Period doubling bifurcations in the family of Chebyshev-Halley type methods

Alicia Cordero, Juan R. Torregrosa and P. Vindel

Instituto de Matemática Multidisciplinar

Universitat Politècnica de València

Instituto de Matemáticas y Aplicaciones de Castellón

Universitat Jaume I Spain

acordero@mat.upv.es, jrtorre@mat.upv.es, vindel@uji.es

Abstract

The choice of a member of a parametric family of iterative methods is not always easy. The family of Chebyshev-Halley schemes is a good example of it. The analysis of bifurcation points of this family allows us to define a real interval in which there exist several problematic behaviors: attracting points that become doubled, other ones that become periodic orbits,... These are aspects to be avoided in an iterative procedure, so it is important to determinate the regions where this conduct takes place.

In this paper we obtain that this family admits attractive 2-cycles in two different intervals, for real values of the parameter.

1 Introduction

The application of iterative methods for solving nonlinear equations $f(z) = 0$, with $f : \mathbf{C} \to \mathbf{C}$, give rise to rational functions whose dynamics are not well-known. The simplest model is obtained when $f(z)$ is a quadratic polynomial and the iterative process is Newton's method. This dynamical study has been extended to other point-to-point iterative methods used for solving nonlinear equations, with higher order of convergence (see, for example [1], [2] and, more recently, [4] and [5]).

The most of the well-known point-to-point cubically convergent method belongs to the one-parameter family called of Chebyshev-Halley. This set of iterative schemes has been widely analyzed under different points of view. In this work we focus our attention in the dynamical behavior of the rational function associated to this family. From the numerical point of view, this dynamical behavior give us important information about its stability and reliability. In this line, Varona in [6] described the dynamical behavior of several well-known iterative methods. The dynamics of some third order iterative methods is also studied in [2], [3] and, more recently, in [4] and [7].

In this paper we are interested in the study of period doubling bifurcations in the Chebyshev-Halley family when it is applied on quadratic polynomials. In a previous paper [10], we have obtained that the corresponding rational function has different fixed points depending on the values on the parameter. As we have proved, two of these fixed points are always superattractive, but the stability of the others (called strange fixed points) depends on the parameter values.

In fact, we check in Section 4 that there are, at least, two real intervals for which attractive 2-cycles appear and therefore three period doubling and on pitchfork bifurcation occur changing the stability of the strange fixed points.

1.1 Basic dynamical concepts

Now, let us recall some basic concepts on complex dynamics (see [8]). Given a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is defined as:

$$z_0,\ R(z_0),\ R^2(z_0), ..., R^n(z_0), ...$$

We are interested in the study of the asymptotic behavior of the orbits depending on the initial condition z_0, that is, we are going to analyze the phase plane of the map R defined by the different iterative methods.

To obtain these phase spaces, the first of all is to classify the starting points from the asymptotic behavior of their orbits.

A $z_0 \in \hat{\mathbb{C}}$ is called a *fixed point* if it satisfies: $R(z_0) = z_0$. A *periodic point* z_0 of period $p > 1$ is a point such that $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0$, $k < p$. A *pre-periodic point* is a point z_0 that is not periodic but there exists a $k > 0$ such that $R^k(z_0)$ is periodic. A *critical point* z_0 is a point where the derivative of rational function vanishes, $R'(z_0) = 0$.

On the other hand, a fixed point z_0 is called *attractor* if $|R'(z_0)| < 1$, *superattractor* if $|R'(z_0)| = 0$, *repulsor* if $|R'(z_0)| > 1$ and *parabolic* if $|R'(z_0)| = 1$.

