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Additional Information

Steffensen type methods for solving nonlinear equations ⋆

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Abstract

In the present paper, by approximating the derivatives in the well known fourth order Ostrowski's method and in an sixth order improved Ostrowski's method by central-difference quotients, we obtain new free from derivatives modifications of these methods. We prove the important fact that the obtained methods preserve their convergence orders four and six, respectively, without calculating any derivatives. Finally, numerical tests confirm the theoretical results and allow us to compare these variants with the corresponding methods that make use of derivatives and with the classical Newton's method.

Key words: Central approximation, Steffensen's method, derivative free method, convergence order, efficiency index.

1 Introduction

In the last years, a lot of papers have developed the idea of removing derivatives from the iteration function in order to avoid defining new functions as

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the first or second derivative, and calculate iterates only by using the function that describes the problem, obviously trying to preserve the convergence order. In this sense, in the literature of nonlinear equations can be frequently found the expression "derivative free", referring in most cases to the second derivative (see [1–3]). The interest of these methods is to be applied on nonlinear equations $f(x) = 0$, when there are many problems in order to obtain and evaluate the derivatives involved.

The procedure of removing the derivatives usually increases the number of functional evaluations per iteration. Commonly in the literature the efficiency of an iterative method is measured by the *efficiency index* defined as $p^{1/d}$, where p is the order of convergence and d is the total number of functional evaluations per step.

There are different methods for computing a zero α of a nonlinear equation $f(x) = 0$, the most known of these methods is the classical Newton's method (NM)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, \ldots, \tag{1}$$

that, under certain conditions, has quadratic convergence.

Newton's method has been modified in a number of ways to avoid the use of derivatives without affecting the order of convergence. For example, replacing in (1) the derivative by the forward approximation

$$f'(x_n) \approx \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)},$$

Newton's method becomes

$$x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)},$$

which is called Steffensen's method (SM). This method has still quadratic convergence, in spite of being derivative free and using only two functional evaluations per step.

When an iterative method is free from first derivative, authors refer to it as a "Steffensen like method". Some of these methods use forward differences for approximating the derivatives. For example, in [4] Jain proposed a Steffensen-secant method (JM) deformed from Newton-secant. This method only uses three functional evaluations per step and gets third order convergence. Other Steffensen like method of third order, based on the homotopy perturbation theory, is presented by Feng and He in [5], (FM). It uses three functional evaluations per step.

By applying forward-difference approximation to Weerakoon-Fernando's formula [6], Zheng et al. derived in [7] a family of Steffensen like methods (ZM) which have order of convergence three and use four functional evaluations per iteration.

In order to control the approximation of the derivative and the stability of the iteration, a Steffensen type method, with quadratic convergence and two functional evaluations per step, has been proposed by Amat and Busquier in [8], (AM). The recent paper [9] has extended it to Banach spaces, obtaining its semilocal and local convergence theorems.

If we try to use forward-difference approximation, with the fourth order Ostrowski's method [10]:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}, \end{aligned} \tag{2}$$

the order of convergence of the new method goes down to three. For this reason, we have used central-difference in (2), obtaining a variant of Ostrowski's method that preserves the convergence order four and is derivative free. Recently, Dehghan and Hajarian [11] proposed a free derivative iterative method (DM) by replacing the forward-difference approximation in Steffensen's method by the central-difference approximation. However, it is still a method of third order and requires four functional evaluations per iteration.

In the same way, we consider the sixth order method proposed by M. Grau et al. in [12] as an improvement to Ostrowski root-finding method, which iteration scheme is:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{y_n - x_n}{2f(y_n) - f(x_n)} f(y_n), \\ x_{n+1} &= z_n - \frac{y_n - x_n}{2f(y_n) - f(x_n)} f(z_n). \end{aligned} \tag{3}$$

We are going to replace in (3) the first derivative by a symmetric-difference in order to obtain a new method that preserves the sixth convergence order and is derivative-free.

The rest of this paper is organized as follows. In Section 2, we describe our free from derivatives methods as a variants of Ostrowski's method and the improved Ostrowski's method, respectively. In Section 3, we establish the convergence order of these methods. Finally, in Section 4 different numerical tests confirm the theoretical results and allow us to compare these variants with the original methods (which make use of derivatives) and also with Newton's method.

