A domain-theoretic approach to fuzzy metric spaces

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Abstract

We introduce a partial order $\sqsubseteq_M$ on the set $\mathbf{B}X$ of formal balls of a fuzzy metric space $(X, M, \wedge)$ in the sense of Kramosil and Michalek, and discuss some of its properties. We also characterize when the poset $(\mathbf{B}X, \sqsubseteq_M)$ is a continuous domain by means of a new notion of fuzzy metric completeness introduced here. The well-known theorem of Edalat and Heckmann that a metric space is complete if and only if its poset of formal balls is a continuous domain, is deduced from our characterization.

Keywords: Fuzzy metric space; Formal ball; Poset; Continuous domain; Standard complete

MSC [2010]: 54A40, 54E50, 06A06, 06B75

1 Introduction and preliminaries

Throughout this paper the letter $\mathbb{N}$ will denote the set of all positive integer numbers.

Let us recall ([19]) that a continuous t-norm is a binary operation $* : [0,1] \times [0,1] \to [0,1]$ that satisfies the following conditions: (i) $*$ is associative and commutative; (ii) $a * 1 = a$ for all $a \in [0,1]$; (iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$; (iv) $*$ is continuous on $[0,1] \times [0,1]$.

Typical examples of continuous t-norms are the minimum, denoted by $\wedge$, and the product denoted by $\cdot$, i.e., $a \wedge b = \min\{a, b\}$ and $a \cdot b = ab$ for all $a, b \in [0,1]$. It is well known and easy to see that $* \leq \wedge$ for any continuous t-norm $*$.

Definition 1 ([11, Definition 7]). A fuzzy metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a function from $X \times X \times [0, \infty)$ to $[0,1]$, such that for all $x, y, z \in X$:

(KM1) $M(x, y, 0) = 0$;
(KM2) $x = y$ if and only if $M(x, y, t) = 1$ for all $t > 0$;
(KM3) $M(x, y, t) = M(y, x, t)$;
(KM4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ for all $t, s \geq 0$;
(KM5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous.

A fuzzy metric space is a triple $(X, M, *)$ such that X is a set and $(M, *)$ is a fuzzy metric on X.

From (KM2) and (KM4) it follows that for all $x, y \in X$, $M(x, y, .)$ is a non-decreasing function.

Each fuzzy metric $(M, *)$ on a set X induces a topology τ_M on X which has as a base the family of open balls $\{B_M(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$.

If $(x_n)_n$ is a sequence in $(X, M, *)$ which converges to a point $x \in X$ with respect to τ_M, we shall write $\lim_{n\to\infty} x_n = x$. Observe that $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} M(x, x_n, t) = 1$ for all $t > 0$.

A sequence $(x_n)_n$ in a fuzzy metric space $(X, M, *)$ is called a Cauchy sequence if for each $t > 0$ and each $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$. $(X, M, *)$ is said to be complete if every Cauchy sequence converges with respect to τ_M (see e.g. [5]).

Remark 1. It is well known (see e.g. [8]) that every (complete) fuzzy metric space $(X, M, *)$ is (completely) metrizable, i.e., there exists a (complete) metric d on X whose induced topology coincides with τ_M. Conversely, if (X, d) is a (complete) metric space and we define $M_d : X \times X \times [0, \infty) \to [0, 1]$ by $M_d(x, y, 0) = 0$ and

$$M_d(x, y, t) = \frac{t}{t + d(x, y)},$$

for all $t > 0$, then $(X, M_d, \wedge)$ is a (complete) fuzzy metric space called the standard fuzzy metric space of (X, d) (compare [4, 5]). Moreover, the topology τ_{M_d} coincides with the topology induced by d.

Next we recall several concepts from the theory of domains which will be useful later on (see e.g. [6]).

A partial order on a set X is a reflexive, antisymmetric and transitive relation $\sqsubseteq$ on X. In this case, we say that the pair $(X, \sqsubseteq)$ is a partially ordered set (a poset, in short).

An element x of a poset $(X, \sqsubseteq)$ is called maximal if condition $x \sqsubseteq y$ implies $x = y$. The set of maximal elements of $(X, \sqsubseteq)$ will be denoted by $\mathrm{Max}((X, \sqsubseteq))$.

A subset D of a poset $(X, \sqsubseteq)$ is directed provided that it is non-empty and any pair of elements of D has an upper bound in D. The least upper bound of a subset D of X is denoted by $\sqcup D$ if it exists. A poset $(X, \sqsubseteq)$ is directed complete, and is called a dcpo, if every directed subset of $(X, \sqsubseteq)$ has a least upper bound.

