On Improved Three-Step Schemes with High Efficiency Index and Their Dynamics

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Abstract This paper presents an improvement of the sixth-order method of Chun and Neta as a class of three-step iterations with optimal efficiency index, in the sense of Kung-Traub conjecture. Each member of the presented class reaches the highest possible order using four functional evaluations. Error analysis will be studied and numerical examples are also made to support the theoretical results. We then present results which describe the dynamics of the presented optimal methods for complex polynomials. The basins of attraction of the existing optimal methods and our methods are presented and compared to illustrate their performances.

Keywords Basin of attraction · Kung-Traub conjecture · fractal · multi-point iterations · Julia set

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1 Introduction

Nonlinear equations arise in many real problems in Science and Engineering. The subject of root-finding of nonlinear equations play an important role in Numerical Analysis and Optimization. In particular, Newton's scheme and its modifications can be used to solve the variational inequality and the nonlinear complementary problems (see, for example [1] and [2]). Also efficient iterative schemes are needed to solve nonlinear equations coming from the application of shooting methods to solve boundary value problems.

Throughout this paper we consider iterative methods to find a simple root α in an open interval D, i.e., $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$.

Recenly, Chun and Neta in [3] presented a sixth-order method as follows

$$
\begin{cases}
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n = y_n - \frac{f(y_n)}{f'(x_n)}\left(1 - \frac{f(y_n)}{f(x_n)}\right)^{-2}, \\
x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\left(1 - \frac{f(y_n)}{f(x_n)} - \frac{f(z_n)}{f(x_n)}\right)^{-2}.
\end{cases} \tag{1}
$$

This scheme without memory includes three evaluations of the function and one evaluation of the first-order derivative per computing step to reach the order six and thus its classical efficiency index is $6^{\frac{1}{4}} \approx 1.565$, which is not as high as possible with four functional evaluations according to the hypothesis of Kung and Traub on the optimality of multi-point iterations without memory [4].

To improve the local order of convergence different techniques have been used getting iterative schemes with and without memory (see, for instance [5–7]) and to upgrade the efficiency index, many optimal methods without memory have been proposed; see, for example, [3,8–16], and the references therein.

The prime motivation of this paper is to optimize (1) in terms of the local convergence order and the efficiency index, as well as to study the dynamics of some optimal eighth-order methods using fractal picture. Thus, we aim at modifying (1) to reach a wide class of three-step methods using four evaluations, i.e. the same as (1), but with higher order of convergence and efficiency. This purpose will be studies by using weight function technique.

The dynamical study of the rational function associated to an iterative method gives important information about the convergence and stability of the scheme. In this sense, we should not forget the early work [17] and more recent studies, as for example [18–23] and the references therein). In order to do this, we recall some concepts. Thus, now we shortly present them. Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map on the Riemann sphere. For $z \in \hat{\mathbb{C}}$, we define its orbit as the set $orb(z) = \{z, R(z), R^2(z), \cdots, R^n(z), \cdots\}$, and subsequently, a point z_0 is a fixed point of R if $R(z_0) = z_0$.

A periodic point z_0 of period m is such that $R^m(z_0) = z_0$ where m is the smallest such integer. And also a point z_0 is called attracting if $|R'(z_0)| < 1$,

repelling if $|R'(z_0)| > 1$, and neutral if $|R'(z_0)| = 1$. The Julia set of a nonlinear map $R(z)$, denoted $\mathcal{J}(R)$, is the closure of the set of its repelling periodic points. The complement of $\mathcal{J}(R)$ is the Fatou set $\mathcal{F}(R)$, where the basin of attraction of the different roots lie.

The rest of the paper is organized in what follows. In the next section, we derive a new family of three-step eighth-order convergent methods, which is consistent with the Kung-Traub conjecture. Section 3 covers some concrete optimal eighth order of convergence methods of the class. Next, in Section 4, we give a number of numerical tests to clarify the effectiveness of the new attained methods. Section 5 successfully discusses the dynamics of the presented optimal iterations. The concluding remarks of the paper is furnished in Section 6.

