eight and sixteen). Bi et al. in [14] developed a new family of eighth-order iterative methods for solving nonlinear equations, which is denoted by BRW8 and whose iterative expression is

$$
\begin{aligned}
y_m &= x_m - \frac{f(x_m)}{f'(x_m)}, \\
z_m &= y_m - \frac{f(x_m) - (1/2) f(y_m)}{f(x_m) - (5/2) f(y_m)} \frac{f(y_m)}{f'(x_m)}, \\
x_{m+1} &= z_m - \frac{f(x_m) + (\theta + 2) f(z_m)}{f(x_m) + \theta f(z_m)} \frac{f(z_m)}{f[z_m, y_m] + f[z_m, x_m, x_m](z_m - y_m)},
\end{aligned}
\tag{1.1}
$$

where $\theta \in \mathbb{R}$.

Also Cordero et al. in [15] developed a parametric family of eighth-order methods based on Ostrowski's scheme, that we will denote by M8. It can be expressed as

$$
\begin{aligned}
y_m &= x_m - \frac{f(x_m)}{f'(x_m)}, \\
z_m &= x_m - \frac{f(x_m)}{f'(x_m)} \frac{f(x_m) - f(y_m)}{f(x_m) - 2f(y_m)}, \\
u_m &= z_m - \frac{f(z_m)}{f'(x_m)} \left(\frac{f(x_m) - f(y_m)}{f(x_m) - 2f(y_m)} + \frac{1}{2} \frac{f(z_m)}{f(y_m) - 2f(z_m)} \right)^2, \\
x_{m+1} &= u_m - \frac{f(z_m)}{f'(x_m)} \frac{3(\beta_2 + \beta_3)(u_m - z_m)}{\beta_1 (u_m - z_m) + \beta_2 (y_m - x_m) + \beta_3 (z_m - x_m)},
\end{aligned}
\tag{1.2}
$$

where $\beta_2 + \beta_3 \neq 0$. Other family of variants of Ostrowski's method with eighth-order convergence was developed by Liu and Wang in [16]. We will denote it by LW8 and its iterative expression is

$$
\begin{aligned}
y_m &= x_m - \frac{f(x_m)}{f'(x_m)}, \\
z_m &= x_m - \frac{f(x_m)(f(x_m) - f(y_m))}{f'(x_m)(f(x_m) - 2f(y_m))}, \\
x_{m+1} &= z_m - \frac{f(z_m)}{f'(x_m)} \left[\left(\frac{f(x_m) - f(y_m)}{f(x_m) - 2f(y_m)} \right)^2 + \frac{f(z_m)}{f(x_m) - \alpha f(y_m)} + G\left(\frac{f(z_m)}{f(x_m)} \right) \right],
\end{aligned}
\tag{1.3}
$$

where α is constant and G denotes a real-valued function.

Let us note that all these methods are optimal in the sense of Kung-Traub's conjecture (see [7]) for methods without memory; that is, they reach eighth-order convergence with only four functional evaluations.

It is usual to design high-order methods from Ostrowski-type schemes, as they are optimal and use few operations per step, trying to obtain procedures as efficient as possible by using different techniques (see, for instance [13, 15, 16]). If we compose Newton and Ostrowski's methods, and estimate the last derivative by divided differences, it is necessary to use divided differences of second order to reach eighth order of convergence (see [14]). Nevertheless, as we will see in the next section, it is possible to obtain an optimal eighth-order scheme by composing King and Newton's methods by using only divided differences of first order.

In this paper, we design a family of eighth-order iterative methods to find a simple root x^* of the nonlinear equation $f(x) = 0$, where $f : D \to \mathbb{R}$ is a smooth function, and D is an open interval. We present in Section 2 a family of eighth-order iterative methods, based on Chun's method. In Section 3, different numerical examples confirm the theoretical results and allow us to compare the new methods with other known methods mentioned in the introduction. Finally, some conclusions are presented in Section 4.