Let x and y be two elements of a poset $(X, \sqsubseteq)$. We say that x is way below y, in symbols $x \ll y$, if for each directed subset D of $(X, \sqsubseteq)$ for which $\sqcup D$ exists, the relation $y \sqsubseteq \sqcup D$ implies $x \sqsubseteq z$ for some $z \in D$. A poset $(X, \sqsubseteq)$ is continuous if for each $x \in X$, the set $\Downarrow x = \{u \in X : u \ll x\}$ is directed, and $x = \sqcup(\Downarrow x)$. A continuous poset which is also a dcpo is called a continuous domain, or simply, a domain if no confusion arises.

In this paper we are interested in the problem of establishing relationships between the theory of complete fuzzy metric spaces and domain theory. Our study is motivated, in part, by the previous researches about the construction of computational models for metric spaces and other related structures by using domains (see the New notes of Chapter V-6 in [6] and the references given therein. See also [1, 2, 3, 10, 12, 14, 15, 16, 17, 18, 20, 21]). In particular, Lawson ([12]) proved that a metric space is a maximal point space if and only if it is complete and separable. Later on, Edalat and Heckmann ([2]) established, in a nice and explicit way, several connections between (complete) metric spaces and domain theory by using the notion of a (closed) formal ball.

A formal ball in a (non-empty) set X is simply a pair (x, r), with $x \in X$ and $r \in [0, \infty)$. The set of formall balls of X is the Cartesian product $X \times [0, \infty)$ which will be denoted by $\mathbf{B}X$ in the sequel.

Edalat and Heckmann showed in [2] that if (X, d) is a metric space, then the binary relation $\sqsubseteq_d$ defined on $\mathbf{B}X$ by

$$(x, r) \sqsubseteq_d (y, s) \Longleftrightarrow d(x, y) \leq r - s, \tag{1}$$

for all $(x, r), (y, s) \in \mathbf{B}X$, is a partial order. Moreover, they proved, among other interesting results, that $(\mathbf{B}X, \sqsubseteq_d)$ is a domain if and only if (X, d) is complete [2, Theorem 6 and Corollary 10].

In order to extending the constructions of Edalat and Heckmann to the fuzzy metric framework, two initial procedures seem to be quiet natural. The first one consists in noting that condition (1) can be formulated as

$$(x, r) \sqsubseteq_d (y, s) \Longleftrightarrow y \in \overline{B}_d(x, r - s),$$

(where $\overline{B}_d(x,0) = \{x\}$), and then to adapt this equivalence to the fuzzy metric context. The second one consists in noting that condition (1) can be formulated in terms of the standard fuzzy metric $(M_d, \wedge)$ as

$$(x,r) \sqsubseteq_d (y,s) \Longleftrightarrow M_d(x,y,t) \geq \frac{t}{t+r-s} \quad \text{for all } t > 0, \tag{2}$$

and then take this equivalence as a starting point to define a possible suitable partial order on the set of formal balls of any fuzzy metric space. Here we shall study this second approach, whereas the first one will be discussed elsewhere. In fact, we shall show that for any fuzzy metric space of type $(X, M, \wedge)$, the binary relation $\sqsubseteq_M$ suggested by (2) is a partial order on $\mathbf{B}X$ (this result was announced in [13]). Moreover, we characterize when the poset $(\mathbf{B}X, \sqsubseteq_M)$ is a domain. This will be done by means of a new notion of fuzzy metric completeness that generalizes the usual one.

2 The results

We begin this section with the following notion which is suggested by the equivalence (2) given in Section 1.

Definition 2 ([13]). For a fuzzy metric space $(X, M, *)$ we define a binary relation $\sqsubseteq_M$ on the set $\mathbf{B}X$ of formal balls of X, by

$$(x,r) \sqsubseteq_M (y,s) \Longleftrightarrow M(x,y,t) \geq \frac{t}{t+r-s} \quad \text{for all } t > 0.$$

Remark 2. Note that if $(x,r) \sqsubseteq_M (y,s)$, then $r \geq s$. Indeed, choose $t_0 > 0$ such that $t_0 + r - s > 0$. Then

$$1 \geq M(x,y,t_0) \geq \frac{t_0}{t_0+r-s} > 0,$$

so $t_0 + r - s \geq t_0$, and thus $r \geq s$.

Next we show that for $* = \wedge$, $(\mathbf{B}X, \sqsubseteq_M)$ is a poset, and give an example of a fuzzy metric space $(X, M, \cdot)$ for which $\sqsubseteq_M$ is not a partial order on $\mathbf{B}X$.

