# On the construction of domains of formal balls for uniform spaces

S. Romaguera, M. A. Sánchez-Granero, M. Sanchis*

*Dedicated to the memory of Professor Sergio Salbany*

**Abstract**

In the spirit of the well-known constructions of Edalat and Heckmann for metric spaces, we endow the set of formal (closed) balls of a given uniform space with a structure of poset and prove several of its properties, which extend to the uniform framework the corresponding ones of metric spaces. In particular, to show under what conditions this poset is a dcpo we introduce and discuss a weak notion of uniform completeness. Some illustrative examples are also given.

*Keywords*: Uniform space; Formal ball; Weakly complete; Continuous poset; Weightable quasi-uniform structure.

# 1 Introduction

Throughout this paper the letters $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Q}$ and $\mathbb{N}$ denote the set of all real numbers, the set of all nonnegative real numbers, the set of all rational numbers and the set of all positive integer numbers, respectively.

In their celebrated paper [2], Edalat and Heckmann established nice and direct links between the theory of (complete) metric spaces and domain theory by means of the notion of a formal ball.

Let us recall that the set of formal (closed) balls of a metric space $(X, d)$ is simply the set $\mathbf{B}X := X \times \mathbb{R}^+$. Each element $(x, r)$ of $\mathbf{B}X$ is called a formal ball.

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Edalat and Heckmann showed that the pair $(\mathbf{B}X, \sqsubseteq)$ is a poset where

$$(x, r) \sqsubseteq (y, s) \Leftrightarrow d(x, y) \leq r - s,$$

for all $(x, r), (y, s) \in \mathbf{B}X$.

In fact, they proved, among other, the following important results for a metric space $(X, d)$ (see [2, Theorems 6 and 13, and Corollary 10]).

(A) $(\mathbf{B}X, \sqsubseteq)$ is a continuous poset.

(B) $(X, \tau_d)$ is homeomorphic to $\mathrm{Max}(\mathbf{B}X)$ when it is endowed with the restriction of the Scott topology of $(\mathbf{B}X, \sqsubseteq)$.

(C) $(X, d)$ is separable if and only if $(\mathbf{B}X, \sqsubseteq)$ is an $\omega$-continuous poset.

(D) $(X, d)$ is complete if and only if $(\mathbf{B}X, \sqsubseteq)$ is a dcpo.

Note that from (A) and (D) it follows that $(X, d)$ is complete if and only if $(\mathbf{B}X, \sqsubseteq)$ is a continuous domain.

Later on, Heckmann [7] improved result (B) showing that the Scott topology of $(\mathbf{B}X, \sqsubseteq)$ admits a compatible weightable quasi-metric $Q$ such that $(X, d)$ is isometric to $(\mathrm{Max}(\mathbf{B}X), Q\mid_{\mathrm{Max}(\mathbf{B}X)})$.

Edalat and Heckmann's approach, which is motivated in part by the work of Lawson on maximal point spaces [12], was continued and extended by several authors to ultrametric spaces, Banach spaces, hyperspaces, partial metric spaces, quasi-metric spaces, etc (see e.g. [1, 3, 8, 9, 14, 15, 16, 17, 18, 20]).

The purpose of this paper is to study the natural problem of constructing a suitable structure of poset when the formal balls are defined on a uniform space and then to generalize Edalat and Heckmann's constructions to the uniform setting. In fact, we shall obtain uniform versions of results (A)-(D) above. In particular, the uniform counterpart of (D) requires a weak notion of completeness which will be introduced here. Finally, the extension to our framework of Heckmann's quasi-metric construction will be also discussed. Our methods and techniques are inspired on the ones developed in [2].

# 2 Background

We start this section with several notions and facts on domain theory which will be useful later on. Our basic reference is [5].

A partially ordered set, or poset for short, is a (nonempty) set $X$ equipped with a (partial) order $\sqsubseteq$ . It will be denoted by $(X, \sqsubseteq)$ or simply by $X$ if no confusion arises.

A subset $D$ of a poset $X$ is directed provided that it is nonempty and every finite subset of $D$ has upper bound in $D$.

A poset $X$ is said to be directed complete, and is called a dcpo, if every directed subset of $X$ has a least upper bound.

An element $x$ of $X$ is said to be maximal if the condition $x \sqsubseteq y$ implies $x = y$. The set of all maximal points of $X$ will be denoted by $\mathrm{Max}((X, \sqsubseteq))$, or simply by $\mathrm{Max}(X)$ if no confusion arises.

Let $X$ be a poset and $x, y \in X$; we say that $x$ is way below $y$, in symbols $x \ll y$, if for each directed subset $D$ of $X$ having least upper bound $z$, the relation $y \sqsubseteq z$ implies the existence of some $u \in D$ with $x \sqsubseteq u$.

A poset $X$ is continuous if it has a basis $B$, where $B$ is said to be a basis for $X$ if for all $x \in X$, the set $\{b \in B : b \ll x\}$ is directed with least upper bound $x$.

A continuous poset which is also a dcpo is called a continuous domain or, simply, a domain.

A continuous poset having a countable basis is said to be an $\omega$-continuous poset.

The Scott topology $\sigma(X)$ of a continuous poset $(X, \sqsubseteq)$ is the topology that has as a base the collection of sets $\{y \in X : x \ll y\}$, $x \in X$.

If $D$ is a subset of $X$, we denote by $\sigma(X)_{|D}$ the restriction of $\sigma(X)$ to $D$.

Now we recall the notion of a uniform structure and of a uniform space as introduced by Gillman and Jerison [6, Chapter 15]

**Definition 1.** A uniform structure on a set $X$ is a nonempty family $\mathcal{D}$ of pseudometrics on $X$ such that:

(1) if $d_1, d_2 \in \mathcal{D}$, then $d_1 \vee d_2 \in \mathcal{D}$;

(2) if $e$ is a pseudometric, and if for every $\varepsilon > 0$, there exists $d \in \mathcal{D}$ and $\delta > 0$ such that $d(x, y) \leq \delta$ implies $e(x, y) \leq \varepsilon$ for all $x, y \in X$, then $e \in \mathcal{D}$.