2 Description of the methods

By using a symmetric difference quotient

$$f'(x_n) \simeq \frac{f(x_n + f(x_n)) - f(x_n - f(x_n))}{2f(x_n)},$$

to approximate the derivative in the fourth order Ostrowski's method (2), we obtain a new method free from derivatives, that we call *modified Ostrowski's method free from derivatives* (ODF):

$$\begin{aligned} y_n &= x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}, \\ x_{n+1} &= x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} \frac{f(y_n) - f(x_n)}{2f(y_n) - f(x_n)}. \end{aligned} \tag{4}$$

As we have said, in [12], Grau et al. proposed an improvement of Ostrowski's method (3) and proved that it has sixth order of convergence. By approximating the derivative by central-difference we obtain a new method free from derivatives, that we call *improved Ostrowski's method free from derivatives* (IODF):

$$\begin{aligned} y_n &= x_n - \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))}, \\ z_n &= y_n - \frac{y_n - x_n}{2f(y_n) - f(x_n)} f(y_n), \\ x_{n+1} &= z_n - \frac{y_n - x_n}{2f(y_n) - f(x_n)} f(z_n). \end{aligned} \tag{5}$$

In the next section, we are going to prove that the methods ODF and IODF have order of convergence four and six, respectively.

3 Analysis of Convergence

In this section we analyze the order of convergence of the methods described previously.

Theorem 1 *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ in an open interval I. If x_0 is sufficiently close to α, then the modified Ostrowski's method free from derivatives defined by (4) has order of convergence four and satisfies the error equation*

$$e_{n+1} = -c_2 \left(-\frac{c_2^2}{c_1^3} + c_3 + \frac{c_3}{c_1^2} \right) e_n^4 + O(e_n^5).$$

Proof: Let $e_n = x_n - \alpha$. The Taylor series of $f(x_n)$ about α is:

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5), \qquad (6)$$

where $c_k = \dfrac{f^{(k)}(\alpha)}{k!}, k = 1, 2, \ldots$
Computing the Taylor series of $f(x_n + f(x_n))$ and substituting $f(x_n)$ by (6) we have

$$\begin{aligned} & f(x_n + f(x_n)) = \\ & = c_1(1+c_1)e_n + (c_1c_2 + (1+c_1)^2 c_2)e_n^2 + (2(1+c_1)c_2^2 + c_1c_3 + (1+c_1)^3 c_3)e_n^3 + \\ & +(3(1+c_1)^2 c_2c_3 + c_2(c_2^2 + 2(1+c_1)c_3) + c_1c_4 + (1+c_1)^4 c_4)e_n^4 + O(e_n^5). \qquad (7) \end{aligned}$$

Analogously, the Taylor series of $f(x_n - f(x_n))$ is:

$$\begin{aligned} & f(x_n - f(x_n)) = \\ & = (1-c_1)c_1 e_n + ((1-c_1)^2 c_2 - c_1c_2)e_n^2 + (-2(1-c_1)c_2^2 + (1-c_1)^3 c_3 - c_1c_3)e_n^3 + \\ & +(-3(1-c_1)^2 c_2c_3 + c_2(c_2^2 - 2(1-c_1)c_3) + (1-c_1)^4 c_4 - c_1c_4)e_n^4 + O(e_n^5). \qquad (8) \end{aligned}$$

Then, the quotient that appears in the expression of y_n in (4) is:

$$\begin{aligned} & \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} = e_n - \frac{c_2 e_n^2}{c_1} + \frac{(2c_2^2 - c_1(2+c_1^2)c_3)e_n^3}{c_1^2} + \\ & + \left(-\frac{4c_2^3}{c_1^3 + c_2c_3} + \frac{7c_2}{c_3}c_1^2 - \frac{3c_4}{c_1 - 4c_1c_4} \right) e_n^4 + O(e_n^5). \qquad (9) \end{aligned}$$

We obtain $y_n - \alpha$ taking into account (9)

$$
\begin{aligned}
y_n - \alpha = e_n - & \frac{2f(x_n)^2}{f(x_n + f(x_n)) - f(x_n - f(x_n))} = \\
= & \frac{c_2 e_n^2}{c_1} - \frac{(2c_2^2 - c_1(2 + c_1^2)c_3)e_n^3}{c_1^2} + \\
+ & \left(\frac{4c_2^3}{c_1^3 - c_2 c_3} - \frac{7c_2c_3}{c_1^2} + \frac{3c_4}{c_1 + 4c_1c_4}\right) e_n^4 + O(e_n^5).
\end{aligned} \tag{10}
$$