Proposition 1. *Let $(X, M, \wedge)$ be a fuzzy metric space. Then $(\mathbf{B}X, \sqsubseteq_M)$ is a poset.*

Proof. Let $(x,r), (y,s), (z,u) \in \mathbf{B}X$. Then we have

- Reflexivity: $(x,r) \sqsubseteq_M (x,r)$ because $M(x,x,t)=1 \quad$ for all $t>0$.
- Antisymmetry: Let $(x,r) \sqsubseteq_M (y,s)$ and $(y,s) \sqsubseteq_M (x,r)$. Then $(x,r) = (y,s)$, because under the above assumption, $r=s$, by Remark 2, and hence $M(x,y,t)=1 \quad$ for all $t>0$ and thus $x=y$.
- Transitivity: Let $(x,r) \sqsubseteq_M (y,s)$ and $(y,s) \sqsubseteq_M (z,u)$. Then $(x,r) \sqsubseteq_M (z,u)$ because, assuming without loss of generality that $r>u$, and putting for each $t>0$, $v=t/(r-u)$, we obtain

$$\begin{aligned} M(x,z,t) &= M(x,z,v(r-u)) \\ &\geq M(x,y,v(r-s)) \wedge M(y,z,v(s-u)) \\ &\geq \frac{v(r-s)}{v(r-s)+r-s} \wedge \frac{v(s-u)}{v(s-u)+s-u} \\ &= \frac{v}{v+1} = \frac{t}{t+r-u}. \end{aligned}$$

Thus, we have proved that $\sqsubseteq_M$ is a partial order on $\mathbf{B}X$. $\square$

Remark 3. Note that $\mathrm{Max}((X,\sqsubseteq_M)) = \{(x,0) : x \in X\}$.

Remark 4. It is clear that if (X,d) is a metric space, the partial orders $\sqsubseteq_d$ and $\sqsubseteq_{M_d}$ coincide.

The following example shows that we cannot guarantee that the binary relation $\sqsubseteq_M$ is a partial order when $*$ is the product.

Example 1. Let $X=\{a,b,c\}$ and $M : X \times X \times [0,\infty) \to [0,1]$ defined by

$M(x,y,0) = M(y,x,0) = 0 \quad$ for all $x,y \in X$,

$M(a,a,t) = M(b,b,t) = M(c,c,t) = 1 \quad$ for all $t>0$,

$M(a,b,t) = M(b,a,t) = M(b,c,t) = M(c,b,t) = t/(t+1) \quad$ for all $t>0$,

$M(a,c,t) = M(c,a,t) = t^2/(t+2)^2 \quad$ for all $t>0$.

It was proved in [9, Example 1] that $(X,M,\cdot)$ is a fuzzy metric space.

Now observe that for $r=2$, $s=1$ and $u=0$, one has

$$M(a,b,t) = \frac{t}{t+1} = \frac{t}{t+r-s},$$

and

$$M(b,c,t) = \frac{t}{t+1} = \frac{t}{t+s-u},$$

for all $t > 0$. So $(a,r) \sqsubseteq_M (b,s)$ and $(b,s) \sqsubseteq_M (c,u)$. However, for $0 < t < 1$, we obtain

$$M(a,c,t) = \frac{t^2}{(t+2)^2} < \frac{t}{t+r-u}.$$

Therefore, the binary relation $\sqsubseteq_M$ is not transitive, and thus $(\mathbf{B}X, \sqsubseteq_M)$ is not a poset.

Remark 5. In a first moment one can think that the following alternative definition of $\sqsubseteq_M$, also could provide a partial order on the set $\mathbf{B}X$:

$$(x,r) \sqsubseteq_M (y,s) \Longleftrightarrow M(x,y,t) \geq \frac{t}{t+r-s} \quad \text{for some } t > 0,$$

However, this is not the case as the next example shows.

Example 2. Let (X,d) be a metric space, with $|X| \geq 2$, and consider the fuzzy metric $(M, \wedge)$ on X given by $M(x,y,t) = 1$ if $d(x,y) < t$, and $M(x,y,t) = 0$ if $d(x,y) \geq t$. Then, for $x \neq y$ and $r = t > d(x,y)$, we have $(x,r) \sqsubseteq_M (y,r)$ and $(y,r) \sqsubseteq_M (x,r)$, so $\sqsubseteq_M$ as defined above is not antisymmetric.

Now we establish some basic properties that will be useful later on. In fact, Lemmas 1 and 2 will provide fuzzy counterparts of [2, Theorem 2] and of a part of [2, Theorem 5], respectively.

Let us recall that a sequence $(x_n)_n$ in a poset $(X, \sqsubseteq)$ is called ascending provided that $x_n \sqsubseteq x_{n+1}$ for all $n \in \mathbb{N}$. Hence, if $(x_n)_n$ is an ascending sequence, the set $\{x_n : n \in \mathbb{N}\}$ is a directed subset of $(X, \sqsubseteq)$, and by an upper bound of $(x_n)_n$ we will mean an upper bound of $\{x_n : n \in \mathbb{N}\}$.