The basin of attraction of an attractor α is defined as the set of pre-images of any order:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}}\ :\ R^n(z_0) \to \alpha,\ n \to \infty\}.$$

The set of points $z \in \hat{\mathbb{C}}$ such that their families $\{R^n(z)\}_{n \in \mathbb{N}}$ are normal in some neighborhood $U(z)$, is the *Fatou set,* $\mathcal{F}(R)$, that is, the Fatou set is composed by the set of points whose orbits tend to an attractor (fixed point, periodic orbit or infinity). Its complement in $\hat{\mathbb{C}}$ is the *Julia set,* $\mathcal{J}(R)$; therefore, the Julia set includes all repelling fixed points, periodic orbits and their pre-images. That means that the basin of attraction of any fixed point belongs to the Fatou set. On the contrary, the boundaries of the basins of attraction belong to the Julia set. The invariant Julia set for Chebyshev's method applied to quadratic polynomials is more complicated than for Newton's method and it has been studied in [9].

The rest of this paper is organized as follows: in Section 2 we introduce the previous results needed to develop the present study. In Section 3 we show the parametric space associated to the parametric family. Section 4 is devoted to characterize the bifurcation points and to classify them. We finish the paper with some remarks and conclusions.

2 Previous results

The family of Chebyshev-Halley type methods can be written as the iterative method

$$z_{n+1} = z_n - \left(1 + \frac{1}{2}\frac{L_f(z_n)}{1 - \alpha L_f(z_n)}\right)\frac{f(z_n)}{f'(z_n)}, \qquad L_f(z) = \frac{f(z)\, f''(z)}{(f'(z))^2}.$$

Our interest is focused on the study of the dynamics of the corresponding fixed point operator when it is applied on the quadratic polynomial $p(z) = z^2 + c$. For this polynomial, the operator corresponds to the rational function:

$$G_p(z) = \frac{z^4(-3 + 2\alpha) + 6cz^2 + c^2(1 - 2\alpha)}{4z(z^2(-2 + \alpha) + \alpha c)},$$

depending on two parameters: α and c.

The parameter c can be obviated by considering the conjugacy map

$$h(z) = \frac{z - i\sqrt{c}}{z + i\sqrt{c}},$$

with the following properties:

$$h(\infty) = 1, h\left(i\sqrt{c}\right) = 0, h\left(-i\sqrt{c}\right) = \infty.$$

Accordingly, we are interested in the dynamics of the operator

$$O_p(z) = \left(h \circ G_p \circ h^{-1}\right)(z) = z^3 \frac{z - 2(\alpha - 1)}{1 - 2(\alpha - 1)z}. \qquad (1)$$

2.1 Fixed and critical points

We have began the study of the dynamics of this operator in function of the parameter α in [10]. As we have proved, the number and stability of fixed points depends on the parameter α.

$$O_p(z) = z \Rightarrow z = 0, \infty, z = 1, z = \frac{-3 + 2\alpha \pm \sqrt{5 - 12\alpha + 4\alpha^2}}{2}. \tag{2}$$

The last two fixed points are the roots of $z^2 + (3 - 2\alpha)z + 1 = 0$, denoted by s_1 and s_2. Furthermore, these two points are not independent as $s_1 = \dfrac{1}{s_2}$.

We need the derivative of the operator (1) to study the stability of fixed points:

$$O_p'(z) = 2z^2 \frac{3(1-\alpha) + 2z(3 - 4\alpha + 2\alpha^2) + 3z^2(1-\alpha)}{(1 - 2(\alpha - 1)z)^2}. \tag{3}$$

From (3) we obtain that the origin and ∞ are always super-attractive fixed points, but the stability of the other fixed points change depending on the values of the parameter α. These points are called *strange fixed points*. Their stability satisfies the following statements:

Proposition 1 *(see [10]) The stability of the fixed point $z = 1$ satisfies the following statements:*

- *If $\left|\alpha - \frac{13}{6}\right| < \frac{1}{3}$, then $z = 1$ is an attractor. In particular, it is a superattractor if $\alpha = 2$.*
- *For $\left|\alpha - \frac{13}{6}\right| = \frac{1}{3}$, $z = 1$ is a parabolic point.*
- *Finally, $z = 1$ is a repulsive fixed point for any other value of α.*

Proposition 2 *(see [10]) The stability of the fixed points $z = s_i$, $i = 1, 2$ satisfies the following statements:*