2 The improved scheme

To build an optimal eighth-order method consuming four evaluations according to (1), we suggest the following iterative class by using weight function approach at the end of the first and third steps

$$
\begin{cases}
y_n = x_n - \frac{f(x_n)}{f'(x_n)} A(\delta), \\
z_n = y_n - \frac{f(y_n)}{f'(x_n)} \left(1 - \frac{f(y_n)}{f(x_n)}\right)^{-2}, \\
x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \left(1 - \frac{f(y_n)}{f(x_n)} - \frac{f(z_n)}{f(x_n)}\right)^{-2} \{G(t) + H(\mu)\},
\end{cases}
\tag{2}
$$

where $\delta = \frac{f(x)}{f'(x)}$, $t = \frac{f(y)}{f(x)}$, and $\mu = \frac{f(z)}{f(y)}$, without the index n. The three weight functions $A(\delta)$, $G(t)$, and $H(\mu)$, should be chosen such that the order arrives at local eight. This is done and illustrated in Theorem 1.

Theorem 1 *Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval D, which contains x_0 as an initial approximation of α. Then the three-step iteration (2) without memory, which contains four evaluations per full cycle has the optimal convergence rate eight and satisfies the error equation below*

$$
\begin{aligned}
e_{n+1} = &\frac{1}{24c_1^7} c_2 (-2c_2^2 + c_1 c_3)(24 c_1 c_2^2 c_3 (7 - 2H''(0)) + 12 c_1^2 c_3^2 (-2 + H''(0)) \\
&+ 12 c_1^2 c_2 (-2c_4 + c_1 A^{(3)}(0)) + c_2^4 (-120 + 48 H''(0) + G^{(4)}(0))) e_n^8 \\
&+ O(e_n^9),
\end{aligned}
\tag{3}
$$

when $A(0) = 1$, $A'(0) = 0$, $A''(0) = 0$, $|A^{(3)}(0)| < \infty$, $|G(0)| < \infty$, $G'(0) = 0$, $G''(0) = 2$, $G^{(3)}(0) = 0$, $|G^{(4)}(0)| < \infty$, $H(0) = 1 - G(0)$, $H'(0) = 1$ and $|H''(0)| < \infty$.

Proof. By defining $e_n = x_n - \alpha$ as the error of the iterative scheme in each iteration, applying the Taylor's series expansion and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9), \quad (4)$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!}, k \geq 1$.

Furthermore, we have

$$f'(x_n) = c_1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + 8c_8 e_n^7 + O(e_n^8). \quad (5)$$

Dividing (4) by (5) gives us

$$\frac{f(x_n)}{f'(x_n)} = e_n - \frac{c_2 e_n^2}{c_1} + \frac{2\left(c_2^2 - c_1 c_3\right) e_n^3}{c_1^2} \quad (6)$$

$$+ \frac{\left(-4c_2^3 + 7c_1 c_2 c_3 - 3c_1^2 c_4\right) e_n^4}{c_1^3} + \cdots + O(e_n^8).$$

Substituting (6) in y_n and writing the Taylor's expansion for $f(y_n)$, we obtain by considering $A(0) = 1$, $A'(0) = 0$, $A''(0) = 0$, and $|A^{(3)}(0)| < \infty$, respectively

$$y_n = \alpha + \frac{c_2 e_n^2}{c_1} + \frac{2\left(-c_2^2 + c_1 c_3\right) e_n^3}{c_1^2} \quad (7)$$

$$+ \left(\frac{4c_2^3 - 7c_1 c_2 c_3 + 3c_1^2 c_4}{c_1^3} - \frac{1}{6} A^{(3)}(0)\right) e_n^4 + \cdots + O(e_n^8),$$

$$f(y_n) = c_2 e_n^2 + \left(-\frac{2c_2^2}{c_1} + 2c_3\right) e_n^3 \quad (8)$$

$$+ \left(\frac{5c_2^3}{c_1^2} - \frac{7c_2 c_3}{c_1} + 3c_4 - \frac{1}{6} c_1 A^{(3)}(0)\right) e_n^4 + \cdots + O(e_n^8).$$