The topology induced by a uniform structure $\mathcal{D}$ will be denoted by $\tau_{\mathcal{D}}$.

A uniform structure $\mathcal{D}$ is Hausdorff if whenever $x \neq y$, there exists $d \in \mathcal{D}$ with $d(x, y) > 0$.

A (Hausdorff) uniform space is a pair $(X, \mathcal{D})$ such that $X$ is a (nonempty) set and $\mathcal{D}$ is a (Hausdorff) uniform structure on $X$.

The intersection of any collection of uniform structures on $X$ is a uniform structure, so if $\mathcal{S}$ is any nonempty family of pseudometrics on $X$, then there exists a smallest uniform structure $\mathcal{D}$ containing $\mathcal{S}$. Then $\mathcal{S}$ is called a subbase of $\mathcal{D}$ (or, equivalently, a subbase of $(X, \mathcal{D})$), and we say that $\mathcal{D}$ is generated by $\mathcal{S}$. In fact, $\mathcal{D}$ can be constructed as follows: A pseudometric $d$ belongs to $\mathcal{D}$ if and only if for each $\varepsilon > 0$ there exist $d_1, \ldots, d_n \in \mathcal{S}$ and $\delta > 0$ such that if $d_i(x, y) \leq \delta$ for $i = 1, \ldots, n$, then $d(x, y) \leq \varepsilon$.

We conclude this section by recalling some notions on asymmetric structures which will be helpful in the last section of this paper. Our basic references are [4, 10, 19].

A quasi-pseudometric on a set $X$ is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$ : (i) $d(x, x) = 0$; (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

Each quasi-pseudometric $d$ on $X$ induces a topology $\tau_d$ on $X$ which has as a base the family of open $d$-balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Following the modern terminology by a quasi-metric on $X$ we mean a quasi-pseudometric $d$ on $X$ such that $d(x, y) = d(y, x) = 0$ if and only if $x = y$. Quasi-metrics were called semi-quasimetrics by Salbany [19].

According to Matthews [13] (see also [11]), a quasi-(pseudo)metric $d$ on a set $X$ is called weightable if there is a function $w : X \to \mathbb{R}^+$ such that $d(x, y) + w(x) = d(y, x) + w(y)$ for all $x, y \in X$. In this case we say that $w$ is a weight for $(X, d)$.

Finally, notice that the notion of a quasi-uniform structure (quasi-uniformity in [4, 19]) arises by replacing in Definition 1 "pseudometric" by "quasi-pseudometric".

According to [11], a quasi-uniform structure is said to be weightable if it is generated by a (nonempty) family of weightable quasi-pseudometrics.

# 3 The poset of formal balls for uniform structures and $\mathcal{S}$-weak completeness

In the sequel all uniform spaces are assumed to be Hausdorff.

As in the metric case, we denote by $\mathbf{B}X$ the set of formal balls of a uniform space $(X, \mathcal{D})$, i.e., $\mathbf{B}X := X \times \mathbb{R}^+$.

As a matter of mathematical interest, we wish to construct a consistent theory on formal balls for uniform spaces from which the metric case can be deduced directly. The natural way to achieve this aim is to work with the concept of subbase of uniform structure.

Let $\mathcal{S}$ be a subbase of a uniform space $(X, \mathcal{D})$. Define a binary relation $\sqsubseteq_{\mathcal{S}}$ on $\mathbf{B}X$ by

$$(x, r) \sqsubseteq_{\mathcal{S}} (y, s) \Leftrightarrow d(x, y) \leq r - s \text{ for all } d \in \mathcal{S}.$$

Then, it is immediate to show (compare [2, Proposition 1]) the following.

**Proposition 1.** $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ *is a poset. Furthermore* $\mathrm{Max}((\mathbf{B}X, \sqsubseteq_{\mathcal{S}})) = \{(x, 0) : x \in X\}$.

Now our first purpose is to extend, to the uniform setting, Theorem 6 of [2] that a metric space $(X, d)$ is complete if and only if $(\mathbf{B}X, \sqsubseteq)$ is a dcpo (see the result (D) in Section 1). To this end we introduce the following notions.

**Definition 2.** Let $\mathcal{S}$ be a subbase of a uniform space $(X, \mathcal{D})$.

(a) A net $(x_\alpha)_{\alpha \in \Lambda}$ in $X$ is called $\mathcal{S}$-equiCauchy if for each $\varepsilon > 0$ there exists $\alpha_0 \in \Lambda$ such that $d(x_\alpha, x_\beta) \leq \varepsilon$ for all $\alpha, \beta \geq \alpha_0$ and all $d \in \mathcal{S}$.

(b) A net $(x_\alpha)_{\alpha \in \Lambda}$ in $X$ $\mathcal{S}$-equiconverges to $x \in X$ if for each $\varepsilon > 0$ there exists $\alpha_0 \in \Lambda$ such that $d(x_\alpha, x) \leq \varepsilon$ for all $\alpha \geq \alpha_0$ and all $d \in \mathcal{S}$. In this case we say that $(x_\alpha)_{\alpha \in \Lambda}$ is $\mathcal{S}$-equiconvergent.

(c) $(X, \mathcal{D})$ is said to be $\mathcal{S}$-weakly complete if each $\mathcal{S}$-equiCauchy net is convergent.

**Remark 1.** (a) Taking $\mathcal{S} = \mathcal{D}$ in Definition 2, we have the notions of $\mathcal{D}$-equiCauchy net, $\mathcal{D}$-equiconvergence and $\mathcal{D}$-weakly completeness.

(b) Note also that each complete uniform space $(X, \mathcal{D})$ is $\mathcal{D}$-weakly complete.

(c) It is clear that, for any subbase $\mathcal{S}$ of $(X, \mathcal{D})$, a net in $X$ is $\mathcal{S}$-equiCauchy if and only if it is $\mathcal{D}$-equiCauchy, and thus $(X, \mathcal{D})$ is $\mathcal{D}$-weakly complete if and only if it is $\mathcal{S}$-weakly complete.