- *If $|\alpha - 3| < \frac{1}{2}$, then s_1 and s_2 are two different attractive fixed points. In particular, for $\alpha = 3$ then s_1 and s_2 are superattractors.*
- *If $|\alpha - 3| = \frac{1}{2}$, then s_1 and s_2 are parabolic. In particular, for $\alpha = \frac{5}{2}$ $s_1 = s_2 = 1$.*
- *For any other value of $\alpha \in$ C, they are repulsive fixed points.*

The *critical points* are those points where the first derivative of the rational operator vanishes, that is

$$z = 0, z = \infty, z = \frac{3 - 4\alpha + 2\alpha^2 \pm \sqrt{-6\alpha + 19\alpha^2 - 16\alpha^3 + 4\alpha^4}}{3(\alpha - 1)}, \qquad (4)$$

the last two critical points are denoted by c_1 and c_2. In addition, they are not independent as $c_1 = \frac{1}{c_2}$.

3 The parameter space

The dynamical behavior of the operator (1) depends on the values of the parameter α. It can be seen in the parameter space, shown in Figure 1.

In this parameter space we observe a black figure (let us to call it *the cat set*), with a certain similarity with the Mandelbrot set (see [11]): for values of α outside this cat set the Julia set is disconnected. The two disks in the main body of the cat set correspond to the α values for those the fixed points $z = 1$ (the head) and s_1 and s_2 became attractive (the body). We also observe a curve similar to a circle that passes through the cat's neck, we call it *the necklace*. As we have proved in [10] the parameter space inside this curve is topologically equivalent to a disk.

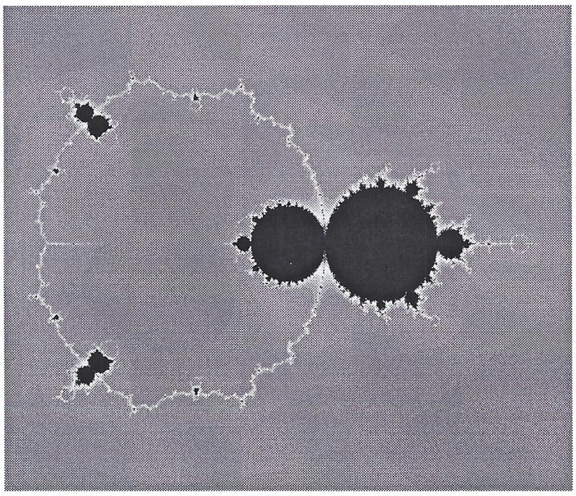

Figure 1: Parameter plane

The head of the cat corresponds to $\left|\alpha - \frac{13}{6}\right| < \frac{1}{3}$, for which the fixed point $z = 1$ is attractive. The body of the cat set corresponds to values of the parameter such that $|\alpha - 3| < \frac{1}{2}$. In this case s_1 and s_2 are attractors and have their own basin of attraction, as there exists one critical point in each basin. The intersection point of both disks is in their common boundary and corresponds to $\alpha = \frac{5}{2}$. For $\alpha = \frac{5}{2}$ the three strange fixed points coincide $z = 1 = s_1 = s_2$ and it is parabolic, $\left|O'_p(1)\right| = 1$, with multiplicity 3. We know, by the flower theorem of Latou (see [12], for example), that this parabolic point is in the common boundary of two attractive regions. The points of an orbit inside each region approach to $z = 1$ without leaving the region.

The boundary of the cat set is exactly the bifurcation locus of the family of Chebyshev-Halley type family acting on quadratic polynomial; that is, the set of parameters for which the dynamics changes abruptly under small changes of α. In this paper we are interested in the study of some of these bifurcations, those that involve cycles of period 2.

