# Derivative-free high-order methods applied to preliminary orbit determination

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**Abstract**

From position and velocity coordinates for several given instants, it is possible to determine orbital elements for the preliminary orbit, taking only into account mutual gravitational attraction forces between the Earth and the satellite. Nevertheless it should be refined with later observations from ground stations, whose geographic coordinates are previously known. Different methods developed for this purpose need, in their process, to find a solution of a nonlinear function. In classical methods it is usual to employ fixed point or secant methods. The later case is often used when it is not possible to obtain the derivative of the nonlinear function. Nowadays, there exist efficient numerical methods that are able to highly improve the results obtained by the classical schemes. We will focus our attention in the method of iteration of the true anomaly, in which the secant method is replaced by more efficient methods, as the second-order Steffensen's method, as well as other high-order derivative-free methods.

*Key words:* Two-body problem, orbit determination, derivative-free, nonlinear equations, order of convergence, efficiency index.

## 1 Introduction

Brahe, Kepler, Newton. This is maybe the main starrings sequence to understand how satellites move nowadays.
Brahe, with his observations and measurements about the planetary movement, remade the prediction tables of his days, getting a minute arc precision.

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At his death, Kepler went on with Brahe's measurements. He published his famous first and second laws at 1609; the third one was published in 1618. Newton published *Philosophiæ naturalis principia mathematica* at 1687, that includes the law of universal gravitation. Joining Newton's law and Kepler's results, the scientist tackled the two-body problem, which solution is that the orbit of a planet describes a conic form. Different observations help us to guess that the orbit is bounded, so that the conic is an ellipse.

Specifically, our two body problem is set by the Earth and the satellite. Once we know the shape of the satellite's orbit, the next goal is the position and velocity search. With the knowledge of these two vectors, we are able to get an approximation of the orbital elements and, thereafter, to improve these orbital elements by analyzing the difference between calculated and real positions.

So, let it be assumed that two satellite position vectors are known. Furthermore, let the time interval by the two observations is also known. The determination of an orbit satisfying these boundary conditions can be made by different techniques. Some of these methods need to find a solution of a nonlinear equation. Our aim is improving the classical schemes of calculation, by means of efficient iterative methods provided by the recent research in the area.

In the original method of orbit determination – the iteration of true anomaly – the nonlinear equation to be solved is the difference between the real and the estimated epoch of the perigee. So, the classical scheme involves the use of the secant method. In our modified iteration of true anomaly, described in Section 3, different high-order derivative-free schemes are used.

Nowadays, position applications are extremely important. Some examples are navigation control, tracking and supervision of artificial satellites, exploration of the gravity field or the sea-level study. For instance, the knowledge of where a satellite is affected by a Solar eclipse is very important to provide the satellite the appropiate self-batteries.

## 2 The two-body problem

We will show a short summary of the two-body problem, solved by Newton. Let $M$ and $m$ be the masses of the primary and secondary bodies, respectively, and $G$ the universal gravitational constant. Moreover, let $\mu$ be the gravitational parameter, defined by $\mu = G(M + m) \approx GM$. Applying second Newton's law for each mass, and adding both expressions, we obtain:

$$\ddot{\vec{r}} + \frac{\mu}{r^3}\vec{r} = 0, \tag{1}$$

where the magnitude of $\vec{r}$ is:

$$r = \frac{p}{1 + e\cos\nu}. \tag{2}$$

Equation (2) can be interpreted as a conic, where $p$ is the semiparameter of the conic, $e$ is its eccentricity and the polar angle $\nu$ is the true anomaly. The position of a satellite can be appointed by that expression.
The scalar velocity of the satellite, given by Vis-Viva Equation [1], can be obtained by

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}, \tag{3}$$

where $a$ is the semimajor axis of the conic.
Many observations of the satellites motion determines that the orbit is bounded. In order to agree with Newton calculations, expression (2) should satisfy $0 < e < 1$, and $0 < a < \infty$, for elliptic motion.
The true anomaly $\nu$ (see Figure 1) is the angular distance between the perigee and the current position of the satellite. If the ellipse is inside a circumference of radius the semimajor axis of the ellipse, passing through perigee, and a vertical projection from the current position of the satellite to the circumference is made (see point Q in Figure 1), the eccentric anomaly $E$ is the angular distance between the perigee and the projection.

The orbital elements set $(a,e,i,\omega,\Omega,\tau)$ determinate the position and the velocity of a satellite in space.