The following are illustrative examples of $\mathcal{D}$-weakly complete uniform spaces.

**Example 1.** Let $X$ be a Tychonoff topological space. For each $f \in C(X)$, let $d_f$ be the pseudometric defined by $d_f(x, y) = |f(x) - f(y)|$, and let $\mathcal{D}$ be the uniform structure generated by $\{d_f : f \in \mathcal{A}\}$, where $\mathcal{A} \subseteq C(X)$ is a family that separates points (given $x \neq y$ there exists $f \in \mathcal{A}$ with $f(x) \neq f(y)$) and is closed for product with reals. Then $(X, \mathcal{D})$ is $\mathcal{D}$-weakly complete. Indeed, let $(x_\alpha)_{\alpha \in \Lambda}$ be a $\mathcal{D}$-equiCauchy net. We prove that it is eventually constant.

If not, for each $\alpha_0 \in \Lambda$ there exists $\alpha, \beta \geq \alpha_0$ with $x_\alpha \neq x_\beta$. Let $\alpha_0 \in \Lambda$ be such that $d(x_\alpha, x_\beta) < 1/2$ for all $\alpha, \beta \geq \alpha_0$ and all $d \in \mathcal{D}$. Let $\alpha, \beta \geq \alpha_0$ with $x_\alpha \neq x_\beta$. Since $\mathcal{A}$ separates points, there exists $f \in \mathcal{A}$ such that $f(x_\alpha) \neq f(x_\beta)$. Since $\mathcal{A}$ is closed for products with reals, we can assume that $|f(x_\alpha) - f(x_\beta)| = 1$, but then $d_f(x_\alpha, x_\beta) = 1 > 1/2$, a contradiction. We conclude that $(X, \mathcal{D})$ is $\mathcal{D}$-weakly complete.

It follows that, in particular, the uniform structure $\mathcal{C}$ generated by $\{d_f : f \in C(X)\}$ is $\mathcal{C}$-weakly complete, the uniform structure $\mathcal{C}^*$ generated by $\{d_f : f \in C^*(X)\}$ is $\mathcal{C}^*$-weakly complete, and the fine uniform structure $\mathcal{FN}$ of $X$ is $\mathcal{FN}$-weakly complete.

**Example 2.** Let $(X, d)$ be a metric space and $\mathcal{A}$ an equicontinuous family of $C(X)$. Let $\mathcal{D}$ generated by $\{d_f : f \in \mathcal{A}\}$ and inducing the same topology than $d$. It easily follows that each Cauchy net is $\mathcal{D}$-equiCauchy. Therefore, if $(X, \mathcal{D})$ is $\mathcal{D}$-weakly complete then $(X, d)$ is complete. If, in addition, the constant function 1 belongs to $\mathcal{A}$, then $(X, \mathcal{D})$ is $\mathcal{D}$-weakly complete if and only if $(X, d)$ is complete. As a consequence, the uniform space $(\mathbb{Q}, \mathcal{D})$ is not $\mathcal{D}$-weakly complete, when $\mathbb{Q}$ is endowed with the restriction of the Euclidean metric, and $\mathcal{A}$ is the family of real valued contractive functions on $\mathbb{Q}$.

**Proposition 2.** *Let $\mathcal{S}$ be a subbase of a uniform space $(X, \mathcal{D})$ and $A$ a directed set in $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$. For each $(x, r) \in A$ define $x_{(x,r)} = x$. Then*

(1) *The net $(x_{(x,r)})_{(x,r)\in A}$ is $\mathcal{S}$-equiCauchy.*

(2) *$A$ has least upper bound, say $(z, t)$, if and only if $t = \inf\{r : (x, r) \in A\}$ and $(x_{(x,r)})_{(x,r)\in A}$ $\mathcal{S}$-equiconverges to $z$.*

**Proof.** We first show (1). Let $r_0 = \inf\{r : (x, r) \in A\}$. If there exists $(y, s) \in A$ such that $s = r_0$ then $(x, r) = (y, s)$ whenever $(y, s) \sqsubseteq_{\mathcal{S}} (x, r)$, and hence $(x_{(x,r)})_{(x,r)\in A}$ is $\mathcal{S}$-equiconvergent to $y$. Consequently $(x_{(x,r)})_{(x,r)\in A}$ is $\mathcal{S}$-equiCauchy.

So we can assume that $r_0 < r$ for each $(x, r) \in A$. Given $\varepsilon > 0$ there exists $(x_\varepsilon, r_\varepsilon) \in A$ such that $r_0 < r_\varepsilon < r_0 + \varepsilon/2$. Let $(x, r) \in A$ and $(y, s) \in A$ be such that $(x_\varepsilon, r_\varepsilon) \sqsubseteq_{\mathcal{S}} (x, r)$ and $(x_\varepsilon, r_\varepsilon) \sqsubseteq_{\mathcal{S}} (y, s)$. For each $d \in \mathcal{S}$, $d(x_{(x,r)}, x_{(y,s)}) = d(x, y) \leq d(x, x_\varepsilon) + d(x_\varepsilon, y) \leq r_\varepsilon - r + r_\varepsilon - s < \varepsilon$. It follows that $(x_{(x,r)})_{(x,r)\in A}$ is $\mathcal{S}$-equiCauchy.

Now we show (2). Suppose that $(z, t)$ is the least upper bound of $A$. First we prove that $t = r_0$. Indeed, since $(z, t)$ is an upper bound of $A$, it is clear that $t \leq r_0$. Suppose that there exists $\varepsilon > 0$ such that $t + \varepsilon \leq r_0$. Since $(x_{(x,r)})_{(x,r)\in A}$ is $\mathcal{S}$-equiCauchy, there exists $(x_\varepsilon, r_\varepsilon) \in A$ such that

$d(a,b) < \varepsilon/2$ for $(x_\varepsilon, r_\varepsilon) \sqsubseteq_{\mathcal{S}} (a,u)$, $(x_\varepsilon, r_\varepsilon) \sqsubseteq_{\mathcal{S}} (b,v)$ and $d \in \mathcal{S}$. It follows that $d(a, x_\varepsilon) < \varepsilon/2 \leq u - (t + \varepsilon/2)$ whenever $(x_\varepsilon, r_\varepsilon) \sqsubseteq_{\mathcal{S}} (a,u)$ and $d \in \mathcal{S}$, so $(a,u) \sqsubseteq_{\mathcal{S}} (x_\varepsilon, t + \varepsilon/2)$.