4 Local bifurcations

For a discrete dynamical system $z_{n+1} = F(z_n, \alpha)$ where F is a smooth function, a necessary condition for (z_0, α_0) to be a bifurcation point (see, for example [13]) is $\left|\frac{\partial F}{\partial z}(z_0, \alpha_0)\right| = 1$. If $\frac{\partial F}{\partial z}(z_0, \alpha_0) = 1$, the bifurcation is either a saddle-node, transcritical or pitchfork bifurcation. If $\frac{\partial F}{\partial z}(z_0, \alpha_0) = -1$, it is a period-doubling (or flip) bifurcation; otherwise, it is a Hopf bifurcation.

In this paper we are interested in the study of bifurcations for this family of iterative methods. As we will see in the following section, this family suffers various period-doubling and one pitchfork bifurcations for different values of the parameter. It is known that period-doubling bifurcation is characterized by one attractive (or repulsive) fixed point that becomes repulsive (attractive) after the bifurcation point, and simultaneously one attractive (repulsive) 2-cycle appears. For the pitchfork bifurcation one attractive (or repulsive) fixed point becomes repulsive (attractive) after the bifurcation point, and simultaneously two attractive (repulsive) fixed points appear.

4.1 The bulb of period 2 of the head

It is easy to check that $z = 1$ is an hyperbolic point for all these values of α belonging to the circle $\left|\alpha - \frac{13}{6}\right| = \frac{1}{3}$, as

$$O'_p(1) = \frac{2e^{i\theta} + 1}{2 + e^{i\theta}}, \quad \left|O'_p(1)\right| = 1.$$

As we have seen in [10], if $\alpha > \frac{11}{6}$ then $O'_p(1) < 1$, if $\alpha = \frac{11}{6}$ then $O'_p(1) = 1$ and when $\alpha < \frac{11}{6}$ then $O'_p(1) > 1$.

We are going to show that there is a doubling period bifurcation for $\alpha = \frac{11}{6}$. For $\alpha < \frac{11}{6}$ the periodic point $z = 1$ become repulsive and one attractive cycle of period 2 appears. So, $(z = 1, \alpha = \frac{11}{6})$ will be a doubling period bifurcation point.

Proposition 3 *For $\alpha \in (\alpha^*, \frac{11}{6})$, $\alpha \in \boldsymbol{R}$, the dynamical plane of $O_p(z)$ contains an attractive cycle of period 2, where:*

$$\alpha^* = \frac{1}{6}\sqrt[3]{\left(134 + 18\sqrt{57}\right)} - \frac{4}{3\sqrt[3]{(134 + 18\sqrt{57})}} + \frac{5}{6} \approx 1.7041.$$

Proof. It is known that a 2-cycle verifies $O_p^2(z) - z = 0$. This implies that

$$z(-1 + z)\left(1 + 3z - 2az + z^2\right) f(\alpha, z)\, g(\alpha, z) = 0,$$

where

$$f(z, \alpha) = 1 + (3 - 2\alpha)z + (3 - 2\alpha)z^2 + (3 - 2\alpha)z^3 + z^4$$

and

$$\begin{aligned} g(z, \alpha) &= 1 + (3 - 4\alpha)z + \left(2 - 6\alpha + 4\alpha^2\right)z^2 + \\ &\quad \left(3 - 6\alpha + 4\alpha^2\right)z^3 + \left(9 - 22\alpha + 20\alpha^2 - 8\alpha^3\right)z^4 + \\ &\quad \left(3 - 6\alpha + 4\alpha^2\right)z^5 + \left(2 - 6\alpha + 4\alpha^2\right)z^6 + (3 - 4\alpha)z^7 + z^8. \end{aligned}$$

As we have seen, the product $z(-1 + z)(1 + 3z - 2\alpha z + z^2)$ yields to the fixed points. So, 2-periodic points come from $f(z, \alpha) = 0$ or $g(z, \alpha) = 0$. We observe that $f\left(z, \frac{11}{6}\right) = \frac{1}{3}(3z^2 + 4z + 3)(z - 1)^2$ so that the periodic points that collapse with the fixed point $z = 1$ for $\alpha = \frac{11}{6}$ come from the zeros of this function. In fact, we will focus our attention on function $f(z, \alpha)$, as it yields periodic orbits in the bulb of the head, while roots of $g(z, \alpha)$ give rise to 2-orbits in the bulb of the body whose intersection point is $\alpha = \frac{7}{2}$.