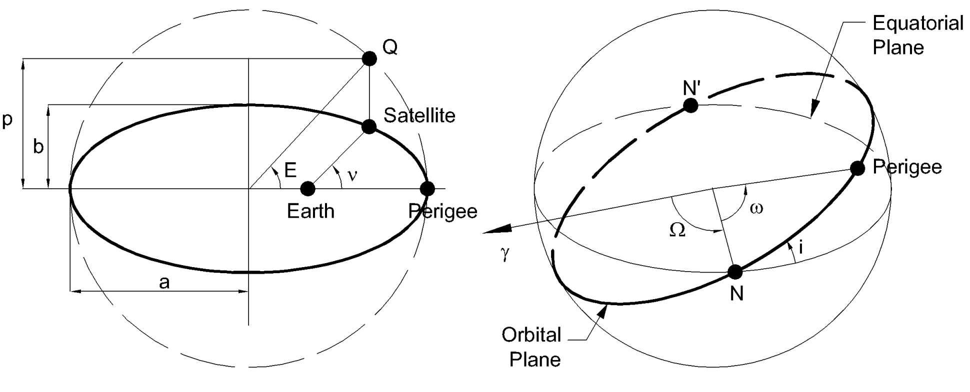


Fig. 1. The Orbital Plane and Space

The line of nodes is the intersection of the orbital and the equatorial planes. The endpoints of the line of nodes are $N$ and $N'$, the ascending and descending nodes, respectively. The inclination $i$ is the angular distance between the equatorial and orbital planes. The vernal equinox $\gamma$ points the direction where the Sun crosses directly over the Earth's equator, and sets the origin of the right ascension. The ascending node $\Omega$ is the right ascension of $N$, i.e., the angular distance between the direction of $\gamma$ and $N$, on the equatorial plane. The argument of perigee $\omega$ is the angular distance between $N$ and the perigee, on the orbital plane. The perigee is the closest point of the ellipltical orbit to the focus, so the perigee epoch $\tau$ is the time when the satellite crosses the perigee.

## 3 Determination of a preliminary orbit from two positions and time

In Section 2 we have stated that an orbit can be established by the position and velocity of the satellite. But, how can we achieve this data? There are several methods in [1] that use different observational information, such as two position vectors and time, angles or mixed data information.
From two position vectors of the satellite, and the time interval between the measurements, $\vec{r}_1(t_1)$ and $\vec{r}_2(t_2)$, we can find the velocity of the satellite by:

$$\vec{v}_1 = \frac{\vec{r}_2 - f\vec{r}_1}{g} \tag{4}$$

where $f$ and $g$ series (described in [1]) will be shown later, in their closed form.

### *3.1 True Anomaly Iteration*

Let $d_\nu$ be the angular distance between the true anomalies of $\vec{r}_1$ and $\vec{r}_2$, that can be obtained from the data and (2). Let $\Delta t$ be the time interval between both observations. The algorithm of this method, in order to determinate a preliminary orbit, is as follows:

(1) Set an initial estimation of $\nu_1$, denoted by $\hat{\nu}_1$
(2) Calculate $\hat{\nu}_2 = \hat{\nu}_1 + d_\nu$
(3) Compute $e = \frac{r_2 - r_1}{r_1 \cos\hat{\nu}_1 - r_2 \cos\hat{\nu}_2}$. If $e \notin (0,1)$, return to step 2, incrementing $\hat{\nu}_1$ in 10 degrees.
(4) Determine $a = \frac{r_1(1+e\cos\hat{\nu}_1)}{1-e^2}$. If $a < 0$, return to step 2, incrementing $\hat{\nu}_1$ in 10 degrees.
(5) Proceed with the obtention of the estimated eccentric anomalies, $E_1$ and $E_2$:
$\sin E_i = \frac{\sqrt{1-e^2}\sin\hat{\nu}_i}{1+e\cos\hat{\nu}_i} \quad \cos E_i = \frac{\cos\hat{\nu}_i + e}{1+e\cos\hat{\nu}_i}, \quad i = 1,2$
(6) Evaluate the nonlinear function $F = k\cdot\Delta t - \sqrt{\frac{a^3}{\mu}}\,[E_2 - E_1 + e(\sin E_1 - \sin E_2)]$, where $k = 0.07436574\ (e.r)^{1/2}/min$.