Now, substituting (10) in the Taylor series of $f(y_n)$, we have

$$
\begin{aligned}
f(y_n) = c_2 e_n^2 - & \frac{(2c_2^2 - c_1(2 + c_1^2)c_3)e_n^3}{c_1} + \\
+ & \left(\frac{c_2^3}{c_1^2} + c_1\left(\frac{4c_2^3}{c_1^3 - c_2c_3} - \frac{7c_2c_3}{c_1^2} + \frac{3c_4}{c_1 + 4c_1c_4}\right)\right) e_n^4 + O(e_n^5).
\end{aligned} \tag{11}
$$

From (6) and (11) we obtain

$$
\begin{aligned}
f(y_n) - f(x_n) = -c_1 e_n + & \left(-c_3 - \frac{2c_2^2 - c_1(2 + c_1^2)c_3}{c_1}\right) e_n^3 + \\
+ \left(\frac{c_2^3}{c_1^2 - c_4 + c_1}\left(\frac{4c_2^3}{c_1^3 - c_2c_3} - \frac{7c_2c_3}{c_1^2} + \frac{3c_4}{c_1 + 4c_1c_4}\right)\right) & e_n^4 + O(e_n^5)
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
2f(y_n) - f(x_n) = -c_1 e_n + c_2 e_n^2 + & \left(-c_3 - \frac{2(2c_2^2 - c_1(2 + c_1^2)c_3)}{c_1}\right) e_n^3 + \\
+ \left(-c_4 + 2\left(\frac{c_2^3}{c_1^2} + c_1\left(\frac{4c_2^3}{c_1^3} - c_2c_3 - \frac{7c_2c_3}{c_1^2} + \frac{3c_4}{c_1} + 4c_1c_4\right)\right)\right) & e_n^4 + O(e_n^5).
\end{aligned}
$$

Taking into account (9), (12) and the last expression, we finally obtain

$$
e_{n+1} = -c_2\left(-\frac{c_2^2}{c_1^3} + c_3 + \frac{c_3}{c_1^2}\right) e_n^4 + O(e_n^5).
$$

This proves that the method is of fourth order. □

Theorem 2 *Let $\alpha \in I$ be a simple zero of sufficiently differentiable function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ in an open interval I. If x_0 is sufficiently close to α, then the improved Ostrowsky's method free from derivatives defined by (5) has order of convergence six and satisfies the error equation*

$$
e_{n+1} = \frac{(-2c_2^2 + c_1(1 + c_1^2)c_3)(-c_2^3 + c_1(1 + c_1^2)c_2c_3)}{c_1^5} e_n^6 + O(e_n^7).
$$

Proof: Let $e_n = x_n - \alpha$. The Taylor series of $f(x_n)$ about α is:

$$f(x_n) = c_1e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7), \tag{13}$$

where $c_k = \dfrac{f^{(k)}(\alpha)}{k!}, k = 1, 2, \ldots$
Computing the Taylor series of $f(x_n + f(x_n))$ and substituting $f(x_n)$ by (13) we have

$$\begin{aligned} f(x_n + f(x_n)) = {} & c_1(1 + c_1)e_n + \left(c_1 + (1 + c_1)^2\right) c_2e_n^2 + \\ & + \left(2(1 + c_1)c_2^2 + c_1c_3 + (1 + c_1)^3c_3\right) e_n^3 + \\ & + \Big(3(1 + c_1)^2c_2c_3 + c_2 \left(c_2^2 + 2(1 + c_1)c_3\right) + \\ & + c_1c_4 + (1 + c_1)^4c_4\Big) e_n^4 + \\ & + \Big(3(1 + c_1)c_3 \left(c_2^2 + c_3 + c_1c_3\right) + 4(1 + c_1)^3c_2c_4 + \\ & + 2c_2(c_2c_3 + c_4 + c_1c_4) + c_1c_5 + (1 + c_1)^5c_5\Big) e_n^5 + \\ & + \Big(2(1 + c_1)^2 \left(3c_2^2 + 2(1 + c_1)c_3\right) c_4 + \\ & + c_3 \left(c_2^3 + 6(1 + c_1)c_2c_3 + 3(1 + c_1)^2c_4\right) + \\ & + 5(1 + c_1)^4c_2c_5 + c_2 \left(c_3^2 + 2(c_2c_4 + c_5 + c_1c_5)\right) + \\ & + c_1c_6 + (1 + c_1)^6c_6\Big) e_n^6 + O(e_n^7). \end{aligned} \tag{14}$$