Let $(b,v) \in A$. Since $A$ is directed, there exists $(a,u) \in A$ with $(x_\varepsilon, r_\varepsilon) \sqsubseteq_{\mathcal{S}} (a,u)$ and $(b,v) \sqsubseteq_{\mathcal{S}} (a,u)$. Then $(b,v) \sqsubseteq_{\mathcal{S}} (a,u) \sqsubseteq_{\mathcal{S}} (x_\varepsilon, t+\varepsilon/2)$. Therefore $(x_\varepsilon, t + \varepsilon/2)$ is an upper bound of $A$. Since $(z,t)$ is the least upper bound of $A$, then $(z,t) \sqsubseteq (x_\varepsilon, t+\varepsilon/2)$, so, in particular, $t + \varepsilon/2 \leq t$, a contradiction. We conclude that $r_0 = t$.

Now we prove that $(x_{(x,r)})_{(x,r)\in A}$ $\mathcal{S}$-equiconverges to $z$. Given $\varepsilon > 0$ there exists $(x_\varepsilon, r_\varepsilon) \in A$ with $r_0 \leq r_\varepsilon < r_0 + \varepsilon$. If $(x_\varepsilon, r_\varepsilon) \sqsubseteq_{\mathcal{S}} (a,u)$, then $u \leq r_\varepsilon$ and hence $d(a,z) \leq u - r_0 \leq r_\varepsilon - r_0 < \varepsilon$ for each $d \in \mathcal{S}$. Therefore $(x_{(x,r)})_{(x,r)\in A}$ $\mathcal{S}$-equiconverges to $z$.

Conversely, suppose that $t = \inf\{r : (x,r) \in A\}$ and $(x_{(x,r)})_{(x,r)\in A}$ $\mathcal{S}$-equiconverges to $z$. First we prove that $(z,t)$ is an upper bound of $A$. Let $(x,r) \in A$ and $0 < \varepsilon < r - t$, then there exists $(x_\varepsilon, r_\varepsilon) \in A$ such that $t \leq r_\varepsilon < t + \varepsilon/2 \leq r$ and $(x,r) \sqsubseteq_{\mathcal{S}} (x_\varepsilon, r_\varepsilon)$. Thus $d(x,z) \leq d(x, x_\varepsilon) + d(x_\varepsilon, z) \leq (r - r_\varepsilon) + \varepsilon/2 \leq r - t + \varepsilon/2$ for all $d \in \mathcal{S}$. It follows that $(x,r) \sqsubseteq_{\mathcal{S}} (z,t)$. Finally, suppose that there exists $(y,s) \in \mathbf{B}X$ such that $(x,r) \sqsubseteq_{\mathcal{S}} (y,s)$ for all $(x,r) \in A$. Thus $s \leq t$. Let $\varepsilon > 0$, then there exists $(x,r) \in A$ with $t \leq r \leq t + \varepsilon/2$. Therefore $d(z,y) \leq d(z,x) + d(x,y) \leq (r-t) + (r-s) \leq \varepsilon/2 + (t + \varepsilon/2 - s) = t - s + \varepsilon$ for all $d \in \mathcal{S}$. It follows that $(z,t) \sqsubseteq_{\mathcal{S}} (y,s)$. Hence $(z,t)$ is the least upper bound of $A$.

**Theorem 1.** *Let $\mathcal{S}$ be a subbase of a uniform space $(X, \mathcal{D})$. Then, the following are equivalent:*

(1) *$(X, \mathcal{D})$ is $\mathcal{S}$-weakly complete.*

(2) *$(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is a dcpo.*

(3) *Each $\mathcal{S}$-equiCauchy net is $\mathcal{S}$-equiconvergent.*

**Proof.** (1) $\Rightarrow$ (2). Suppose that $(X, \mathcal{D})$ is $\mathcal{S}$-weakly complete, and let $A$ be a directed set in $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$. Let $r_0 = \inf\{r : (x,r) \in A\}$. If there exists $(y,s) \in A$ such that $s = r_0$ then $(x,r) = (y,s)$ whenever $(y,s) \sqsubseteq_{\mathcal{S}} (x,r)$, and hence $(y,s)$ is an upper bound of $A$. Moreover, if $(z,t)$ is another upper bound of $A$ then it is clear that $(y,s) \sqsubseteq_{\mathcal{S}} (z,t)$. Therefore $(y,s)$ is the least upper bound of $A$.

Now, suppose that $r_0 < r$ for each $(x,r) \in A$. By Proposition 2, $(x_{(x,r)})_{(x,r)\in A}$ is a $\mathcal{S}$-equiCauchy net. Since $(X, \mathcal{D})$ is $\mathcal{S}$-weakly complete, the net $(x_{(x,r)})_{(x,r)\in A}$ converges to a point $x_0 \in X$.

Again by Proposition 2, $(x_0, r_0)$ is the least upper bound of $A$, and hence $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is a dcpo.