Moreover for the second step of (2), we find

$$z_n - \alpha = \frac{\left(2c_2^3 - c_1 c_2 c_3\right) e_n^4}{c_1^3} + \frac{\left(-6\left(5c_2^4 - 7c_1 c_2^2 c_3 + c_1^2 c_3^2 + c_1^2 c_2 c_4\right) + c_1^3 c_2 A^{(3)}(0)\right) e_n^5}{3c_1^4}$$

$$+ \frac{1}{12c_1^5}(372c_2^5 - 864c_1 c_2^3 c_3 + 2c_1^2 c_2^2(126c_4 - 13c_1 A^{(3)}(0)) \quad (9)$$

$$+ 2c_1^3 c_3(-42c_4 + 5c_1 A^{(3)}(0)) + c_1^2 c_2(360c_3^2 + c_1(-36c_5 + c_1 A^{(4)}(0))))e_n^6$$

$$+ \cdots + O(e_n^9).$$

Using (9), we now have

$$f(z_n) = \frac{(2c_2^3 - c_1 c_2 c_3)e_n^4}{c_1^2} + (-\frac{2(5c_2^4 - 7c_1 c_2^2 c_3 + c_1^2 c_3^2 + c_1^2 c_2 c_4)}{c_1^3} + \frac{1}{3} c_2 A^{(3)}(0))e_n^5$$

$$+ \frac{1}{12c_1^4}(372c_2^5 - 864c_1 c_2^3 c_3 + 2c_1^2 c_2^2(126c_4 - 13c_1 A^{(3)}(0))$$

$$+ 2c_1^3 c_3(-42c_4 + 5c_1 A^{(3)}(0)) + c_1^2 c_2(360c_3^2 + c_1(-36c_5 + c_1 A^{(4)}(0))))e_n^6$$

$$+ \cdots + O(e_n^9).$$

It is now easy to find

$$
\begin{aligned}
1 - \frac{f(y_n)}{f(x_n)} - \frac{f(z_n)}{f(x_n)} &= 1 - \frac{2c_2 e_n}{c_1} + \frac{(7c_2^2 - 4c_1 c_3)e_n^2}{c_1^2} \\
&+ \left(\frac{-26c_2^3 + 26c_1 c_2 c_3 - 6c_1^2 c_4}{c_1^3} + \frac{1}{3}A^{(3)}(0) \right)e_n^3 \\
&+ \left(\frac{93c_2^4}{c_1^4} - \frac{138c_2^2 c_3}{c_1^3} + \frac{24c_3^2 + 38c_2 c_4}{c_1^2} - \frac{8(3c_5 + c_2 A^{(3)}(0))}{3c_1} + \frac{1}{12}A^{(4)}(0) \right)e_n^4 \\
&+ \frac{1}{60c_1^5}(40(-3(153c_2^5 - 312c_1 c_2^3 c_3 + 99c_1^2 c_2^2 c_4 + 5c_1^2 c_2(24c_3^2 - 5c_1 c_5) \\
&+ 5c_1^3(-7c_3 c_4 + c_1 c_6)) + c_1^3(23c_2^2 - 8c_1 c_3)A^{(3)}(0)) - 45c_1^4 c_2 A^{(4)}(0) \\
&+ c_1^5 A^{(5)}(0))e_n^5 + O(e_n^6).
\end{aligned}
\tag{10}
$$

For the last step of (2), we obtain by using (7)-(10)