The Taylor series of $f(x_n - f(x_n))$ is:

$$\begin{aligned} f(x_n - f(x_n)) = {} & -(-1 + c_1)c_1e_n + \left(1 - 3c_1 + c_1^2\right) c_2e_n^2 + \\ & + \left(2(-1 + c_1)c_2^2 - (-1 + c_1)^3c_3 - c_1c_3\right) e_n^3 + \\ & + \Big(-3(-1 + c_1)^2c_2c_3 + c_2 \left(c_2^2 + 2(-1 + c_1)c_3\right) + \\ & + (-1 + c_1)^4c_4 - c_1c_4\Big) e_n^4 + \\ & + \Big(-3(-1 + c_1)c_3 \left(c_2^2 + (-1 + c_1)c_3\right) + 4(-1 + c_1)^3c_2c_4 + \\ & + 2c_2(c_2c_3 + (-1 + c_1)c_4) - (-1 + c_1)^5c_5 - c_1c_5\Big) e_n^5 + \\ & + \Big(-c_2^3c_3 + 2 \left(4 - 6c_1 + 3c_1^2\right) c_2^2c_4 + \\ & + (-1 + c_1)^2(-7 + 4c_1)c_3c_4 + c_2 \Big((7 - 6c_1)c_3^2 + \\ & + \left(-7 + 22c_1 - 30c_1^2 + 20c_1^3 - 5c_1^4\right) c_5\Big) + \\ & + \left(1 - 7c_1 + 15c_1^2 - 20c_1^3 + 15c_1^4 - 6c_1^5 + c_1^6\right) c_6\Big) e_n^6 + O(e_n^7). \end{aligned} \tag{15}$$

Substituting (14) and (15) in the expression of y_n in (5), gives us

$$
\begin{aligned}
y_n-\alpha = {} & x_n-\alpha-\frac{2f(x_n)^2}{f(x_n+f(x_n))-f(x_n-f(x_n))}= \\
= {} & e_n-\frac{2f(x_n)^2}{f(x_n+f(x_n))-f(x_n-f(x_n))}= \\
= {} & \frac{c_2e_n^2}{c_1}-\frac{\left(2c_2^2-c_1\left(2+c_1^2\right)c_3\right)e_n^3}{c_1^2}+ \\
& +\left(\frac{4c_2^3}{c_1^3}-c_2c_3-\frac{7c_2c_3}{c_1^2}+\frac{3c_4}{c_1}+4c_1c_4\right)e_n^4- \\
& -\frac{1}{c_1^4}\left(8c_2^4-c_1\left(20+3c_1^2\right)c_2^2c_3+2c_1^2\left(5+2c_1^2\right)c_2c_4+\right. \\
& \left.+c_1^2\left(\left(6+3c_1^2+c_1^4\right)c_3^2-c_1\left(4+10c_1^2+c_1^4\right)c_5\right)\right)e_n^5- \\
& -\frac{1}{c_1^5}\left(-16c_2^5+c_1\left(52+7c_1^2\right)c_2^3c_3-4c_1^2\left(7+3c_1^2\right)c_2^2c_4-\right. \\
& -c_1^2c_2\left(\left(33+12c_1^2+c_1^4\right)c_3^2+c_1\left(-13-10c_1^2+c_1^4\right)c_5\right)+ \\
& \quad +c_1^3\left(\left(17+17c_1^2+8c_1^4\right)c_3c_4-\right. \\
& \quad \left.\left.-\ c_1\left(5+20c_1^2+6c_1^4\right)c_6\right)\right)e_n^6+O(e_n^7).
\end{aligned}
\tag{16}
$$