(2) $\Rightarrow$ (3). Suppose that $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is a dcpo and let $(x_\alpha)_{\alpha\in\Lambda}$ be a $\mathcal{S}$-equiCauchy net in $(X, \mathcal{D})$. For each $\varepsilon > 0$ there exists $\alpha_\varepsilon \in \Lambda$ such that $d(x_\alpha, x_\beta) < \varepsilon/2$ for all $\alpha, \beta \geq \alpha_\varepsilon$ and all $d \in \mathcal{S}$. Put $A = \{(x_\alpha, \varepsilon) : \alpha \geq \alpha_\varepsilon\}$. Let us prove that $A$ is a directed set in $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$. Given $(x_\alpha, \varepsilon), (x_\beta, \delta) \in A$ with $\alpha \geq \alpha_\varepsilon$ and $\beta \geq \alpha_\delta$, choose $\mu > 0$ with $\mu < \min\{\varepsilon, \delta\}/2$, and $\gamma \in \Lambda$ with $\gamma \geq \alpha, \beta$. Then $d(x_\alpha, x_\gamma) < \varepsilon/2 < \varepsilon - \mu$ and $d(x_\beta, x_\gamma) < \delta/2 < \delta - \mu$ for all $d \in \mathcal{S}$. Hence $(x_\alpha, \varepsilon) \sqsubseteq_{\mathcal{S}} (x_\gamma, \mu)$ and $(x_\beta, \delta) \sqsubseteq_{\mathcal{S}} (x_\gamma, \mu)$, so $A$ is a directed set.

Since $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is a dcpo, $A$ has least upper bound $(y, s)$. Now we prove that the net $(x_\alpha)_{\alpha\in\Lambda}$ $\mathcal{S}$-equiconverges to $y$. Let $\varepsilon > 0$ and $\alpha \geq \alpha_\varepsilon$. Since $(x_\alpha, \varepsilon) \in A$ and $(y, s)$ is an upper bound of $A$, it follows that $(x_\alpha, \varepsilon) \sqsubseteq_{\mathcal{S}} (y, s)$, so $d(x_\alpha, y) \leq \varepsilon - s \leq \varepsilon$ for all $d \in \mathcal{S}$.

(3) $\Rightarrow$ (1). Obviously every $\mathcal{S}$-equiconvergent net is convergent.

Taking $\mathcal{S} = \{d\}$, where $d$ is a metric on $X$, we immediately deduce from Theorem 1 the following.

**Corollary** [2, Theorem 6]**.** *A metric space $(X, d)$ is complete if and only if $(\mathbf{B}X, \sqsubseteq)$ is a dcpo.*

# 4 Continuity of $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$

In this section we shall show that for any subbase $\mathcal{S}$ of a uniform space $(X, \mathcal{D})$ the poset $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is continuous.

To this purpose, the following characterization of the way-below relation in $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ will be crucial (compare [2, Proposition 7]).

**Proposition 3.** *Let $\mathcal{S}$ be a subbase of a uniform space $(X, \mathcal{D})$. Then $(x, r) \ll (y, s)$ in $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ if and only if there exists $\varepsilon > 0$ such that $d(x, y) \leq r - s - \varepsilon$ for all $d \in \mathcal{S}$.*

**Proof.** Suppose that $(x, r) \ll (y, s)$. Let $A = \{(y, s + \varepsilon) : \varepsilon > 0\}$. We show that $A$ is a directed set in $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ and $(y, s)$ is the least upper bound of $A$. Indeed if $(y, s+\varepsilon) \sqsubseteq_{\mathcal{S}} (z, t)$ for all $\varepsilon > 0$, then $d(y, z) \leq s + \varepsilon - t$ for all $\varepsilon > 0$ and all $d \in \mathcal{S}$, so $d(y, z) \leq s - t$ for all $d \in \mathcal{S}$ and thus $(y, s) \sqsubseteq_{\mathcal{S}} (z, t)$. Since $(x, r) \ll (y, s)$, there exists $\varepsilon > 0$ with $(x, r) \sqsubseteq_{\mathcal{S}} (y, s + \varepsilon)$, so $d(x, y) \leq r - s - \varepsilon$ for all $d \in \mathcal{S}$.

Conversely, suppose that there exists $\varepsilon > 0$ such that $d(x, y) \leq r - s - \varepsilon$ for all $d \in \mathcal{S}$. Let $A$ be a directed set with least upper bound $(z, t)$ such that $(y, s) \sqsubseteq_{\mathcal{S}} (z, t)$. By Proposition 2, $t = \inf\{r : (x, r) \in A\}$, and $(x_{(x,r)})_{(x,r)\in A}$ $\mathcal{S}$-equiconverges to $z$. If there exists $(a, u) \in A$ such that $u = t$ then $(b, v) = (a, u)$ whenever $(a, u) \sqsubseteq_{\mathcal{S}} (b, v)$. It easily follows that $(x, r) \ll (y, s)$, so we assume that $t < u$ for all $(a, u) \in A$. In that case there exists $(x_\varepsilon, r_\varepsilon) \in A$ such that $t \leq r_\varepsilon < t + \varepsilon/2$ and $d(a, z) < \varepsilon/2$ whenever $(x_\varepsilon, r_\varepsilon) \sqsubseteq_{\mathcal{S}} (a, u)$ and $d \in \mathcal{S}$. If $(a, u) = (x_\varepsilon, r_\varepsilon)$, then

$$\begin{aligned} d(x, a) &\leq d(x, y) + d(y, z) + d(z, a) \leq (r - s - \varepsilon) + (s - t) + \varepsilon/2 \\ &= r - (t + \varepsilon/2) < r - r_\varepsilon \leq r - u, \end{aligned}$$

for all $d \in \mathcal{D}$. Therefore $(x, r) \sqsubseteq_{\mathcal{S}} (a, u)$ and hence $(x, r) \ll (y, s)$.

**Theorem 2.** *Let $\mathcal{S}$ be a subbase of a uniform space $(X, \mathcal{D})$. Then $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is a continuous poset.*

**Proof.** We shall show that $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is a basis for $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$. Indeed, given $(x, r) \in \mathbf{B}X$ it is easy to check that $A = \{(x, r+\varepsilon) : \varepsilon > 0\}$ is a directed set. Moreover, $(x, r)$ is the least upper bound of $A$ by Proposition 2, and it is clear by Proposition 3 that $(x, r + \varepsilon) \ll (x, r)$. Therefore $A \subseteq B$ where $B = \{(y, s) \in \mathbf{B}X : (y, s) \ll (x, r)\}$. Observe that $r = \inf\{s : (y, s) \in B\}$.