2. Development of Eighth-Order Algorithm

Let us consider the family of optimal methods proposed by Chun in [9], that is a variant of King's family,

$$\begin{aligned} y_m &= x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m &= y_m - \frac{f^2(x_m)}{f^2(x_m) - 2f(x_m)f(y_m) + 2\beta f^2(y_m)} \frac{f(y_m)}{f'(x_m)}, \end{aligned} \tag{2.1}$$

where $\beta \in \mathbb{R}$. We consider now a three-step iterative scheme composed by Chun's scheme (2.1) and a third step designed by using Newton's method and weight functions.

$$x_{m+1} = z_m - H(\mu_m) \frac{f(z_m)}{f'(z_m)}, \tag{2.2}$$

where $\mu_m = f(z_m)/f(x_m)$ and $H(t)$ represents a real-valued function.

However, this procedure is not optimal, as it uses two new functional evaluations in the last step. So, we express $f'(z_m)$ as a linear combination of $f[y_m, x_m]$, $f[z_m, y_m]$, and $f[z_m, x_m]$.

$$f'(z_m) = \theta_1 f[y_m, x_m] + \theta_2 f[z_m, y_m] + (1 - \theta_1 - \theta_2) f[z_m, x_m], \tag{2.3}$$

where θ_1 and θ_2 are real numbers.

By using (2.3), we propose the following iterative scheme:

$$\begin{aligned} y_m &= x_m - \frac{f(x_m)}{f'(x_m)}, \\ z_m &= y_m - \frac{f^2(x_m)}{f^2(x_m) - 2f(x_m)f(y_m) + 2\beta f^2(y_m)} \frac{f(y_m)}{f'(x_m)}, \\ x_{m+1} &= z_m - H(\mu_m) \frac{f(z_m)}{\theta_1 f[y_m, x_m] + \theta_2 f[z_m, y_m] + (1 - \theta_1 - \theta_2) f[z_m, x_m]}, \end{aligned} \tag{2.4}$$

whose convergence analysis will be made in the following result.

Theorem 2.1. *Let x^* be a simple zero of a sufficiently differentiable function $f : D \subseteq \mathbb{R} \to \mathbb{R}$. If the initial point x_0 is sufficiently close to x^*, then the sequence $\{x_m\}_{m\geq 0}$ generated by any method of the family* (2.4) *converges to x^*. If $H(t)$ is any function with $H(0) = 1$, $H'(0) = 1$, $H''(0) < \infty$, and $\beta = 1/2$, then the convergence order of any method of the family* (2.4) *is eight if and only if $\theta_1 = -1$, $\theta_2 = 1$.*

Proof. Let $e_m = x_m - x^*$ be the error at the mth iteration and $c_m = (1/m!)(f^{(m)}(x^*)/f'(x^*))$, $m = 2, 3, \ldots$. By using Taylor expansions, we have

$$\begin{aligned} f(x_m) &= f'(x^*)\left[e_m + c_2 e_m^2 + c_3 e_m^3 + c_4 e_m^4 + c_5 e_m^5 + c_6 e_m^6 + c_7 e_m^7 + c_8 e_m^8 + O\left(e_m^9\right)\right], \\ f'(x_m) &= f'(x^*)\left[1 + 2c_2 e_m + 3c_3 e_m^2 + 4c_4 e_m^3 + 5c_5 e_m^4 + 6c_6 e_m^5 + 7c_7 e_m^6 + 8c_8 e_m^7 + O\left(e_m^8\right)\right]. \end{aligned} \tag{2.5}$$

Now, from (2.5), we have

$$\begin{aligned} y_m = x^* &+ c_2 e_m^2 + \left(2c_3 - 2c_2^2\right) e_m^3 + \left(3c_4 - 3c_2 c_3 - 2\left(2c_3 - 2c_2^2\right) c_2\right) e_m^4 \\ &+ \left(4c_5 - 10c_2 c_4 - 6c_3^2 + 20c_3 c_2^2 - 8c_2^4\right) e_m^5 \\ &+ \left(-17c_4 c_3 + 28c_4 c_2^2 - 13c_2 c_5 + 33c_2 c_3^2 + 5c_6 - 52c_3 c_2^3 + 16c_2^5\right) e_m^6 + O\left(e_m^7\right), \end{aligned} \tag{2.6}$$

and then, we get

$$\begin{aligned} f(y_m) = f'(x^*)\Big[&c_2 e^2 + \left(2c_3 - 2c_2^2\right) e_m^3 + \left(3c_4 - 7c_2 c_3 + 5c_2^3\right) e_m^4 \\ &+ \left(-6c_3^2 + 24c_3 c_2^2 - 10c_2 c_4 + 4c_5 - 12c_2^4\right) e^5 \\ &+ \left(-17c_4 c_3 + 34c_4 c_2^2 - 13c_2 c_5 + 5c_6 + 37c_2 c_3^2 - 73c_3 c_2^3 + 28c_2^5\right) e_m^6 + O\left(e_m^7\right)\Big]. \end{aligned} \tag{2.7}$$