Now, substituting (16) in the Taylor series of $f(y_n)$ we have

$$
\begin{aligned}
f(y_n) = {} & c_2e_n^2+\left(-\frac{2c_2^2}{c_1}+2c_3+c_1^2c_3\right)e_n^3+ \\
& +\left(\frac{5c_2^3}{c_1^2}-\frac{7c_2c_3}{c_1}-c_1c_2c_3+3c_4+4c_1^2c_4\right)e_n^4+ \\
& +\frac{1}{c_1^3}\left(-12c_2^4+c_1\left(24+5c_1^2\right)c_2^2c_3-2c_1^2\left(5+2c_1^2\right)c_2c_4+\right. \\
& \left.+c_1^2\left(-\left(6+3c_1^2+c_1^4\right)c_3^2+c_1\left(4+10c_1^2+c_1^4\right)c_5\right)\right)e_n^5+ \\
& +\frac{1}{c_1^4}\left(28c_2^5-c_1\left(73+13c_1^2\right)c_2^3c_3+2c_1^2\left(17+10c_1^2\right)c_2^2c_4+\right. \\
& +c_1^2c_2\left(\left(37+16c_1^2+2c_1^4\right)c_3^2+c_1\left(-13-10c_1^2+c_1^4\right)c_5\right)+ \\
& +c_1^3\left(-\left(17+17c_1^2+8c_1^4\right)c_3c_4+\right. \\
& \left.\left.+\ c_1\left(5+20c_1^2+6c_1^4\right)c_6\right)\right)e_n^6+O(e_n^7).
\end{aligned}
\tag{17}
$$

Using (13), (16) and (17) into (5), gives

$$
\begin{aligned}
z_n-\alpha = y_n-\mu_n f(y_n) &= \frac{c_2\left(c_2^2-c_1\left(1+c_1^2\right)c_3\right)e_n^4}{c_1^3}-\\
&-\frac{\left(4c_2^4-2c_1\left(4+c_1^2\right)c_2^2c_3+c_1^2\left(2+3c_1^2+c_1^4\right)c_3^2+2c_1^2\left(1+2c_1^2\right)c_2c_4\right)e_n^5}{c_1^4}-\\
&-\frac{1}{c_1^5}\left(-10c_2^5+2c_1\left(15+2c_1^2\right)c_2^3c_3-4c_1^2\left(3+2c_1^2\right)c_2^2c_4+c_1^3\left(7+17c_1^2+8c_1^4\right)c_3c_4+\right.\\
&\left.+c_1^2c_2\left(\left(-18-8c_1^2+c_1^4\right)c_3^2+c_1\left(3+10c_1^2+c_1^4\right)c_5\right)\right)e_n^6+O(e_n^7)
\end{aligned}
\tag{18}
$$

and substituting (18) in the Taylor series of $f\left(z_n\right)$ we have

$$
\begin{aligned}
f(z_n) &= \frac{c_2\left(c_2^2-c_1\left(1+c_1^2\right)c_3\right)e_n^4}{c_1^2}-\\
&-\frac{\left(4c_2^4-2c_1\left(4+c_1^2\right)c_2^2c_3+c_1^2\left(2+3c_1^2+c_1^4\right)c_3^2+2c_1^2\left(1+2c_1^2\right)c_2c_4\right)e_n^5}{c_1^3}-\\
&-\frac{1}{c_1^4}\left(-10c_2^5+2c_1\left(15+2c_1^2\right)c_2^3c_3-4c_1^2\left(3+2c_1^2\right)c_2^2c_4+c_1^3\left(7+17c_1^2+8c_1^4\right)c_3c_4+\right.\\
&\left.+c_1^2c_2\left(\left(-18-8c_1^2+c_1^4\right)c_3^2+c_1\left(3+10c_1^2+c_1^4\right)c_5\right)\right)e_n^6+O(e_n^7).
\end{aligned}
\tag{19}
$$

Taking into account (18) and (19), we finally obtain

$$
\begin{aligned}
e_{n+1} &= z_n-\alpha-\mu_n f(z_n) = \\
&= \frac{\left(-2c_2^2+c_1\left(1+c_1^2\right)c_3\right)\left(-c_2^3+c_1\left(1+c_1^2\right)c_2c_3\right)}{c_1^5}e_n^6+O(e_n^7).
\end{aligned}
\tag{20}
$$

This proves that the method is of sixth order. □

It is easy to observe that the method ODF uses four functional evaluations per step, whereas the IODF needs five. There are many techniques for obtaining high order iterative methods, but the complexity of the iterative expressions increase considerably. So we have introduced in [13], in the context of nonlinear systems, a new index in order to compare the different methods, taking into account not only the number of functional evaluations, but also the number of products and quotients involved in each step of the iterative process. The *computational efficiency index* is defined as $CI = p^{1/(d+op)}$, where p is the order of convergence, d is the number of functional evaluations per step and op is the number of products and quotients per iteration.