Now given $(a, u) \ll (x, r)$ and $(b, v) \ll (x, r)$, by Proposition 3 there exists $\varepsilon > 0$ such that $d(a, x) \leq u - r - \varepsilon = u - (r + \varepsilon)$ and $d(b, x) \leq v - r - \varepsilon = v - (r + \varepsilon)$ for all $d \in \mathcal{S}$. So $(a, u) \sqsubseteq_{\mathcal{S}} (x, r + \varepsilon)$ and $(b, v) \sqsubseteq_{\mathcal{S}} (x, r + \varepsilon)$, and hence $B$ is a directed set. On the other hand, given $\varepsilon > 0$, if $(a, u) \in B$ satisfies that $(x, r+\varepsilon) \sqsubseteq_{\mathcal{S}} (a, u)$, then $u \leq r+\varepsilon$ and thus $d(x, a) \leq u-r \leq \varepsilon$ for all $d \in \mathcal{S}$, so the net $(x_{(a,u)})_{(a,u)\in B}$ is $\mathcal{S}$-equiconvergent to $x$. By Proposition 2, $(x, r)$ is the least upper bound of $B$.

We conclude that $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is a basis for $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$, and thus the poset $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is continuous.

Proposition 3 and Theorem 2 provide generalizations to the uniform framework of Proposition 7 and the first part of Corollary 10 of [2], respectively. Moreover, from Theorems 1 and 2 we deduce the following.

**Corollary.** *Let $\mathcal{S}$ be a subbase of a uniform space $(X, \mathcal{D})$. Then $(X, \mathcal{D})$ is $\mathcal{S}$-weakly complete if and only if $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is a continuous domain.*

**Definition 3.** Let $\mathcal{S}$ be a subbase of a uniform space $(X, \mathcal{D})$. We say that $(X, \mathcal{D})$ is $\mathcal{S}$-equiseparable if there exists a countable $\mathcal{S}$-equidense subset $D$ of $X$, where $\mathcal{S}$-equidense means that for each $x \in X$ and $\varepsilon > 0$ there exists $y \in D$ such that $d(x, y) < \varepsilon$ for all $d \in \mathcal{S}$.

We finish this section by characterizing when $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is $\omega$-continuous. This characterization extends the corresponding result to metric spaces obtained by Edalat and Heckmann in [2, Corollary 10].

**Theorem 3.** *Let $\mathcal{S}$ be a subbase of a uniform space $(X, \mathcal{D})$. Then $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is an $\omega$-continuous poset if and only if $(X, \mathcal{D})$ is $\mathcal{S}$-equiseparable.*

**Proof.** Suppose that $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$ is $\omega$-continuous. Then there exists a countable basis $B$ of $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$. This means that for each $(x, r) \in \mathbf{B}X$ the set $A = \{(y, s) \in B : (y, s) \ll (x, r)\}$ is directed with least upper bound $(x, r)$. By Proposition 2, $(x_{(y,s)})_{(y,s) \in A}$ $\mathcal{S}$-equiconverges to $x$ and $r = \inf\{s : (y, s) \in A\}$. It follows that $D = \{y \in X : (y, s) \in B\}$ is a countable $\mathcal{S}$-equidense subset of $\mathbf{B}X$.

Conversely, let $D$ be a countable $\mathcal{S}$-equidense subset of $\mathbf{B}X$. Let $B = \{(y, q) \in \mathbf{B}X : y \in D, q \in \mathbb{Q}, q > 0\}$. We prove that $B$ is a (countable) basis for $(\mathbf{B}X, \sqsubseteq_{\mathcal{S}})$.

Let $(x, r) \in \mathbf{B}X$ and $A = \{(y, q) \in B : (y, q) \ll (x, r)\}$. In order to prove that $A$ is a directed set with least upper bound $(x, r)$, we prove that for each $\varepsilon > 0$ there exists $(y, q) \in A$ such that $(x, r + \varepsilon) \sqsubseteq_{\mathcal{S}} (y, q)$.

Let $\varepsilon > 0$ and let $q \in \mathbb{Q}$ with $r + \varepsilon < q < r + 2\varepsilon$. Take $y \in D$ with $d(y, x) \leq q - (r + \varepsilon)$ for all $d \in \mathcal{S}$. It follows that $(x, r + \varepsilon) \sqsubseteq_{\mathcal{S}} (y, q)$ and $(y, q) \ll (x, r)$ (and hence $(y, q) \in A$).

# 5 The Scott topology and weightable quasi-uniform structures

Edalat and Heckmann proved in [2, Theorem 13] (see the result (B) in Section 1) that for any metric space $(X, d)$, the mapping $i : X \to \mathbf{B}X$ given by $i(x) = (x, 0)$ for all $x \in X$, is a homeomorphism between $(X, \tau_d)$ and $(\mathrm{Max}(\mathbf{B}X), \sigma(\mathbf{B}X)_{|\mathrm{Max}(\mathbf{B}X)})$.

Heckmann improved this result in [7] (see also [17, Section 4]), showing that for any metric space $(X, d)$ the function $Q : \mathbf{B}X \times \mathbf{B}X \to \mathbb{R}^+$ defined as

$$Q((x, r), (y, s)) = \max\{d(x, y), |r - s|\} + s - r,$$

for all $(x,r),(y,s) \in \mathbf{B}X$, is a weightable quasi-metric on $\mathbf{B}X$ that induces the Scott topology on $\mathbf{B}X$ and such that $(X,d)$ and $(\mathrm{Max}(\mathbf{B}X), Q_{|\mathrm{Max}(\mathbf{B}X)})$ are isometric via the mapping $i$ defined above (actually, Heckmann's construction was accomplished in the realm of partial metric spaces in the sense of Matthews [13], a class of spaces "equivalent" to the class of weightable quasi-metric spaces as proved in [13, Theorems 4.1 and 4.2]).

Next we extend Heckmann's construction to uniform spaces.