$$
\begin{aligned}
e_{n+1} &= \frac{c_2(-2c_2^2 + c_1 c_3)(-1 + G(0) + H(0))e_n^4}{c_1^3} \\
&+ \frac{(6(5c_2^4 - 7c_1 c_2^2 c_3 + c_1^2 c_3^2 + c_1^2 c_2 c_4)(-1 + G(0) + H(0))}{3c_1^4}e_n^5 \\
&+ \frac{3c_2^2(-2c_2^2 + c_1 c_3)G'(0) - c_1^3 c_2(-1 + G(0) + H(0))A^{(3)}(0)}{3c_1^4}e_n^5 \\
&+ \frac{1}{12c_1^5}(-12c_2^5(-31 + 25G(0) + 25H(0) - 16G'(0) + 4H'(0) + G''(0)) \\
&+ 6c_1 c_2^3 c_3(-144 + 134G(0) + 134H(0) - 42G'(0) + 8H'(0) + G''(0)) \\
&+ 2c_1^3 c_3(-1 + G(0) + H(0))(42c_4 - 5c_1 A^{(3)}(0)) \\
&+ 2c_1^2 c_2^2(6c_4(-21(-1 + G(0) + H(0)) + 2G'(0)) \\
&+ c_1(13(-1 + G(0) + H(0)) - 2G'(0))A^{(3)}(0)) - c_1^2 c_2(12c_3^2(-30 + 29G(0) \\
&+ 29H(0) - 4G'(0) + H'(0)) + c_1(-1 + G(0) + H(0))(-36c_5 + c_1 A^{(4)}(0))))e_n^6 \\
&+ \frac{1}{360c_1^6}(-60c_1^2 c_2^3(-6c_4(-100 + 90G(0) + 90H(0) - 33G'(0) \\
&+ 8H'(0) + G''(0)) + c_1(-56 + 46G(0) + 46H(0) - 21G'(0) + 8H'(0) + G''(0))A^{(3)}(0)) \\
&+ 120c_2^6(-222 + 60G(0) + 60H(0) - 213G'(0) + 96H'(0) + 33G''(0) - G^{(3)}(0)) \\
&- 60c_1 c_2^4 c_3(-1476 + 948G(0) + 948H(0) - 942G'(0) + 360H'(0) + 84G''(0) \\
&- G^{(3)}(0)) + 5c_1^3(-144c_3^3(9G(0) + 9H(0) - 2(5 + G'(0)) + H'(0)) + 4c_1(-1 + G(0) + H(0)) \\
&(-18c_4 + c_1 A^{(3)}(0))(-6c_4 + c_1 A^{(3)}(0)) - 15c_1 c_3(-1 + G(0) + H(0)) \\
&(-48c_5 + c_1 A^{(4)}(0))) + 15c_1^2 c_2^2(24c_3^2(-188 + 156G(0) + 156H(0) \\
&- 73G'(0) + 26H'(0) + 3G''(0)) + c_1(-672c_5(-1 + G(0) + H(0))) \\
&+ 72c_5 G'(0 + c_1(15(-1 + G(0) + H(0)) - 2G'(0))A^{(4)}(0))) - 6c_1^3 c_2(20c_3 \\
&(6c_4(-44 + 42G(0) + 42H(0) - 7G'(0) + 2H'(0)) + c_1(26 - 24G(0) \\
&- 24H(0) + 5G'(0) - 2H'(0))A^{(3)}(0))c_1(-1 + G(0) + H(0))(-240c_6 + c_1 A^{(5)}(0))))e_n^7 + O(e_n^8).
\end{aligned}
$$

Table 1 Some typical forms of the weight functions in (2), $\omega \in \mathbb{R}$

Methods	$A(\delta)$	$G(t)$	$H(\mu)$
Forms	$1 + \delta^3 + \omega\delta^5$	$t^2 + \omega t^4$	$1 + \mu + \omega\mu^2$

By choosing $|G(0)| < \infty$, $G'(0) = 0$, $G''(0) = 2$, $G^{(3)}(0) = 0$, $|G^{(4)}(0)| < \infty$, $H(0) = 1 - G(0)$, $H'(0) = 1$ and $|H''(0)| < \infty$, the error equations in the last step will be attained as comes next $e_{n+1} = \frac{1}{24c_1^7}c_2(-2c_2^2 + c_1c_3)(24c_1c_2^2c_3(7 - 2H''(0)) + 12c_1^2c_3^2(-2 + H''(0)) + 12c_1^2c_2(-2c_4 + c_1A^{(3)}(0)) + c_2^4(-120 + 48H''(0) + G^{(4)}(0)))e_n^8 + O(e_n^9)$. This shows that the iterative scheme (2) can attain at least eighth order of convergence. Hence, the proof is complete. $\square$

Remark 1. The suggested class (2) includes only four evaluations per full cycle to achieve the local convergence order eight. Hence, its classical efficiency index is $8^{\frac{1}{4}} \approx 1.682$, which is much better than the efficiency of its origin, i.e. $6^{\frac{1}{4}} \approx 1.565$ of the method (1).

Remark 2. The weight function $A(\delta)$ at the end of the first step gives a generality to this step. Since, its third derivative around zero has a clear effect on the final error equation (3). Such an effect provides a more generality to the proposed scheme along with furnishing a better condition for making with memory schemes out to (2).