Lemma 1. *Let $(X, M, \wedge)$ be a fuzzy metric space and let D be a directed subset of $(\mathbf{B}X, \sqsubseteq_M)$. Then, there is an ascending sequence in D which has the same upper bounds as D.*

Proof. Let $s = \inf\{r : (x,r) \in D\}$. Then, for each $n \in \mathbb{N}$ there is $(y_n, s_n) \in D$ such that $s_n \leq s + 1/n$. Put $(x_1, r_1) = (y_1, s_1)$. As D is directed there is $(x_2, r_2) \in D$ such that $(x_1, r_1) \sqsubseteq_M (x_2, r_2)$ and $(y_2, s_2) \sqsubseteq_M (x_2, r_2)$. Applying the same reasoning successively we obtain that for each $n > 1$

there is $(x_n, r_n) \in D$ which is an upper bound of (x_{n-1}, r_{n-1}) and (y_n, s_n). Then $((x_n, r_n))_n$ is an ascending sequence in D.

We shall show that any upper bound of $((x_n, r_n))_n$ is an upper bound of any element of D.

Indeed, let $(z, u) \in D$ be such that $(x_n, r_n) \sqsubseteq_M (z, u)$ for all $n \in \mathbb{N}$ and let (a, v) be an arbitrary element of D. Since D is directed, for each $n \in \mathbb{N}$ there is $(b_n, v_n) \in D$ which is an upper bound of (a, v) and (x_n, r_n).

Note that for each $n \in \mathbb{N}$, $r_n \leq s_n$, so

$$r_n - v_n \leq s_n - v_n \leq s - v_n + \frac{1}{n} \leq \frac{1}{n},$$

for all $n \in \mathbb{N}$. Now given $t > 0$, put

$$t_n = \frac{t}{v - u + 2/n},$$

for all $n \in \mathbb{N}$. (Note that, indeed, $t_n > 0$ because from $(x_n, r_n) \sqsubseteq_M (z, u)$ it follows that $u \leq r_n$, and from $(x_n, r_n) \sqsubseteq_M (b_n, v_n)$ it follows that $v_n \leq r_n \leq v_n + 1/n \leq v + 1/n$. Hence $v - u + 1/n \geq 0$, and thus, $v - u + 2/n > 0$, for all $n \in \mathbb{N}$).

Therefore

$$t_n(v - u + 2r_n - 2v_n) \leq \frac{t(v - u + 2/n)}{v - u + 2/n} = t,$$

for all $n \in \mathbb{N}$, so

$$\begin{aligned} M(a, z, t) &\geq M(a, b_n, t_n(v - v_n)) \wedge M(b_n, x_n, t_n(r_n - v_n)) \wedge M(x_n, z, t_n(r_n - u)) \\ &\geq \frac{t_n(v - v_n)}{t_n(v - v_n) + v - v_n} \wedge \frac{t_n(r_n - v_n)}{t_n(r_n - v_n) + r_n - v_n} \wedge \frac{t_n(r_n - u)}{t_n(r_n - u) + r_n - u} \\ &= \frac{t_n}{t_n + 1} = \frac{t}{t + v - u + 2/n}, \end{aligned}$$

for all $n \in \mathbb{N}$. Hence

$$M(a, z, t) \geq \frac{t}{t + v - u}.$$

We conclude that (z, u) is an upper bound of D. □

Lemma 2. *Let $(X, M, \wedge)$ be a fuzzy metric space. If $((x_n, r_n))_n$ is an ascending sequence in $(\mathbf{B}X, \sqsubseteq_M)$, with $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} r_n = r$,* *then* $(x, r) = \sqcup D$, *where* $D = \{(x_n, r_n) : n \in \mathbb{N}\}$.

Proof. We first prove that (x, r) is an upper bound of D.

Indeed, fix $k \in \mathbb{N}$. We want to show that $(x_k, r_k) \sqsubseteq_M (x, r)$.

Since $(r_n)_n$ is a decreasing sequence, then $r \leq r_n$ for all $n \in \mathbb{N}$, so, in particular, $r \leq r_k$.

- If $r = r_k$, we deduce that $r = r_n$ for all $n \geq k$. Hence, from the fact that $(x_k, r_k) \sqsubseteq_M (x_n, r_n)$ for all $n \geq k$, it follows that
$$M(x_k, x_n, t) \geq \frac{t}{t + r_k - r_n} = \frac{t}{t + r - r} = 1,$$
for all $n \geq k$ and $t > 0$. Therefore $x_n = x_k$ for all $n \geq k$, and thus, $x = x_n$ for all $n \geq k$. Consequently (x, r) is an upper bound of D.