Combining (2.5), (2.6) and (2.7), we obtain

$$\begin{aligned} z_m = x^* &+ \left(-c_2c_3 + c_2^3 + 2\beta c_2^3\right)e_m^4 - 2\left((2+6\beta)c_2^4 - 2(2+3\beta)c_2^2c_3 + c_3^2 + c_2c_4\right)e_m^5 \\ &+ \left(\left(10 + 44\beta - 4\beta^2\right)c_2^5 - 6(5+14\beta)c_2^3c_3 + 6(2+3\beta)c_2^2c_4\right. \\ &\left. -7c_3c_4 + 3c_2\left((6+8\beta)c_3^2 - c_5\right)\right)e_m^6 + O\left(e_m^7\right). \end{aligned} \tag{2.8}$$

So, from (2.8), we get

$$\begin{aligned} f(z_m) &= f'(x^*)\left[\left((1+2\beta)c_2^3 - c_2c_3\right)e_m^4 - 2\left(\left((2+6\beta)c_2^4 - 2(2+3\beta)c_2^2c_3 + c_3^2 + c_2c_4\right)\right)e_m^5\right. \\ &\quad + \left(\left(10 + 44\beta - 4\beta^2\right)c_2^5 - 6(5+14\beta)c_2^3c_3 + 6(2+3\beta)c_2^2c_4\right. \\ &\quad \left.\left. -7c_3c_4 + 3c_2\left((6+8\beta)c_3^2 - c_5\right)\right)e_m^6 + O\left(e_m^7\right)\right]. \end{aligned} \tag{2.9}$$

Using the Taylor expansion of H around 0 and considering $H''(0) < \infty$, we get

$$\begin{aligned} H(\mu_m) &= H(0) + H'(0)\frac{f(z_m)}{f(x_m)} + O\left((\mu_m)^2\right) \\ &= H(0) + H'(0)c_2\left(-c_3 + c_2^2 + 2\beta c_2^2\right)e_m^3 \\ &\quad + H'(0)\left(-2c_2c_4 + 9c_3c_2^2 - 5c_2^4 + 12\beta c_3c_2^2 - 14\beta c_4^4 - 2c_3^2\right)e_m^4 + O\left(e_m^5\right). \end{aligned} \tag{2.10}$$

In these terms, the error equation of the method can be expressed as

$$\begin{aligned} e_{m+1} = &- (-1 + H(0))c_2\left((1+2\beta)c_2^2 - c_3\right)e_m^4 \\ &+ \left(-(4 + 12\beta + H(0)(-5 + 2\beta(-7+\theta_2) + \theta_2))c_2^4\right. \\ &\quad + (8 + 12\beta + H(0)(-9 - 12\beta + \theta_2))c_2^2c_3 \\ &\quad \left. +2(-1 + H(0))c_3^2 + 2(-1 + H(0))c_2c_4\right)e_m^5 + O\left(e_m^6\right), \end{aligned} \tag{2.11}$$

which shows that the convergence order of any method of the family (2.4) is at least five if $H(0) = 1$. Then,

$$\begin{aligned} e_{m+1} = & -(-1+\theta_2)c_2^2\left((1+2\beta)c_2^2 - c_3\right)e_m^5 \\ & + c_2\left(\left(-5+\theta_1+7\theta_2-\theta_2^2+2\beta\left(-7+\theta_1+9\theta_2-\theta_2^2\right)\right)c_2^4\right. \\ & + \left(10-\theta_1-14\beta(-1+\theta_2)-12\theta_2+\theta_2^2\right)c_2^2c_3 \\ & \left. +3(-1+\theta_2)c_3^2+2(-1+\theta_2)c_2c_4\right)e_m^6+O\left(e_m^7\right). \end{aligned} \tag{2.12}$$

and it is necessary that $\theta_1 = -1$ and $\theta_2 = 1$ in order to reach order of convergence seven. Then, the error equation is