3 Some optimal concrete methods

Some new methods can now be constructed using Theorem 1 and the class (2). Note that *some typical* forms of the weight functions satisfying Theorem 1 are illustrated in Table 1. By applying $A^{(3)}(0) = 0$, $H''(0) = 0$, and $G^{(4)}(0) = 120$ we have, for instance

$$
\begin{cases}
y_n = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \left(\frac{f(x_n)}{f'(x_n)}\right)^5\right), \\
z_n = y_n - \frac{f(y_n)}{f'(x_n)}\left(1 - \frac{f(y_n)}{f(x_n)}\right)^{-2}, \\
x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\frac{\left(1 + \left(\frac{f(y_n)}{f(x_n)}\right)^2 + 5\left(\frac{f(y_n)}{f(x_n)}\right)^4 + \frac{f(z_n)}{f(y_n)}\right)}{\left(1 - \frac{f(y_n)}{f(x_n)} - \frac{f(z_n)}{f(x_n)}\right)^2},
\end{cases}
\tag{11}
$$

with the following error equation

$$
e_{n+1} = -\frac{c_2\left(-2c_2^2 + c_1c_3\right)\left(-7c_2^2c_3 + c_1c_3^2 + c_1c_2c_4\right)}{c_1^6}e_n^8 + O(e_n^9).
\tag{12}
$$

By using $A^{(3)}(0) = 6$, $H''(0) = 0$, and $G^{(4)}(0) = 120$, we have

$$
\begin{cases}
y_n = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 + \left(\frac{f(x_n)}{f'(x_n)}\right)^3\right), \\
z_n = y_n - \frac{f(y_n)}{f'(x_n)}\left(1 - \frac{f(y_n)}{f(x_n)}\right)^{-2}, \\
x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\frac{\left(1 + \left(\frac{f(y_n)}{f(x_n)}\right)^2 + 5\left(\frac{f(y_n)}{f(x_n)}\right)^4 + \frac{f(z_n)}{f(y_n)}\right)}{\left(1 - \frac{f(y_n)}{f(x_n)} - \frac{f(z_n)}{f(x_n)}\right)^2},
\end{cases}
\tag{13}
$$

where its error relation reads

$$
e_{n+1} = \frac{c_2\left(-2c_2^2 + c_1 c_3\right)\left(3c_1^2 c_2 + 7c_2^2 c_3 - c_1\left(c_3^2 + c_2 c_4\right)\right)}{c_1^6} e_n^8 + O(e_n^9). \tag{14}
$$

Finally, by applying $A^{(3)}(0) = 0$, $H''(0) = 0$, and $G^{(4)}(0) = 0$, we attain

$$
\begin{cases}
y_n = x_n - \frac{f(x_n)}{f'(x_n)}\left(1 - \left(\frac{f(x_n)}{f'(x_n)}\right)^5\right), \\
z_n = y_n - \frac{f(y_n)}{f'(x_n)}\left(1 - \frac{f(y_n)}{f(x_n)}\right)^{-2}, \\
x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\frac{\left(1 + \left(\frac{f(y_n)}{f(x_n)}\right)^2 + \frac{f(z_n)}{f(y_n)}\right)}{\left(1 - \frac{f(y_n)}{f(x_n)} - \frac{f(z_n)}{f(x_n)}\right)^2},
\end{cases}
\tag{15}
$$

where its error equation is

$$
e_{n+1} = \frac{c_2\left(2c_2^2 - c_1 c_3\right)\left(5c_2^4 - 7c_1 c_2^2 c_3 + c_1^2 c_3^2 + c_1^2 c_2 c_4\right)}{c_1^7} e_n^8 + O(e_n^9). \tag{16}
$$

4 Numerical examples

In this section, we check the effectiveness of the iterative class (2) by choosing their members (11), (13), and (15). Due to this, we have compared them with the following scheme, described in [9]

$$
\begin{cases}
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n = x_n - \frac{f(x_n)}{f'(x_n)}\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)}, \\
x_{n+1} = u_n - \frac{3(\alpha+\beta)(u_n - z_n)}{\gamma(u_n - z_n) + \alpha(y_n - x_n) + \beta(z_n - x_n)}\frac{f(z_n)}{f'(x_n)},
\end{cases}
\tag{17}
$$

where $\alpha, \beta, \gamma \in \mathbb{R}, \alpha+\beta \neq 0$ and $u_n = z_n - \left(\frac{f(x_n) - f(y_n)}{f(x_n) - 2f(y_n)} + \frac{1}{2}\frac{f(z_n)}{f(y_n) - 2f(z_n)}\right)^2 \frac{f(z_n)}{f'(x_n)}$. We will choose $\alpha = \beta = \gamma = 1$, in Table 3.