Indeed, let $(X,\mathcal{D})$ be a uniform space. For each $d \in \mathcal{D}$ define a function $Q_d : \mathbf{B}X \times \mathbf{B}X \to \mathbb{R}^+$ by

$$Q_d((x,r),(y,s)) = \max\{d(x,y), |r-s|\} + s - r,$$

for all $(x,r),(y,s) \in \mathbf{B}X$.

It is routine to check that $Q_d$ is a weightable quasi-pseudometric with weight $w_d$ given by $w_d((x,r)) = 2r$ for all $(x,r) \in \mathbf{B}X$.

Hence the quasi-uniform structure $\mathcal{U}_{\mathbf{B}\mathcal{D}}$ on $\mathbf{B}X$ generated by $\{Q_d : d \in \mathcal{D}\}$ is weightable.

Furthermore, from the fact that $Q_d((x,0),(y,0)) = d(x,y)$ for all $x,y \in X$ and all $d \in \mathcal{D}$, we deduce that the mapping $i : X \to \mathbf{B}X$ defined by $i(x) = (x,0)$ for all $x \in X$, is a uniform isomorphism between $(X,\mathcal{D})$ and $(\mathrm{Max}((\mathbf{B}X, \sqsubseteq_{\mathcal{D}})), \mathcal{U}_{\mathbf{B}\mathcal{D}|\mathrm{Max}(\mathbf{B}X)})$.

Finally, we have $\tau_{\mathcal{U}_{\mathbf{B}\mathcal{D}}} \subseteq \sigma(\mathbf{B}X)$ as an easy consequence of Proposition 3.

In this context it seems interesting to discuss the case that $\sup_{d\in\mathcal{D}} d(x,y) < \infty$ for all $x,y \in X$. To this end, we first establish the following well-known fact.

**Lemma 1.** *Let $(X,\mathcal{D})$ be a uniform space such that for each $x,y \in X$, $\sup_{d\in\mathcal{D}} d(x,y) < \infty$. Then the function $D : X \times X \to \mathbb{R}^+$ given by*

$$D(x,y) = \sup_{d\in\mathcal{D}} d(x,y)$$

*is a metric on $X$ such that $\tau_{\mathcal{D}} \subseteq \tau_D$.*

With the help of Lemma 1 we obtain the following.

**Proposition 4.** *Let $(X,\mathcal{D})$ be a uniform space such that for each $x,y \in X$, $\sup_{d\in\mathcal{D}} d(x,y) < \infty$. Then, the function $Q_{\mathcal{D}} : \mathbf{B}X \times \mathbf{B}X \to \mathbb{R}^+$ given by*

$$Q_{\mathcal{D}}((x,r),(y,s)) = \max\left\{\sup_{d\in\mathcal{D}} d(x,y), |r-s|\right\} + s - r,$$

*is a weightable quasi-metric on* $\mathbf{B}X$ *such that* $\tau_{Q_{\mathcal{D}}} = \sigma(\mathbf{B}X)$*. Moreover,* $\tau_{\mathcal{D}} \subseteq \sigma(\mathbf{B}X)_{|\mathrm{Max}(\mathbf{B}X)}$*.*

**Proof.** Since the proof of the first part is almost obvious we only prove that $\sigma(\mathbf{B}X) = \tau_{Q_{\mathcal{D}}}$.

Let $(x, r) \in \mathbf{B}X$. Suppose $(z, t) \ll (x, r)$. Then there is $\varepsilon > 0$ such that $d(z, x) \leq t - r - \varepsilon$ for all $d \in \mathcal{D}$. Thus, it easily follows that

$$B_Q((x, r), \varepsilon/2) \subseteq \{(y, s) : (z, t) \ll (y, s)\}.$$

Therefore $\sigma(\mathbf{B}X) \subseteq \tau_{Q_{\mathcal{D}}}$. Conversely, we have

$$(x, r) \in \{(y, s) : (x, r + \varepsilon) \ll (y, s)\} \subseteq B_Q((x, r), 2\varepsilon),$$

for all $\varepsilon > 0$ (note, in particular, that if $r < s$ and $\sup_{d \in \mathcal{D}} d(x, y) \leq |r - s|$, then from $d(x, y) \leq r + \varepsilon - s - \delta$, it follows $s + \delta \leq r + \varepsilon$). Hence $\tau_{Q_{\mathcal{D}}} \subseteq \sigma(\mathbf{B}X)$.

Finally, since $Q_{\mathcal{D}}((x, 0), (y, 0)) = D(x, y)$, and $\tau_{Q_{\mathcal{D}}|\mathrm{Max}(\mathbf{B}X)} = \sigma(\mathbf{B}X)_{|\mathrm{Max}(\mathbf{B}X)}$, we deduce from Lemma 1 that $\tau_{\mathcal{D}} \subseteq \sigma(\mathbf{B}X)_{|\mathrm{Max}(\mathbf{B}X)}$.

From Lemma 1 and Proposition 4, a natural question to ask is whether the inclusion $\tau_{\mathcal{D}} \subseteq \sigma(\mathbf{B}X)_{|\mathrm{Max}(\mathbf{B}X)}$ can be a proper inclusion. To illustrate this situation we conclude the paper with two examples. In the first one, the equality $\tau_{\mathcal{D}} = \sigma(\mathbf{B}X)_{|\mathrm{Max}(\mathbf{B}X)}$ holds but in the second one it fails to be true in a dramatic way. In fact, the topology $\tau_{\mathcal{D}}$ has no isolated points but $\sigma(\mathbf{B}X)_{|\mathrm{Max}(\mathbf{B}X)}$ is the discrete topology.