- If $r < r_k$, we have that $t/(t+r_k-r) < 1$ for all $t > 0$. Then, there exists $\varepsilon_0 > 0$ such that $t/(t+r_k-r) < 1-\varepsilon_0$ for all $t > 0$. Now fix $t > 0$. For each $\varepsilon \in (0, \varepsilon_0 \wedge t)$, there exists $m > k$ such that $M(x, x_m, \varepsilon) > 1 - \varepsilon$ because $\lim_{n\to\infty} x_n = x$. Hence
$$\begin{aligned} M(x, x_k, t) &\geq M(x, x_m, \varepsilon) \wedge M(x_m, x_k, t - \varepsilon) > \\ &> (1 - \varepsilon) \wedge \frac{t - \varepsilon}{t - \varepsilon + r_k - r_m} \geq (1 - \varepsilon) \wedge \frac{t - \varepsilon}{t - \varepsilon + r_k - r}. \end{aligned}$$
Taking limits as $\varepsilon \to 0$, we obtain
$$M(x, x_k, t) \geq \frac{t}{t + r_k - r}.$$
Since $t > 0$ is arbitrary, we conclude that $(x_k, r_k) \sqsubseteq_M (x, r)$, so (x, r) is an upper bound of D.

Finally, suppose that there is $(z, u) \in \mathbf{B}X$ such that $(x_n, r_n) \sqsubseteq_M (z, u)$ for all $n \in \mathbb{N}$. This implies that $r_n \geq u$ for all $n \in \mathbb{N}$, and since $\lim_{n\to\infty} r_n = r$, we deduce that $r \geq u$. We distinguish two cases again.

- If $u = r$, we have $(x_n, r_n) \sqsubseteq_M (z, r)$ for all $n \in \mathbb{N}$, so
$$M(z, x_n, t) \geq \frac{t}{t + r_n - r},$$
for all $n \in \mathbb{N}$ and $t > 0$. Since $\lim_{n\to\infty} r_n = r$, it follows that, for each $t > 0$, $\lim_{n\to\infty} M(z, x_n, t) = 1$, so $z = x$. We have shown that $(x, r) = (z, u)$.

- If $u < r$, we shall suppose that $r < r_n$ for all $n \in \mathbb{N}$ (otherwise, there is $k \in \mathbb{N}$ such that $r = r_n$ and $x = x_n$ for all $n \geq k$, and thus $(x, r) \sqsubseteq_M (z, u)$). Take an arbitrary $t > 0$. For each $n \in \mathbb{N}$ put $v_n = t/(r_n - u)$. Then

$$\begin{aligned} M(x, z, t) &\geq M(x, x_n, v_n(r_n - r)) \wedge M(x_n, z, v_n(r - u)) \\ &\geq \frac{v_n(r_n - r)}{v_n(r_n - r) + r_n - r} \wedge \frac{v_n(r - u)}{v_n(r - u) + r - u} \\ &= \frac{v_n}{v_n + 1} = \frac{t}{t + r_n - u}. \end{aligned}$$

  Taking limits as $n \to \infty$, we obtain

$$M(x, z, t) \geq \frac{t}{t + r - u}.$$

Therefore $(x, r) \sqsubseteq_M (z, u)$. We conclude that $(x, r) = \sqcup D$. $\square$

In the rest of the paper we discuss the problem of obtaining a fuzzy counterpart of the aforementioned theorem of Edalat and Heckmann that a metric space (X, d) is complete if and only if $(\mathbf{B}X, \sqsubseteq_d)$ is a domain. In this direction, we showed in [13] that if $(X, M, \wedge)$ is a complete fuzzy metric space, then $(\mathbf{B}X, \sqsubseteq_M)$ is a domain, but the converse is not true. In the following we introduce a new notion of completeness which is suitable to characterize those fuzzy metric spaces $(X, M, \wedge)$ such that $(\mathbf{B}X, \sqsubseteq_M)$ is a domain.

Definition 3. A sequence $(x_n)_n$ in a fuzzy metric space $(X, M, *)$ is called a standard Cauchy sequence if for each $\varepsilon \in (0, 1)$ there exists $n_0 \in \mathbb{N}$ such that

$$M(x_n, x_m, t) > \frac{t}{t + \varepsilon},$$

for all $n, m \geq n_0$ and $t > 0$.

Definition 4. A fuzzy metric space $(X, M, *)$ is called standard complete if every standard Cauchy sequence converges.

It is easy to see that every standard Cauchy sequence in a fuzzy metric space is a Cauchy sequence, and, hence, every complete fuzzy metric space is standard complete.