The following optimal three-step eighth-order iteration [11] will also be used

$$
\begin{cases}
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n = y_n - \frac{f(y_n)}{f'(x_n)}\{1 + 2\frac{f(y_n)}{f(x_n)} + 4(\frac{f(y_n)}{f(x_n)})^2\}, \\
x_{n+1} = z_n - \{(\frac{f(y_n)}{f(x_n)})^4 + (\frac{f(y_n)}{f(x_n)})^5\}\frac{f(z_n)}{f'(x_n)} \\
\qquad\quad - \{1 + \frac{f(z_n)}{f(x_n)} + (\frac{f(z_n)}{f(x_n)})^2\}\frac{f[x_n, y_n]f(z_n)}{f[x_n, z_n]f[y_n, z_n]}.
\end{cases}
\tag{18}
$$

For comparisons, we have also used the following methods presented in [14]

$$
\begin{cases}
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n = y_n - \frac{f(y_n)}{f'(x_n)}\{\frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)}\}, \\
x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\{\frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)}\}\{1 + 2\frac{f(z_n)}{f(x_n)} + \frac{f(z_n)}{f(y_n)} \\
+(\frac{f(z_n)}{f(y_n)})^2 + (\frac{f(y_n)}{f(x_n)})^2 + 4(\frac{f(y_n)}{f(x_n)})^3 + 12(\frac{f(y_n)}{f(x_n)})^4 - \frac{f(z_n)}{f'(x_n)} - (\frac{f(y_n)}{f'(x_n)})^2\},
\end{cases}
\tag{19}
$$

and

$$
\begin{cases}
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n = y_n - \frac{f(y_n)}{f'(x_n)}\{\frac{f(x_n)}{f(x_n)-2f(y_n)}\}, \\
x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}\{\frac{f(x_n)}{f(x_n)-2f(y_n)}\}\{1 + 2\frac{f(z_n)}{f(x_n)} + \frac{f(z_n)}{f(y_n)} + (\frac{f(y_n)}{f(x_n)})^2 \\
+2(\frac{f(y_n)}{f(x_n)})^3 - \frac{f(z_n)}{f'(x_n)} + (\frac{f(y_n)}{f'(x_n)})^3\}.
\end{cases}
\tag{20}
$$

The nonlinear test functions are furnished in Table 2.

Table 2 Test functions and their zeros

Test Nonlinear Functions	Simple Zeros
$f_1(x) = 10xe^{-x^2} - 1$	$\alpha_1 \approx 1.6796306104284499406749203388379703$
$f_2(x) = x^5 + x^4 + 4x^2 - 15$	$\alpha_2 \approx 1.3474280989683049815067153807148212$
$f_3(x) = xe^{x^2} - (\sin(x))^2 + 3\cos(x) + 5$	$\alpha_3 \approx -1.2076478271309189270094167585835608$
$f_4(x) = x^4 + \sin(\frac{\pi}{x^2}) - 5$	$\alpha_4 = \sqrt{2}$
$f_5(x) = x^2 e^x - \sin(x)$	$\alpha_5 = 0$
$f_6(x) = (\sin(x) - \frac{\sqrt{2}}{2})^2(x+1)$	$\alpha_6 = -1$
$f_7(x) = 11x^{11} - 1$	$\alpha_7 \approx 0.8041330975036643237414634983732973$
$f_8(x) = \sin(3x) + x\cos(x)$	$\alpha_8 \approx 1.1977695352162711659385794729509898$

The results of comparisons are given in Table 3 in terms of the number significant digits for each test function after 3 iterations (with the same Total Number of Evaluations; TNE=12), that is, e.g. $0.1e - 680$ shows that the absolute value of the given nonlinear function (f_1) after three iterations is zero up to 680 decimal places. For numerical comparisons, the stopping criterion is $|f(x_n)| < 1.E - 1000$. We used MATLAB 7.5 for numerical computations and variable precision arithmetics with 1100 digits of mantissa.