A new factorization of f is obtained

$$f(z, \alpha) = f_1(z, \alpha)\, f_2(z, \alpha),$$

where

$$\begin{aligned} f_1(z, \alpha) &= 1 + \frac{1}{2}\left(3 - 2\alpha - \sqrt{5 - 4\alpha + 4\alpha^2}\right)z + z^2, \\ f_2(z, \alpha) &= 1 + \frac{1}{2}\left(3 - 2\alpha + \sqrt{5 - 4\alpha + 4\alpha^2}\right)z + z^2, \end{aligned}$$

and we observe that $f_1\left(\frac{11}{6}, z\right) = (1 - z)^2$ and $f_2\left(\frac{11}{6}, z\right) = \frac{1}{3}(3 - 4z + 3z^2)$. So, the cycle of period 2 that becomes attractive comes from $f_1(\alpha, z) = 0$.

The points of the 2-cycle in the bulb of period 2 on the head are:

$$z_1 = -\frac{3}{4} + \frac{1}{2}\alpha + \frac{1}{4}\sqrt{5 - 4\alpha + 4\alpha^2} + \frac{1}{4}\sqrt{-2 - 16\alpha + 8\alpha^2 + (-6 + 4\alpha)\sqrt{5 - 4\alpha + 4\alpha^2}}$$
$$z_2 = -\frac{3}{4} + \frac{1}{2}\alpha + \frac{1}{4}\sqrt{5 - 4\alpha + 4\alpha^2} - \frac{1}{4}\sqrt{-2 - 16\alpha + 8\alpha^2 + (-6 + 4\alpha)\sqrt{5 - 4\alpha + 4\alpha^2}}$$

because

$$O_p(z_1) = z_2 \quad \text{and} \quad O_p(z_2) = z_1.$$

The function of stability is

$$S(\alpha) = O_p'(z_1) \cdot O_p'(z_2),$$

that is,

$$S(\alpha) = \frac{-54 + 132\alpha - 166\alpha^2 + 112\alpha^3 - 40\alpha^4 + 6(\alpha - 1)(3 - \alpha + 2\alpha^2)\sqrt{5 - 4\alpha + 4\alpha^2}}{-9 + 26\alpha - 34\alpha^2 + 23\alpha^3 - 8\alpha^4 + 2(\alpha - 1)(2 - 3\alpha + 2\alpha^2)\sqrt{5 - 4\alpha + 4\alpha^2}}.$$

To know the range where this cycle is attractor we demand

$$|S(\alpha)| = \left|O_p'(z_1) \cdot O_p'(z_2)\right| = 1.$$

For $\alpha \in \mathbf{R}$ this implies that

$$(1 - 2\alpha)^2 (6\alpha - 11)\left(-19 + 22\alpha - 20\alpha^2 + 8\alpha^3\right) = 0.$$

The only real root inside the head of the cat (corresponding to the real root of the third-degree polynomial in the last equation) is

$$\alpha^* = \frac{1}{6}\sqrt[3]{\left(134 + 18\sqrt{57}\right)} - \frac{4}{3\sqrt[3]{\left(134 + 18\sqrt{57}\right)}} + \frac{5}{6} \approx 1.7041$$

and

$$|S(\alpha^*)| = \left|S\left(\frac{11}{6}\right)\right| = 1.$$

Moreover, we observe that this cycle is attractive in the interval $\alpha^* < \alpha < \frac{11}{6}$ by drawing the function $O_p'(z_1) \cdot O_p'(z_2)$ in this interval (see Figure 2).

■

We observe that there is one value where this cycle is super-attractive, that coincides with the minimum of the function $O_p'(z_1) \cdot O_p'(z_2)$

$$O_p'(z_1) \cdot O_p'(z_2) = 0, \alpha \approx 1.7738.$$

Let us point out that these points are complex for values of α in the interval $(\alpha^*, \frac{11}{6})$.