Remark 6. Although the notion of standard Cauchy sequence certainly yields a strong property, it is not hard to construct fuzzy metric spaces having non-eventually constant standard Cauchy sequences. For instance, let $(X, M, \cdot)$ be a stationary fuzzy metric space (i.e., for each $x, y \in X$, the function $t \to M(x, y, t)$ is constant [7, Definition 2]), having non-eventually constant Cauchy sequences, and let $\varphi : (0, \infty) \to (0, \infty)$ be a non-decreasing and continuous function such that $\varphi(t) \geq t$ for all $t > 0$. According to [7, Example 15], $(X, M_\varphi, \cdot)$ is a fuzzy metric space, where $M_\varphi(x, y, 0) = 0$ for all $x, y \in X$, and

$$M_\varphi(x, y, t) = \frac{M(x, y, t) + \varphi(t)}{1 + \varphi(t)},$$

for all $x, y \in X$ and $t > 0$. Finally, it is routine to check that any Cauchy sequence in $(X, M, \cdot)$ is a standard Cauchy sequence in $(X, M_\varphi, \cdot)$.

The following result, whose easy proof is omitted, provides a significative class of fuzzy metric spaces for which the notions of completeness and standard completeness coincide, and, in addition, justifies the names of "standard Cauchy" and "standard complete", respectively.

Proposition 2. *Let (X, d) be a metric space and let $*$ be a continuous t-norm. Then:*

(a) *A sequence in $(X, M_d, *)$ is standard Cauchy if and only if it is Cauchy.*

(b) *$(X, M_d, *)$ is standard complete if and only if it is complete.*

In order to prove our main result we also need the following lemmas.

Lemma 3. *Let $(X, M, \wedge)$ be a fuzzy metric space. If $((x_n, r_n))_n$ is an ascending sequence in $(\mathbf{B}X, \sqsubseteq)$, then $(x_n)_n$ is a standard Cauchy sequence in $(X, M, \wedge)$ and $(r_n)_n$ is a Cauchy sequence in $[0, \infty)$.*

Proof. Since the sequence $((x_n, r_n))_n$ is ascending, $(x_n, r_n) \sqsubseteq_M (x_{n+1}, r_{n+1})$, so $r_n \geq r_{n+1}$ for all $n \in \mathbb{N}$. Hence, there exists $r \geq 0$ such that $\lim_{n\to\infty} r_n = r$. So, in particular, $(r_n)_n$ is a Cauchy sequence in $[0, \infty)$.

In order to prove that $(x_n)_n$ is a standard Cauchy sequence in $(X, M, \wedge)$ choose an arbitrary $\varepsilon \in (0, 1)$. Then, there is $n_0 \in \mathbb{N}$ such that $0 \leq r_n - r_m <$

ε whenever $n_0 \leq n \leq m$. Since $(x_n, r_n) \sqsubseteq_M (x_m, r_m)$ we deduce that

$$M(x_n, x_m, t) \geq \frac{t}{t + r_n - r_m} > \frac{t}{t + \varepsilon},$$

for $m \geq n \geq n_0$ and $t > 0$. Therefore $(x_n)_n$ is a standard Cauchy sequence in $(X, M, \wedge)$. $\square$

Lemma 4. *Let $(x_n)_n$ be a standard Cauchy sequence in the fuzzy metric space $(X, M, \wedge)$. Then, there is a subsequence $(x_{n_k})_k$ of (x_n) such that $(x_{n_k}, 2^{-k})_k$ is an ascending sequence in $(\mathbf{B}X, \sqsubseteq_M)$.*

Proof. Since $(x_n)_n$ is standard Cauchy, there is $n_1 \in \mathbb{N}$ such that

$$M(x_{n_1}, x_n, t) \geq \frac{t}{t + 2^{-2}},$$

for all $n \geq n_1$ and $t > 0$. Similarly, there is $n_2 > n_1$ such that

$$M(x_{n_2}, x_n, t) \geq \frac{t}{t + 2^{-3}},$$

for all $n \geq n_2$ and $t > 0$. Continuing this process, we construct a subsequence $(x_{n_k})_k$ of (x_n) such that

$$M(x_{n_k}, x_{n_{k+1}}, t) \geq \frac{t}{t + 2^{-(k+1)}},$$

for all $k \in \mathbb{N}$ and $t > 0$. Therefore $(x_{n_k}, 2^{-k}) \sqsubseteq_M (x_{n_{k+1}}, 2^{-(k+1)})$ for all $k \in \mathbb{N}$. This concludes the proof. $\square$

Lemma 5. *Let $(X, M, \wedge)$ be a standard complete fuzzy metric space. Then $(x, r + \varepsilon) \ll (x, r)$ for all $(x, r) \in \mathbf{B}X$ and for all $\varepsilon > 0$.*

Proof. Let D be a directed subset of $\mathbf{B}X$ such that there is $(z, u) = \sqcup D$ with $(x, r) \sqsubseteq_M (z, u)$. By Lemma 1 there exists an ascending sequence $((z_n, u_n))_n$ in D for which (z, u) is its least upper bound. Given $\varepsilon > 0$ we shall show that $(x, r + \varepsilon) \sqsubseteq_M (z_k, u_k)$ for some $k \in \mathbb{N}$.