Although the results shown in Table 3 are not as good as would be desirable for methods (11), (13) and (15), the dynamical study that we show below indicates that there are important aspects in which new methods improve other eighth-order known methods. We are going to study the basins of attraction of the different methods for finding the roots of unity.

There are various criteria involved in choosing an iterative method to approximate the root of an equation. These include the initial value problem, the rate of convergence and the complexity of the calculation. In order to investigate these dynamics with some higher order methods we have improved the method (1) and now study their complex dynamics.

Table 3 Comparison of different methods for solving test functions with $TNE = 12$.

f	Guess		(17)	(18)	(19)	(20)	(11)	(13)	(15)
f_1	1.72	$\|f\|$	0.1e-680	0.6e-663	0.3e-765	0.5e-659	0.2e-660	0.6e-700	0.3e-654
	1		0.3e-118	0.4e-221	0.3e-231	0.4e-167	0.5e-213	0.8e-53	0.6e-86
	1.5		0.2e-391	0.4e-473	0.2e-438	0.1e-376	0.2e-375	0.5e-366	0.8e-361
f_2	1	$\|f\|$	0.1e-190	-	0.1e-49	0.2e-127	0.4e-80	0.2e-132	0.9e-197
	1.1		0.5e-278	0.1e-81	0.2e-168	0.7e-223	0.7e-165	0.2e-197	0.5e-299
	1.8		0.1e-213	0.1e-215	0.2e-227	0.5e-194	0.1e-195	0.2e-189	0.4e-187
f_3	-1.1	$\|f\|$	0.1e-516	0.6e-292	0.3e-322	0.6e-423	0.1e-325	0.1e-325	0.6e-433
	-1.5		0.1e-253	0.4e-238	0.5e-187	0.1e-216	0.2e-253	0.1e-247	0.2e-205
	-1		0.3e-379	0.1e-73	0.8e-151	0.5e-269	0.4e-158	0.1e-143	0.4e-254
f_4	1.3	$\|f\|$	0.3e-470	0.5e-439	0.4e-439	0.1e-507	0.3e-504	0.1e-459	0.3e-528
	1		0.3e-232	0.2e-269	0.6e-263	0.2e-289	0.2e-199	0.2e-146	0.5e-227
	1.6		0.2e-469	0.1e-363	0.3e-394	0.8e-447	0.3e-376	0.3e-351	0.5e-430
f_5	0.1	$\|f\|$	0.8e-360	0.5e-350	0.1e-393	0.1e-343	0.6e-349	0.1e-357	0.1e-338
	0.5		0.6e-400	0.1e-317	0.1e-385	0.3e-366	0.2e-342	0.8e-342	0.8e-382
	-0.1		0.3e-476	0.5e-482	0.3e-476	0.8e-457	0.4e-441	0.6e-438	0.2e-436
f_6	-0.8	$\|f\|$	0.2e-294	0.1e-212	0.9e-398	0.5e-274	0.2e-248	0.7e-242	0.3e-288
	-1.2		0.1e-441	0.1e-421	0.4e-421	0.2e-402	0.1e-385	0.2e-380	0.5e-381
	-0.9		0.2e-483	0.1e-498	0.1e-553	0.6e-469	0.2e-500	0.1e-454	0.2e-457
f_7	0.81	$\|f\|$	0.5e-752	0.5e-695	0.4e-737	0.8e-717	0.4e-697	0.5e-697	0.5e-716
	0.79		0.5e-546	0.3e-461	0.2e-485	0.3e-503	0.1e-460	0.2e-460	0.5e-519
	1		0.5e-76	0.3e-62	0.1e-106	0.3e-65	0.5e-64	0.6e-64	0.2e-61
f_8	1	$\|f\|$	0.2e-372	0.5e-589	0.1e-435	0.5e-434	0.2e-378	0.2e-605	0.7e-374
	0.8		0.1e-150	0.1e-91	0.2e-160	0.7e-156	0.8e-159	0.4e-193	0.2e-116
	1.5		0.1e-188	0.2e-102	0.9e-194	0.6e-175	0.2e-147	0.1e-121	0.3e-224
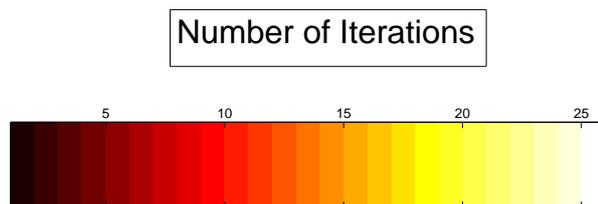
Fig. 1 HOT Colormap in MATLAB

Note that a point z_0 belongs to the Julia set if and only if dynamics in a neighborhood of z_0 displays sensitive dependence on the initial conditions, so that nearby initial conditions lead to wildly different behavior after a number of iterations.