**Example 3.** Given two rational numbers $r_1, r_2$ with $r_1 < r_2$, let $f_{r_1,r_2}$ be the continuous real-valued function defined as

$$f_{r_1,r_2}(x) = \begin{cases} r_1 & \text{if } x \leq r_1 \\ x & \text{if } r_1 < x < r_2 \\ r_2 & \text{if } r_2 \leq x \end{cases}$$

For each such function, we define a pseudometric $d_{r_1,r_2}$ by setting

$$d_{r_1,r_2}(x, y) = |f_{r_1,r_2}(x) - f_{r_1,r_2}(y)|,$$

for all $x, y \in \mathbb{R}$. Let $\mathcal{D}$ denote the uniform structure on $\mathbb{R}$ which has as a subbase the family

$$\mathcal{S} = \{d_{r_1,r_2} \,:\, r_1, r_2 \in \mathbb{Q} \text{ with } r_1 < r_2\}\,.$$

Notice that, since $\mathcal{S}$ is countable, the uniform structure $\mathcal{D}$ is metrizable. The symbol $\tau_e$ stands for the Euclidean topology on $\mathbb{R}$.

Next we show that $\tau_{\mathcal{D}}$ agrees with $\tau_e$. Indeed, by [6, Theorem 15.6], we have that $\tau_{\mathcal{D}}$ is coarser than $\tau_e$. To see that $\tau_{\mathcal{D}}$ is finer than $\tau_e$, suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ $\tau_{\mathcal{D}}$-converges to some $x$. Given $\varepsilon > 0$, choose rationals numbers $r_1, r_2$ with $r_1 < x < r_2$ and $]r_1, r_2[\subset]x - \varepsilon, x + \varepsilon[$. Then, there is $n_0 \in \mathbb{N}$ such that

$$d_{r_1,r_2}(x_n, x) = |f_{r_1,r_2}(x_n) - f_{r_1,r_2}(x)| < r_2 - r_1,$$

for all $n \geq n_0$, that is,

$$d_{r_1,r_2}(x_n, x) = |x_n - x| < r_2 - r_1 < 2\varepsilon,$$

for all $n \geq n_0$. Thus, the sequence $(x_n)_{n\in\mathbb{N}}$ $\tau_e$-converges to $x$. Therefore $\tau_{\mathcal{D}} = \tau_e$.

Taking into account the definition of $f_{r_1,r_2}$, it is routine to check that $|f_{r_1,r_2}(x) - f_{r_1,r_2}(y)|$ takes its maximum value when $r_1 \leq x$ and $y \leq r_2$ and that this value is $|x - y|$. This simple observation yields that, for all $x, y \in \mathbb{R}$,

$$\sup\{|f_{r_1,r_2}(x) - f_{r_1,r_2}(y)| \,:\, r_1, r_2 \in \mathbb{Q} \text{ with } r_1 < r_2\} = |x - y|\,.$$

Consequently, the topology induced by the metric $D$ given as

$$D(x, y) = \sup\{|f_{r_1,r_2}(x) - f_{r_1,r_2}(y)| \,:\, r_1, r_2 \in \mathbb{Q} \text{ with } r_1 < r_2\}\,,$$

for all $x, y \in \mathbb{R}$, coincides with the topology $\tau_e$. We deduce that $\tau_{\mathcal{D}} = \sigma(\mathbf{B}X)_{|\mathrm{Max}(\mathbf{B}X)}$, where $X := (\mathbb{R}, \mathcal{D})$.

**Example 4.** For all $x, y \in ]0, 1[$ with $x < y$, consider the function $f_{x,y}$ defined as

$$f_{x.y}(z) = \begin{cases} x & \text{if } z < x \\ y & \text{if } x \leq z < y \\ y + 1 & \text{if } y \leq z \end{cases}$$

Each such function has associated a pseudometric $d_{x,y}$ defined by letting

$$d_{x,y}(w, s) = |f_{x,y}(w) - f_{x,y}(s)|\,,$$

for all $w, s \in ]0,1[$. It is routine to check that the family $\{d_{x,y} : x, y \in ]0,1[\}$ is a subbase for a uniform structure $\mathcal{D}$ on $]0,1[$.

We show that the topology $\tau_{\mathcal{D}}$ is the Sorgenfrey topology on $]0,1[$. Indeed, by the equality

$$[a, a+\varepsilon[= \{x : |f_{a,a+\varepsilon}(x) - f_{a,a+\varepsilon}(a)| < \varepsilon\},$$

for all $a \in ]0,1[$ and all $\varepsilon$ with $0 < \varepsilon < 1/2$ and $a + \varepsilon < 1$, we deduce that the Sorgenfrey topology is coarser than $\tau_{\mathcal{D}}$. To see the converse, it suffices to notice that, given $a \in ]0,1[$, a basic $\tau_{\mathcal{D}}$-neighborhood of $a$, $\{z : |f_{x,y}(z) - f_{x,y}(a)| < \varepsilon\}$, contains the set $[a, a+\delta[$ where $\delta$ can be chosen in the following way: (1) $\delta < x - a$ if $a < x$, (2) $\delta < y - a$ if $x \leq a < y$, and (3) $a + \delta < 1$ if $y \leq a$.

Observe that, if $w, s \in ]0,1[$ satisfy $w < s$, then

$$\sup\{|f_{x,y}(w) - f_{x.y}(s)| : x < y\} \geq |f_{w,s}(w) - f_{w,s}(s)| = 1.$$

Thus, we have that the topology induced on $]0,1[$ by the metric $D$ given as

$$D(w, s) = \sup\{|f_{x,y}(w) - f_{x.y}(s)| : x, y \in ]0,1[ \text{ with } x < y\},$$

for all $w, s \in ]0,1[$, is the discrete topology. Consequently $\tau_{\mathcal{D}} \subsetneq \sigma(\mathbf{B}X)_{|\mathrm{Max}(\mathbf{B}X)}$, where $X := (]0,1[, \mathcal{D})$.

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Salvador Romaguera
Instituto Universitario de Matemática Pura y Aplicada,
Universitat Politècnica de València, 46022 Valencia, spain
E-mail: sromague@mat.upv.es

Miguel A. Sánchez-Granero
Department of Mathematics, Faculty of Science,
Universidad de Almería , 04120 Almería, Spain
E-mail: misanche@ual.es

Manuel Sanchis
Institut Universitari de Matemàtiques i Aplicacions de Castelló (IMAC),
Universitat Jaume I,Campus del RiuSec. s/n, 12071 Castelló, Spain
E-mail: sanchis@mat.uji.es