Indeed, by Lemma 3, $(z_n)_n$ is a standard Cauchy sequence in $(X, M, \wedge)$ and $(r_n)_n$ is a Cauchy sequence in $[0, \infty)$, so there is $(y, v) \in \mathbf{B}X$ such that $\lim_{n\to\infty} z_n = y$ and $\lim_{n\to\infty} u_n = v$. By Lemma 2, $y = z$ and $v = u$.

Now take $k \in \mathbb{N}$ such that $u_k < u + \varepsilon/2$. We distinguish two cases.

- $r = u$. Then $M(x,z,t) = 1$ for all $t > 0$ because $(x,r) \sqsubseteq_M (z,u)$, so $x = z$. Hence (x,r) is the least upper bound of $((z_n,u))_n$, and, in particular,
$$M(x, z_k, t) \geq \frac{t}{t + u_k - r},$$
for all $t > 0$, so
$$M(x, z_k, t) > \frac{t}{t + (r+\varepsilon) - u_k},$$
for all $t > 0$, i.e., $(x, r+\varepsilon) \sqsubseteq_M (z_k, u_k)$.

- $r > u$. In this case, we have for each $t > 0$,

$$\begin{aligned}
M(x, z_k, t) &\geq M\left(x, z, \frac{r-u}{r+\varepsilon-u_k} t\right) \wedge M\left(z, z_k, \left(1 - \frac{r-u}{r+\varepsilon-u_k}\right) t\right) \\
&\geq \frac{\left(\frac{r-u}{r+\varepsilon-u_k}\right) t}{\left(\frac{r-u}{r+\varepsilon-u_k}\right) t + r - u} \wedge \frac{\left(1 - \frac{r-u}{r+\varepsilon-u_k}\right) t}{\left(1 - \frac{r-u}{r+\varepsilon-u_k}\right) t + u_k - u} \\
&= \frac{(r-u)t}{(r-u)t + (r-u)(r+\varepsilon-u_k)} \\
&\quad \wedge \frac{(\varepsilon - u_k + u)t}{(\varepsilon - u_k + u)t + (u_k - u)(r+\varepsilon-u_k)} \\
&= \frac{t}{t + r + \varepsilon - u_k} \wedge \frac{t}{t + \left(\frac{u_k-u}{\varepsilon-(u_k-u)}\right)(r+\varepsilon-u_k)}.
\end{aligned}$$

Since
$$\frac{u_k - u}{\varepsilon - (u_k - u)} = \frac{1}{\frac{\varepsilon}{u_k-u} - 1} < 1,$$
it follows that
$$\frac{t}{t + r + \varepsilon - u_k} \wedge \frac{t}{t + \left(\frac{u_k-u}{\varepsilon-(u_k-u)}\right)(r+\varepsilon-u_k)} = \frac{t}{t + r + \varepsilon - u_k},$$
so
$$M(x, z_k, t) \geq \frac{t}{t + r + \varepsilon - u_k},$$
for all $t > 0$. We conclude that $(x, r+\varepsilon) \ll (x, r)$. $\square$

Lemma 6. *Let $(X, M, \wedge)$ be a fuzzy metric space. If $(x, r) \ll (y, s)$, then there is $\varepsilon \in (0, 1)$ such that*

$$M(x, y, t) > \frac{t}{t + r - (s + \varepsilon)}.$$

So, in particular, $r > s$.

Proof. Take the ascending sequence $(y, s + 1/n)_n$. Since $\lim_{n\to\infty}(s + 1/n) = s$ and $\{y\}$ may be seen as a constant sequence, by Lemma 2 we deduce that (y, s) is the least upper bound of the sequence $((y, s + 1/n))_n$. Since, by hypothesis, $(x, r) \ll (y, s)$, then there exists $n_0 \in \mathbb{N}$ such that $(x, r) \sqsubseteq_M (y, s + 1/n_0)$, i.e.,

$$M(x, y, t) \geq \frac{t}{t + r - (s + 1/n_0)},$$

for all $t > 0$. Taking $\varepsilon \in (0, 1/n_0)$, we deduce that

$$M(x, y, t) > \frac{t}{t + r - (s + \varepsilon)},$$

and, thus, $r > s + \varepsilon$. □

Theorem 1. *For a fuzzy metric space $(X, M, \wedge)$ the following conditions are equivalent.*

(1) *$(X, M, \wedge)$ is standard complete.*

(2) *$(\mathbf{B}X, \sqsubseteq_M)$ is a domain.*

(3) *$(\mathbf{B}X, \sqsubseteq_M)$ is a dcpo.*

Proof. (1) $\Longrightarrow$ (2) We first show that $(\mathbf{B}X, \sqsubseteq_M)$ is a dcpo. Let D be a directed subset of $(\mathbf{B}X, \sqsubseteq_M)$. By Lemma 1 there is an ascending sequence $((x_n, r_n))_n$ in D which has the same upper bounds as D, and by Lemma 3, $(x_n)_n$ is a standard Cauchy sequence in $(X, M, \wedge)$ and $(r_n)_n$ is a Cauchy sequence in $[0, \infty)$. Sice $(X, M, \wedge)$ is standard complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Then, by Lemma 2, $(x, r) = \sqcup D$, where $r = \lim_{n\to\infty} r_n$. Hence $(\mathbf{B}X, \sqsubseteq_M)$ is a dcpo.