- If $|F| > \epsilon_1$:
  (a) $F_0 = F$, $(\hat{\nu}_1)_0 = \hat{\nu}_1$
  (b) Return to step 2, incrementing $\hat{\nu}_1$ in $2 \cdot 10^{-7}$ degrees. In the step 6, the value of $F$ is called $F_1$
  (c) Apply the secant method: $(\hat{\nu}_1)_1 = (\hat{\nu}_1)_0 - \frac{F_0}{dF}$, where $dF = \frac{F_1 - F_0}{2 \cdot 10^{-7}}$
  (d) While $|(\hat{\nu}_1)_1 - (\hat{\nu}_1)_0| > \epsilon_2$, repeat from step 2, taking $\hat{\nu}_1 = (\hat{\nu}_1)_1$
- If $|F| < \epsilon_1$, go to step 7.

(7) Obtain $f$ and $g$ series, and the velocity:

$$\left.\begin{array}{l} f = 1 - \frac{a}{r_1}[1 - cos(E_2 - E_1)] \\ g = k \cdot \Delta t - \sqrt{\frac{a^3}{\mu}}[E_2 - E_1 - \sin(E_2 - E_1)] \end{array}\right\} \vec{v}_1 = \frac{\vec{r}_2 - f\vec{r}_1}{g} \tag{5}$$

Once the position and velocity vectors are known, orbital elements are easily obtained (see [1]).

### *3.2 Derivative-free high-order methods*

In order to improve the efficiency of true anomaly iteration method, we are going to replace the secant method by derivative-free high-order methods. Secant method is based on Newton's one. The iterative expression of Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

The first method we introduce is Steffensen, denoted by STF. In this method, the derivative $f'(x_k)$ is replaced by the forward-difference approximation $f'(x_k) \approx \frac{f(x_k + f(x_k)) - f(x_k)}{f(x_k)}$. So, the iterative expression is:

$$x_{k+1} = x_k - \frac{[f(x_k)]^2}{f(x_k + f(x_k)) - f(x_k)}. \tag{6}$$

A common guideline used to improve the local order of convergence is the composition of two iterative methods of order $p$ and $q$, respectively. In this way, we can obtain a method of $pq$-order, as it is showed in [2]. Afterwards, some approximations are made in order to avoid functional evaluations, trying to hold the order of convergence. Hence, we are introducing some of this composed methods.

Liu, Zheng and Zhao, designed a fourth-order method [3] by composition of Steffensen and Newton's method, and an estimation of the derivative involved. It is denoted by LZZ and its iterative expression is

$$x_{k+1} = y_k - \frac{f[x_k, y_k] - f[y_k, z_k] + f[x_k, z_k]}{(f[x_k, y_k])^2} f(y_k), \tag{7}$$

where $y_k$ is STF (6), $z_k = x_k + f(x_k)$ and $f[\cdot,\cdot]$ is the divided difference of order one.
Cordero and Torregrosa presented in [4] a one-parameter family of optimal fourth-order derivative-free methods, CT from now on. The iterative expression of this family is

$$x_{k+1} = y_k - \frac{f(y_k)}{\dfrac{f(y_k) - \beta f(z_k)}{y_k - z_k} + \dfrac{f(y_k) - \delta f(x_k))}{y_k - x_k}}, \tag{8}$$

where $y_k$ is STF (6), $z_k = x_k + f(x_k)$, and parameters $\beta$ and $\delta$ must verify $\beta + \delta = 1$. Hereinafter, we will take $\beta = 1$ and $\delta = 0$.
By direct composition of Steffensen's and Newton's methods, using a Padé approximant of degree one, a derivative-free fourth-order method is obtained (see [5]). Composing this fourth-order method with Newton's method again, and using a Padé approximant of degree two, the authors obtain an optimal derivative-free eighth-order method, denoted by M8 and described in [5]. Its iterative expression is

$$x_{k+1} = u_k - \frac{f(u_k)}{b_2 - b_1 b_4}, \tag{9}$$

where $u_k$ is STF and Newton composition with the degree one approximant of Padé, $b_1 = f(u_k)$, $b_2 = f[y_k, u_k] - b_3(y_k - u_k) + f(y_k)b_4$, $b_3 = f[y_k, u_k, z_k] + b_4 f[y_k, z_k]$, $b_4 = \frac{f[y_k,u_k,x_k] - f[y_k,u_k,z_k]}{f[y_k,z_k] - f[y_k,x_k]}$ and $f[\cdot,\cdot,\cdot]$ is the divided difference of order two.