In the next table we present the order of convergence, the efficiency index and the computational efficiency index of the Steffensen's like methods mentioned in Section 1 and our new methods.

Method	**Order**	**Efficiency index**	**Comp. Efficiency index**
Steffensen (SM)	2	$2^{1/2}$	$2^{1/(2+2)}$
Jain (JM)	3	$3^{1/3}$	$3^{1/(3+6)}$
Feng-He (FM)	3	$3^{1/3}$	$3^{1/(3+8)}$
Zheng et al. (ZM)	3	$3^{1/4}$	$3^{1/(4+6)}$
Amat-Busquier (AM)	2	$2^{1/2}$	$2^{1/(2+2)}$
Dehghan-Hajarian (DM)	3	$3^{1/4}$	$3^{1/(4+4)}$
ODF	4	$4^{1/4}$	$4^{1/(4+4)}$
IODF	6	$6^{1/5}$	$6^{1/(5+5)}$

Table 1
Order and efficiency indices of some derivative free methods

We can observe the position of our methods in relation to the other ones, taking into account the efficiency index:

$$I_{FM} = I_{JM} > I_{IODF} > I_{ODF} = I_{SM} = I_{AM} > I_{ZM} = I_{DM}$$

and the computational efficiency index

$$CI_{IODF} > CI_{ODF} = CI_{SM} = CI_{AM} > CI_{DM} > CI_{JM} > CI_{ZM} > CI_{FM}.$$

4 Numerical results

In this section we check the effectiveness of the new methods ODF and $IODF$ applied to obtain the solution of several nonlinear equations. We use equations (a) to (j) to compare the described methods with their counterparts that make use of derivatives, that is, Ostrowski's method (OM) and improved Ostrowski's method (IOM) and the classical Newton's method (NM).

(a) $f(x) = \sin^2 x - x^2 + 1, \;\; \alpha \approx 1.404492,$
(b) $f(x) = x^2 - e^x - 3x + 2, \;\; \alpha \approx 0.257530,$
(c) $f(x) = \cos x - x, \;\; \alpha \approx 0.739085,$
(d) $f(x) = (x-1)^3 - 1, \;\; \alpha = 2,$
(e) $f(x) = x^3 - 10, \;\; \alpha \approx 2.154435,$
(f) $f(x) = \cos(x) - xe^x + x^2, \;\; \alpha \approx 0.639154,$
(g) $f(x) = e^x - 1.5 - \arctan(x), \;\; \alpha \approx 0.767653,$
(h) $f(x) = x^3 + 4x^2 - 10, \;\; \alpha \approx 1.365230,$
(i) $f(x) = 8x - \cos(x) - 2x^2, \;\; \alpha \approx 0.128077,$
(j) $f(x) = \arctan(x), \;\; \alpha = 0,$

Numerical computations have been carried out using variable precision arithmetics with 256 digits in MATLAB 7.1. The stopping criterion used is $|x_{k+1} - x_k| + |f(x_k)| < 10^{-100}$, therefore, we check that the iterates succession converge to an approximation to the solution of the nonlinear equation. For every method, we count the number of iterations needed to reach the wished tolerance and estimate the computational order of convergence (ACOC), according to (see [14])

$$\rho = \frac{\ln(|x_{k+1} - x_k| \,/\, |x_k - x_{k-1}|)}{\ln(|x_k - x_{k-1}| \,/\, |x_{k-1} - x_{k-2}|)}. \tag{21}$$

The value of ρ that appears in Table 2 is the last coordinate of vector ρ when the variation between its values is small. A comparison between methods using derivatives and derivative free methods can be established. The behavior of the new methods is similar to the classical ones of the same order of convergence, as theoretical results show. It can be observed that the new methods need more iterations than their partenaires, in some cases, but when the initial estimation is not good and methods using derivatives diverge, derivative free methods ODF and IODF converge quickly.