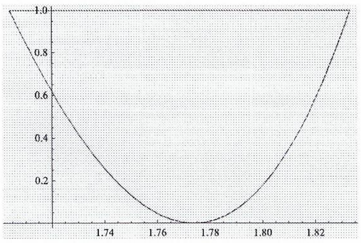

Figure 2: Behavior of the stability function $O'_p(z_1) \cdot O'_p(z_2)$

4.2 The bulb of period 2 in the body of the cat

We know (see [10]) that for $\alpha = \frac{7}{2}$ the strange fixed points, $s_1 = 2 - \sqrt{3}$ and $s_2 = 2 + \sqrt{3}$, become parabolic $|O'(s_1)| = |O'(s_2)| = 1$. For real $\alpha > \frac{7}{2}$ these strange fixed points are repulsive. In this section we show that for real $\alpha > \frac{7}{2}$ there is a bulb where two attractive cycles of period 2 appear. As in the previous section, a doubling period bifurcation occurs for each strange fixed point.

Proposition 4 *For* $\alpha \in (\frac{7}{2}, \alpha^{**})$, $\alpha \in \mathrm{R}$, *the dynamical plane of* $O_p(z)$ *contains two attractive cycles of period 2 where* $\alpha^{**} \approx 3.738271$.

Proof. We can prove that, for $\alpha = \dfrac{7}{2}$, s_1 and s_2 are two roots of $O_p^2(z) - z$, that is, $(z - s_1)(z - s_2) = z^2 - 4z + 1$ is a factor of $O_p^2(z) - z$ since

$$g\left(z, \frac{7}{2}\right) = \left(1 - 4z + z^2\right)^2 \left(1 - 3z - 12z^2 - 3z^3 + z^4\right).$$

Therefore, as happened in the previous section with $f(z, \alpha)$, the cycles of period 2 that collapse with s_1 and s_2, for $\alpha = \dfrac{7}{2}$, come from the roots of $g(z, a)$.

We can factorize $g(z, \alpha) = g_1(z, \alpha) g_2(z, \alpha) g_3(z, \alpha) g_4(z, \alpha)$ where $g_i(z, \alpha)$ are polynomials of degree two, $g_i(z, \alpha) = 1 + b_i z + z^2$. The relationships that the coefficients must satisfy are:

$$\begin{aligned}
b_1 + b_2 + b_3 + b_4 &= 3 - 4\alpha, \\
b_1 b_2 + b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4 &= 4\alpha^2 - 6\alpha - 2, \\
b_1 b_2 b_3 + b_1 b_2 b_4 + b_1 b_3 b_4 + b_2 b_3 b_4 &= 4\alpha^2 - 6\alpha - 6, \\
b_1 b_2 b_3 b_4 &= -8\alpha^3 + 12\alpha^2 - 10\alpha + 7.
\end{aligned}$$

The solution of this system is:

$$
\begin{aligned}
b_1(\alpha) &= \frac{1}{4}\left(3-4\alpha-\sqrt{-3+8\alpha}-\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+\frac{-3+20\alpha-32\alpha^2}{\sqrt{-3+8\alpha}}}\right), \\
b_2(\alpha) &= \frac{1}{4}\left(3-4\alpha-\sqrt{-3+8\alpha}+\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+\frac{-3+20\alpha-32\alpha^2}{\sqrt{-3+8\alpha}}}\right), \\
b_3(\alpha) &= \frac{1}{4}\left(3-4\alpha+\sqrt{-3+8\alpha}-\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+\frac{-3+20\alpha-32\alpha^2}{\sqrt{-3+8\alpha}}}\right), \\
b_4(\alpha) &= \frac{1}{4}\left(3-4\alpha+\sqrt{-3+8\alpha}+\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+\frac{-3+20\alpha-32\alpha^2}{\sqrt{-3+8\alpha}}}\right).
\end{aligned}
$$