In order to compare the applied methods, we will use the efficiency index $I = p^{1/d}$, where $p$ is the order of convergence and $d$ is the total number of functional evaluations per step. Kung and Traub conjectured in [6] that $p \leq 2^{d-1}$.

| Method | NEW | SEC | STF | LZZ | CT | M8 |
|---|---|---|---|---|---|---|
| p | 2 | 1.618 | 2 | 4 | 4 | 8 |
| d | 2 | 2 | 2 | 3 | 3 | 4 |
| I | 1.4142 | 1.2720 | 1.4142 | 1.5874 | 1.5874 | 1.6818 |

Table 1
Comparison of Methods

### *3.3 Numerical Results*

Numerical computations have been carried out using variable precision arithmetic, with 500 significant digits, in MATLAB R2009B. The stopping criterion used is $|x_{k+1} - x_k| < 10^{-500}$.

Each method is applied to a reference orbit. These reference orbits can be found in [1]. The study is based on the worst case, i.e., the initial estimation of $\nu_0$ that causes the maximum number of iterations with the secant method. In Table 2 we show the values $\nu_0$ (degrees) and $\Delta t$ (Julian Days) – the time interval between the two measurements of the position, for each reference orbit.

| RefOrb | $\nu_0$ | $\Delta t$ |
|---|---|---|
| I | 156.8515 | 0.01044412 |
| II | 68.7325 | 0.01527809 |
| III | 165.9299 | 0.01316924 |

Table 2
Parameters of each Reference Orbit

The information displayed in Table 3 is, for each method, the last value of $|x_{k+1}-x_k|$, the number of iterations needed to reach the expected tolerance, the approximated computational order of convergence (ACOC) $\rho$ and the mean elapsed time.The computer specifications are: Intel(R) Core(TM) i7 CPU, 950@3.07GHz with 16.00GB of RAM.
The ACOC is defined in [7] as

$$p \approx \rho = \frac{\ln\left(|x_{k+1}-x_k|/|x_k-x_{k-1}|\right)}{\ln\left(|x_k-x_{k-1}|/|x_{k-1}-x_{k-2}|\right)}. \tag{10}$$

| Ref | Parameter | SEC | STF | LZZ | CT | M8 |
|---|---|---|---|---|---|---|
| I | $\lvert x_{k+1}-x_k\rvert$ | 3.3885e-319 | 1.0233e-186 | 2.8184e-305 | 9.3325e-216 | 2.8949e-136 |
| | iter | 56 | 12 | 7 | 6 | 5 |
| | ACOC | 1.00 | 2.00 | 4.00 | 4.00 | 8.24 |
| | e-time | 18.2864 | 2.7509 | 2.5310 | 2.2068 | 1.4625 |
| II | $\lvert x_{k+1}-x_k\rvert$ | 3.9525e-323 | 1.7378e-258 | 1.3490e-173 | 1.7378e-133 | 1.6239e-74 |
| | iter | 63 | 15 | 7 | 6 | 5 |
| | ACOC | 1.00 | 2.00 | 4.00 | 4.00 | 7.75 |
| | e-time | 15.7550 | 3.0288 | 2.1762 | 1.8823 | 1.4507 |
| III | $\lvert x_{k+1}-x_k\rvert$ | 3.6308e-318 | 1.2023e-233 | 1.2589e-160 | 1.2882e-262 | 1.3317e-260 |
| | iter | 105 | 28 | 7 | 6 | 5 |
| | ACOC | 1.00 | 2.00 | 4.00 | 4.00 | 8.00 |
| | e-time | 22.5885 | 7.0931 | 2.5951 | 2.2656 | 1.6436 |

Table 3
Numerical Results for Reference Orbit I, II and III

The mean elapsed time is calculated by the "tic-toc" command of Matlab, after 100 performances of the program.
Once we have applied each method to the three reference orbits shown in Table 2, we conclude several facts. The composed methods LZZ, CT and M8 reach higher convergence order than simple ones SEC and STF, so the number of iterations needed to get the stopping criterion decreases as the method has higher convergence order. Paying attention to fourth-order methods, CT has a better behaviour than LZZ in terms of the mean elapsed time, so CT is faster than LZZ. The M8 method has the best efficiency index and it is optimal. Is a fact that M8 is the method that needs the minimum number of iterations to get the result and, moreover, the fastest one.

## Acknowledgements

This research was supported by Ministerio de Ciencia y Tecnología MTM2010-18539 and by Vicerrectorado de Investigación, Universitat Politècnica de València PAID-06-2010-2285.

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