$$e_{m+1} = -c_2\left(\left(H'(0)+2\beta H'(0)\right)c_2^2 - \left(-1+H'(0)\right)c_3\right)\left((1+2\beta)c_2^3 - c_2c_3\right)e_m^7 + O\left(e_m^8\right), \tag{2.13}$$

and finally, if $\beta = -1/2$ and $H'(0) = 1$, it can be concluded that the order is eight and

$$e_{m+1} = c_3c_2^2\left(2c_2c_3 - c_4 + 2c_2^3\right)e_m^8 + O\left(e_m^9\right). \tag{2.14}$$

□

Remark 2.2. Any method of the family (2.4) has the efficiency index equals to $8^{1/4} \approx 1.682$, which is better than the Newton's method with efficiency index equals to $2^{1/2} \approx 1.414$ and equal to BRW8, M8, and LW8.

In what follows, we give some concrete optimal iterative methods of family (2.4) for different functions H.

(F1) $H(t) = (1+\beta t)^\gamma, \beta \cdot \gamma = 1$, where $\beta, \gamma \in \mathbb{R}$. Hence we get a new eighth-order method whose last step is

$$x_{m+1} = z_m - (1+\beta\mu_m)^\gamma \frac{f(z_m)}{-f[y_m, x_m] + f[z_m, y_m] + f[z_m, x_m]}. \tag{2.15}$$

(F2) $H(t) = 1 + (t/(1+\lambda t))$, where $\lambda \in \mathbb{R}$. So,

$$x_{m+1} = z_m - \left(1 + \frac{\mu_m}{1+\lambda\mu_m}\right)\frac{f(z_m)}{-f[y_m, x_m] + f[z_m, y_m] + f[z_m, x_m]}. \tag{2.16}$$

(F3) $H(t) = 1/(1 - t + \omega t^2)$, where $\omega \in \mathbb{R}$. Then we obtain a new scheme

$$x_{m+1} = z_m - \left(\frac{1}{1-\mu_m+\omega\mu_m^2}\right)\frac{f(z_m)}{-f[y_m, x_m] + f[z_m, y_m] + f[z_m, x_m]}. \tag{2.17}$$

3. Numerical Results

We present some examples to illustrate the efficiency of the iterative algorithm. All computations were done using MAPLE. We have used as stopping criteria that $|x_{m+1} - x_m| \leq 10^{-200}$ or $|f(x_m)| \leq 10^{-200}$. The test functions are listed below:

(a) $f_1(x) = x^3 + 4x^2 - 15;\ x^* \approx 1.6319808055661;$

(b) $f_2(x) = xe^{x^2} - \sin^2(x) + 3\cos(x) + 5;\ x^* \approx -1.2076478271309;$

(c) $f_3(x) = \sin(x) - (x/2);\ x^* \approx 1.8954942670339;$

(d) $f_4(x) = 10xe^{-x^2} - 1;\ x^* \approx 1.6796306104285;$

(e) $f_5(x) = \cos(x) - x; x^* \approx 0.73908513321516;$

(f) $f_6(x) = \sin^2(x) - x^2 + 1;\ x^* \approx 1.4044916482153.$

(g) $f_7(x) = e^{-x} + \cos(x);\ x^* \approx 1.7461395304080.$

We compare the classical Newtons (CN), BRW8 method with $\theta = 1$; M8 scheme with $\beta_1 = 0$, $\beta_3 = 0$, and $\beta_2 = 1$; LW8 method with $\alpha = 1$ and $G(t) = 4t$, and our methods with $\omega = 1$, $\lambda = 1$, $\beta = 1$, and $\gamma = 1$.

In Table 1, the following elements appear for each test function and each iterative method: the value of the elements involved in the stopping criterium, $|x_{m+1} - x_m|$ and $|f(x_m)|$, the number of iterations, iter, needed to converge to the solution, and the approximated computational order of convergence ρ, that can be calculated by using the formula (see [17])

$$\rho = \frac{\ln(|x_{m+1} - x_m| / |x_m - x_{m-1}|)}{\ln(|x_m - x_{m-1}| / |x_{m-1} - x_{m-2}|)}, \tag{3.1}$$

where x_{m+1}, x_m, x_{m-1}, and x_{m-2} are iterations close to a zero of the nonlinear equation.