We define the function:

$$
\begin{aligned}
h_1(z,\alpha) &= g_1(z,\alpha)\, g_2(z,\alpha) \\
&= 1+\frac{1}{2}\left(3-4\alpha-\sqrt{-3+8\alpha}\right) z+\frac{1}{2}(-1+2\alpha)\left(1+\sqrt{-3+8\alpha}\right) z^2 \\
&\quad +\frac{1}{2}\left(3-4\alpha-\sqrt{-3+8\alpha}\right) z^3+z^4,
\end{aligned}
$$

that coincides with the factorization of the strange fixed points for $\alpha=\frac{7}{2}$.

The four solutions of $h_1(z,\alpha)=0$ are:

$$
\begin{aligned}
z_1(\alpha) &= \frac{1}{8}(-3+4\alpha+\sqrt{-3+8\alpha}+\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}} \\
&\quad -\sqrt{-6+8\alpha+2\sqrt{8\alpha-3}}\sqrt{\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}}-\left(\frac{-8\alpha^2+6\alpha+1+(1+2\alpha)\sqrt{8\alpha-3}}{2\alpha-1}\right)},
\end{aligned}
$$

$$
\begin{aligned}
z_2(\alpha) &= \frac{1}{8}(-3+4\alpha+\sqrt{-3+8\alpha}+\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}} \\
&\quad +\sqrt{-6+8\alpha+2\sqrt{8\alpha-3}}\sqrt{\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}}-\left(\frac{-8\alpha^2+6\alpha+1+(1+2\alpha)\sqrt{8\alpha-3}}{2\alpha-1}\right)}
\end{aligned}
$$

$$
\begin{aligned}
z_3(\alpha) &= \frac{1}{8}(-3+4\alpha+\sqrt{-3+8\alpha}-\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}} \\
&\quad -\sqrt{(6-8\alpha-2\sqrt{8\alpha-3})}\sqrt{\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}}+\left(\frac{-8\alpha^2+6\alpha+1+(1+2\alpha)\sqrt{8\alpha-3}}{2\alpha-1}\right)}
\end{aligned}
$$

$$
\begin{aligned}
z_4(\alpha) &= \frac{1}{8}(-3+4\alpha+\sqrt{-3+8\alpha}-\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}} \\
&\quad +\sqrt{(6-8\alpha-2\sqrt{8\alpha-3})}\sqrt{\sqrt{2}\sqrt{23-16\alpha+8\alpha^2+(1-4\alpha)\sqrt{-3+8\alpha}}+\left(\frac{-8\alpha^2+6\alpha+1+(1+2\alpha)\sqrt{8\alpha-3}}{2\alpha-1}\right)}
\end{aligned}
$$

It easy to see that $z_1\left(\frac{7}{2}\right) = z_3\left(\frac{7}{2}\right) = s_1$ and $z_2\left(\frac{7}{2}\right) = z_4\left(\frac{7}{2}\right) = s_2$. Therefore the cycles of period 2 will be (z_1, z_3) and (z_2, z_4). It can be checked that

$$O_p(z_1) = z_3, \;\; O_p(z_3) = z_1; \quad O_p(z_4) = z_2, \;\; O_p(z_2) = z_4.$$

Moreover, it can be checked graphically that

$$\left|O'_p(z_1)\,O'_p(z_3)\right| \leq 1 \quad \text{and} \quad \left|O'_p(z_2)\,O'_p(z_4)\right| \leq 1.$$