In order to show that $(\mathbf{B}X, \sqsubseteq_M)$ is continuous take $(x, r), (y, s), (z, u) \in \mathbf{B}X$ such that $(y, s) \ll (x, r)$ and $(z, u) \ll (x, r)$. By Lemma 6 there is $\varepsilon \in (0, 1)$ such that

$$M(x, y, t) > \frac{t}{t + s - (r + \varepsilon)} \quad \text{and} \quad M(x, z, t) > \frac{t}{t + u - (r + \varepsilon)},$$

for all $t > 0$. Hence $(y,s) \sqsubseteq_M (x, r+\varepsilon)$, and $(z,u) \sqsubseteq_M (x, r+\varepsilon)$. Since, by Lemma 5, $(x, r+\varepsilon) \in \Downarrow (x,r)$, we deduce that $\Downarrow (x,r)$ is directed. Moreover (x,r) is, obviously, an upper bound of $\Downarrow (x,r)$. Finally, let $(z,u) \in \mathbf{B}X$ be such that $(y,s) \sqsubseteq_M (z,u)$ for all $(y,s) \in \Downarrow (x,r)$. In particular $(x, r+1/n) \sqsubseteq_M (z,u)$ for all $n \in \mathbb{N}$ by Lemma 5, so for each $t > 0$ we have

$$M(x,z,t) \geq \frac{t}{t + r + \frac{1}{n} - u},$$

whenever $n \in \mathbb{N}$. Consequently $M(x,z,t) \geq t/(t+r-u)$, i.e., $(x,r) \sqsubseteq_M (z,u)$. We conclude that $(x,r) = \sqcup (\Downarrow (x,r))$, and hence $(\mathbf{B}X, \sqsubseteq_M)$ is continuous.

(2) $\Longrightarrow$ (3) Obvious

(3) $\Longrightarrow$ (1) Let $(x_n)_n$ be a standard Cauchy sequence in $(X, M, \wedge)$. By Lemma 4 there is a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that $(x_{n_k}, 2^{-k})_k$ is an ascending sequence in $(\mathbf{B}X, \sqsubseteq_M)$. Then, there is $(x,r) \in \mathbf{B}X$ such that $(x,r) = \sqcup D$, where $D = \{(x_{n_k}, 2^{-k}) : k \in \mathbb{N}\}$. Clearly $r = 0$, so $(x_{n_k}, 2^{-k}) \sqsubseteq_M (x, 0)$ for all $k \in \mathbb{N}$, and consequently

$$M(x, x_{n_k}, t) \geq \frac{t}{t + 2^{-k}},$$

for all $k \in \mathbb{N}$ and $t > 0$, which implies $\lim_{k\to\infty} M(x, x_{n_k}, t) = 1$ for all $t > 0$, i.e., $\lim_{k\to\infty} x_{n_k} = x$. Since $(x_n)_n$ is standard Cauchy, it is, in particular, a Cauchy sequence and hence, $\lim_{n\to\infty} x_n = x$. We conclude that $(X, M, \wedge)$ is standard complete. □

Corollary 1 ([2]). *For a metric space (X,d) the following conditions are equivalent.*

(1) *(X,d) is complete.*

(2) *$(\mathbf{B}X, \sqsubseteq_d)$ is a domain.*

(3) *$(\mathbf{B}X, \sqsubseteq_d)$ is a dcpo.*

Proof. (1) $\Longrightarrow$ (2) By Proposition 2, $(X, M_d, \wedge)$ is standard complete, so $(\mathbf{B}X, \sqsubseteq_{M_d})$ is a domain by Theorem 1**.** Now the conclusion follows from Remark 4.

(2) $\Longrightarrow$ (3) Obvious.

(3) $\Longrightarrow$ (1) By Remark 4, $(\mathbf{B}X, \sqsubseteq_{M_d})$ is a dcpo, so $(X, M_d, \wedge)$ is standard complete by Theorem 1. The conclusion follows from Proposition 2. □

Acknowledgements. The authors thank the referee for a careful reading of the paper and several suggestions that have allowed to improve its quality. Salvador Romaguera thanks the support of the Ministry of Economy and Competitiveness of Spain, Grant MTM2012-37894-C02-01.

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