4. Conclusions

In this work, we have constructed a new general eighth-order iterative family of methods without memory for solving nonlinear equations. Convergence analysis shows that the order of convergence of the methods is eight. Per iteration the present methods require three evaluations of the function and one evaluation of its first derivative and therefore have the efficiency index equal to $8^{1/4} = 1.682$. Some of the obtained methods were also compared in their performance and efficiency to various other iteration methods of the same order, and it was observed that they demonstrate at least equal behavior.

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Table 1: Comparison of various iterative methods.

f_1, $x_0 = 2$	CN	RWB8	M8	LW8	F1	F2	F3
$\lvert x_{m+1} - x_m \rvert$	$6.4e-110$	$7.9e-59$	$7.1e-54$	$7.5e-49$	$2.9e-62$	$2.9e-62$	$2.9e-62$
$\lvert f(x_m) \rvert$	$3.7e-218$	0	0	0	0	0	0
iter	8	3	3	3	3	3	3
ρ	2.0	7.9	8.0	8.0	8.0	8.0	8.0
f_2, $x_0 = -1$	CN	RWB8	M8	LW8	F1	F2	F3
$\lvert x_{m+1} - x_m \rvert$	$1.9e-128$	$4.1e-28$	$1.1e-50$	$3.9e-43$	$8.5e-27$	$4.8e-27$	$4.8e-27$
$\lvert f(x_m) \rvert$	$1.1e-254$	$9.7e-217$	0	0	0	$2.9e-208$	$2.1e-206$
iter	9	3	3	3	3	3	3
ρ	2.0	8.0	8.0	8.0	8.1	8.1	8.1
f_3, $x_0 = 1.9$	CN	RWB8	M8	LW8	F1	F2	F3
$\lvert x_{m+1} - x_m \rvert$	$6.1e-166$	$3.5e-168$	$4.8e-161$	$7.1e-155$	$7.5e-170$	$7.5e-170$	$7.5e-170$
$\lvert f(x_m) \rvert$	0	0	0	0	0	0	0
iter	7	3	3	3	3	3	3
ρ	2.0	7.8	7.6	7.8	8.0	8.0	8.0
f_4, $x_0 = 1.5$	CN	RWB8	M8	LW8	F1	F2	F3
$\lvert x_{m+1} - x_m \rvert$	$2.0e-108$	$6.6e-55$	$5.3e-52$	$3.5e-45$	$5.0e-56$	$5.0e-56$	$5.0e-56$
$\lvert f(x_m) \rvert$	$1.1e-215$	0	0	0	0	0	0
iter	8	3	3	3	3	3	3
ρ	2.0	7.9	8.0	8.0	8.0	8.0	8.0
f_5, $x_0 = 1$	CN	RWB8	M8	LW8	F1	F2	F3
$\lvert x_{m+1} - x_m \rvert$	$7.1e-167$	$3.3e-83$	$5.3e-82$	$1.7e-66$	$1.8e-86$	$2.0e-86$	$1.9e-81$
$\lvert f(x_m) \rvert$	0	0	0	0	0	0	0
iter	8	9	3	3	3	3	3
ρ	2.0	8.0	8.0	8.0	8.0	8.0	8.0
f_6, $x_0 = 1.5$	CN	RWB8	M8	LW8	F1	F2	F3
$\lvert x_{m+1} - x_m \rvert$	$2.6e-148$	$6.2e-86$	$3.8e-72$	$2.3e-66$	$4.5e-87$	$4.5e-87$	$4.5e-87$
$\lvert f(x_m) \rvert$	$1.3e-295$	0	0	0	0	0	0
iter	8	3	3	3	3	3	3
ρ	2.0	7.8	8.0	8.0	8.0	8.0	8.0
f_7, $x_0 = 2$	CN	RWB8	M8	LW8	F1	F2	F3
$\lvert x_{m+1} - x_m \rvert$	$9.6e-170$	$2.7e-80$	$5.3e-78$	$2.8e-61$	$1.3e-76$	$1.5e-76$	$1.3e-76$
$\lvert f(x_m) \rvert$	0	0	0	0	0	0	0
iter	8	3	3	3	3	3	3
ρ	2.0	7.9	7.9	8.0	8.0	8.0	8.0

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