$f(x)$	x_0	**Iterations**					**p**				
		NM	OM	IOM	ODF	IODF	NM	OM	IOM	ODF	IODF
a)	1	9	5	5	5	5	2.00	4.00	6.00	4.00	6.00
b)	0.7	7	5	4	5	6	2.00	4.00	6.00	4.00	5.99
c)	1	8	5	4	5	5	2.00	4.00	6.00	3.80	6.00
d)	1.5	11	6	5	6	6	2.00	4.00	6.00	4.00	6.00
e)	2	8	5	4	5	6	2.00	4.00	6.00	4.00	5.99
f)	1	9	5	4	6	NC	2.00	4.00	6.00	4.00	-
g)	1	9	5	4	5	5	2.00	4.00	6.00	4.00	6.00
h)	1.5	8	5	4	6	6	2.00	4.00	6.00	4.00	6.01
i)	1	9	5	4	5	6	2.00	4.00	6.00	4.00	5.99
j)	1	8	5	5	5	5	3.00	5.00	7.00	5.00	7.00
j)	2.5	NC	NC	5	8	6	-	-	7.00	5.00	7.00

Table 2
Numerical results for nonlinear equations from (a) to (j)

5 Conclusions

We have used central-difference approximations for the first derivative in Ostrowski's method, that has order of convergence four and in a improved version of Ostrowski's method with sixth order of convergence, obtaining two new iterative methods for nonlinear equations free from derivatives and we have proven that they preserve their convergence order. The theoretical results have been checked with some numerical examples, comparing our algorithms with Newton's method and with the corresponding methods that make use of derivatives. We have compared some Steffensen like methods with our methods from the point of view of efficiency index and computational efficiency index.

References

[1] Z. Xiaojian, Modified ChebyshevHalley methods free from second derivative, Applied Mathematics and Computation, 203 (2008) 824-827.

[2] C. Chun, Some second derivative free variants of ChebyshevHalley methods, Applied Mathematics and Computation, 191 (2007) 410-414.

[3] C. Chun, Y. Ham, Some second-derivative-free variants of super-Halley method with fourth-order convergence, Applied Mathematics and Computation, 195 (2008) 537-541.

[4] P. Jain, Steffensen type methods for solving nonlinear equations, Applied Mathematics and Computation, 194 (2007) 527-533.

[5] X. Feng, Y. He, High order oterative methods without derivatives for solving nonlinear equations, Applied Mathematics and Computation, 186 (2007) 1617-1623.

[6] S. Weerakoon, T.G.I. Fernando, A variant of Newton's methods with accelerated third-order convergence, Applied Mathematics Letters, 13 (2000) 87-93.

[7] Q. Zheng, J. Wang, P. Zhao, L. Zhang, A Steffensen-like method and its higher-order variants, Applied Mathematics and Computation, 214 (2009) 10-16.

[8] S. Amat, S. Busquier, On a Steffensen's type method and its behavior for semismooth equations, Applied Mathematics and Computation, 177 (2006) 819-823.

[9] V. Alarcón, S. Amat, S. Busquier, D.J. López, A Steffensen's type method in Banach spaces with applications on boundary-value problems, Journal of Computational and Applied Mathematics, 216 (2008) 243-250.

[10] A.M. Ostrowski, Solutions of Equations and Systems of Equations, Academic Press, New York, 1960.

[11] M. Dehghan, M. Hajarian, An Some derivative free quadratic and cubic convergence iterative formulas for solving nonlinear equations, Journal of Computational and Applied Mathematics, 29 (2010) 19-30.

[12] M. Grau, J.L. Díaz-Barrero, An improvement to Ostrowski root-finding method, Applied Mathematics and Computation, 173 (2006) 450-456.

[13] A. Cordero, J.L. Hueso, E. Martínez, J.R. Torregrosa, A modified Newton-Jarratt's composition, Numerical Algorithm. doi: 10.1007/s11075-009-9359-z.

[14] A. Cordero, J.R. Torregrosa, Variants of Newton's method using fifth order quadrature formulas, Applied Mathematics and Computation, 190 (2007) 686-698.