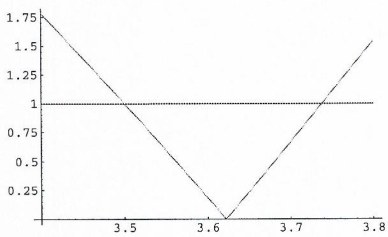

Figure 3: $\left|O'_p(z_1)\,O'_p(z_3)\right|$

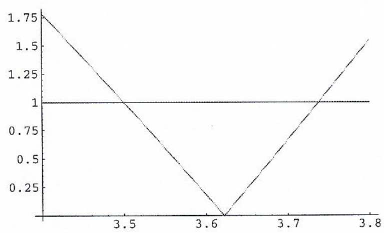

Figure 4: $\left|O'_p(z_2)\,O'_p(z_4)\right|$

So, we observe that there is one interval where these 2-cycles are attractive (see Figures 3 and 4). We also obtain that

$$O'_p(z_1)\,O'_p(z_3) = -1 \Rightarrow \alpha^{**} \approx 3.738271.$$

■

In particular, it can be checked that $O'_p(z_1)\,O'_p(z_3) = 1$ and $O'_p(z_2)\,O'_p(z_4) = 1$ for $\alpha = \frac{7}{2}$. Moreover, the 2-cycle (z_1, z_3) is superattractive for $\alpha \approx 3.6218839102$; and the 2-cycle (z_2, z_4) is superattractive for $\alpha \approx 3.621883885696156$.

4.3 Diagram of bifurcations

Summarizing the results obtained in the previous subsections and in paper [10] (Propositions 1 and 2), we know that for $\alpha \in \left(\alpha^*, \frac{11}{6}\right)$ there is an attractive 2-cycle, that collapses for $\alpha = \frac{11}{6}$ with the repulsive fixed point $z = 1$. For $\alpha \in \left(\frac{11}{6}, \frac{5}{2}\right)$ the fixed point $z = 1$ is attractive. For $\alpha = \frac{5}{2}$ the fixed point $z = 1$ collapses with the repulsive fixed points s_1 and s_2. For $\alpha \in \left(\frac{5}{2}, \frac{7}{2}\right)$ these fixed points s_1 and s_2 become attractive. For $\alpha > \frac{7}{2}$, s_1 and s_2 become repulsive and two attractive 2-cycles appear.

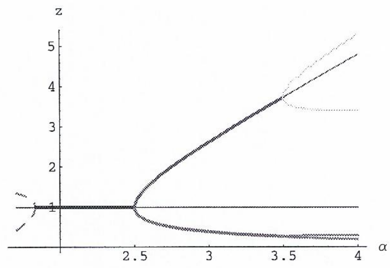

Figure 5: Bifurcation diagram for the strange fixed points, $\alpha \in (\alpha^*, \alpha^{**})$

So, the bifurcations points are $\left(z = 1, \alpha = \frac{11}{6}\right)$, $\left(z = 1, \alpha = \frac{5}{2}\right)$, $\left(z = 2 - \sqrt{3}, \alpha = \frac{7}{2}\right)$ and $\left(z = 2 + \sqrt{3}, \alpha = \frac{7}{2}\right)$. By applying the operator (3) we obtain

$$O_p'\left(z = 1, \alpha = \frac{11}{6}\right) = -1 \quad , \quad O_p'\left(z = 1, \alpha = \frac{5}{2}\right) = 1$$
$$O_p'\left(z = 2 - \sqrt{3}, \alpha = \frac{7}{2}\right) = -1 \quad , \quad O_p'\left(z = 2 + \sqrt{3}, \alpha = \frac{7}{2}\right) = -1.$$

So, as we see in Figure 5 there are three period doubling bifurcations: $\left(z = 1, \alpha = \frac{11}{6}\right)$, $\left(z = 2 - \sqrt{3}, \alpha = \frac{7}{2}\right)$ and $\left(z = 2 + \sqrt{3}, \alpha = \frac{7}{2}\right)$ and one pitchfork bifurcation in $\left(z = 1, \alpha = \frac{5}{2}\right)$.

5 Conclusions

We have characterized bifurcations of Chebyshev-Halley family, for real values of parameter α. It has been stated that some doubling period and pitchfork bifurcations appear for values of the parameter in the interval (α^*, α^{**}). This is a very useful information from the numerical point of view, as the behavior of the iterative methods out of this interval will be more reliable, since there are not attracting elements different from the roots.

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