     (a) (11)   (b) (13)   (c) (15)
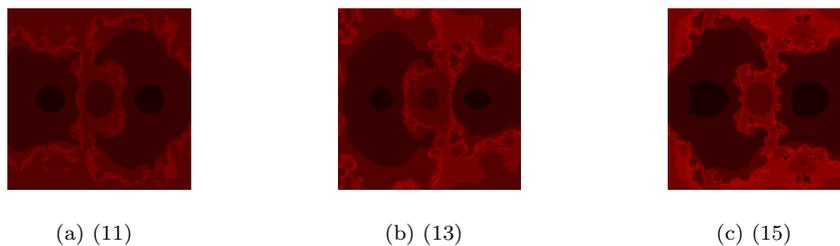
Fig. 2 Basins of attraction of methods (11), (13) and (15) for $f(x) = x^2 - 1$

     (a) (17)   (b) (18)

Fig. 3 Basins of attraction of methods (17) and (18) for $f(x) = x^2 - 1$

     (a) (19)   (b) (20)

Fig. 4 Basins of attraction of methods (19) and (20) for $f(x) = x^2 - 1$

5 Fractal pictures for the basins of attractions of the eighth order methods

We consider the polynomial $f(x) = x^r - 1$, $x \in \mathbb{C}$ for finding the rth roots of unity. The rth roots of unity are given by

$$\alpha_j = \cos\left(\frac{2\pi(j-1)}{r}\right) + i\sin\left(\frac{2\pi(j-1)}{r}\right), \ j = 1, 2...r.$$

The basin of attraction corresponding to a zero α_j of the polynomial $f(x)$ is the set of all starting points x_0 which are attracted to α_j. We use a similar like technique as in [23] to generate the basins of attraction.

    In our numerical experiments, we take a square $\mathbb{R} \times \mathbb{R} = [-2, 2] \times [-2, 2]$ of 256×256 points and we apply the eighth order iterative method starting in every x_0 in the square.

If the sequence generated by iterative method attempt a zero α_j of the polynomial with a tolerance $|f(x_k)| < 1e - 5$ and a maximum of 25 iterations, we decide that x_0 is in the basin of attraction of these zero.

If the iterative method starting in x_0 reaches a zero in n iterations ($n \leq 25$), then we mark this point x_0 with a color depending on n with colormap hot in MATLAB as shown in Figure 1. If $n > 25$, we conclude that the starting point has diverged and we assign a white color. Let us denote μ be the mean number of iterations for the converging points and N_d the number of diverging points. We choose $r = 2, 3, 7, 20$ for our numerical experiments.

(a) (11)   (b) (13)   (c) (15)
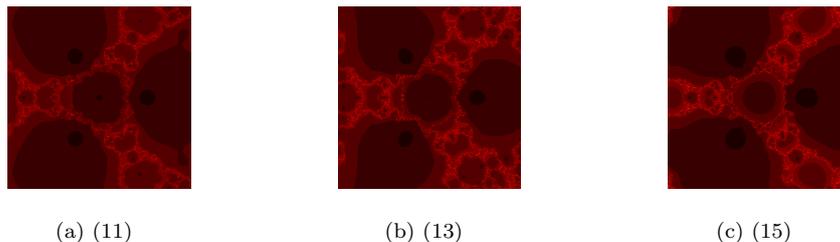
Fig. 5 Basins of attraction of methods (11), (13) and (15) for $f(x) = x^3 - 1$

(a) (17)   (b) (18)

Fig. 6 Basins of attraction of methods (17) and (18) for $f(x) = x^3 - 1$

(a) (19)   (b) (20)

Fig. 7 Basins of attraction of methods (19) and (20) for $f(x) = x^3 - 1$

| (a) (11) | (b) (13) | (c) (15) |
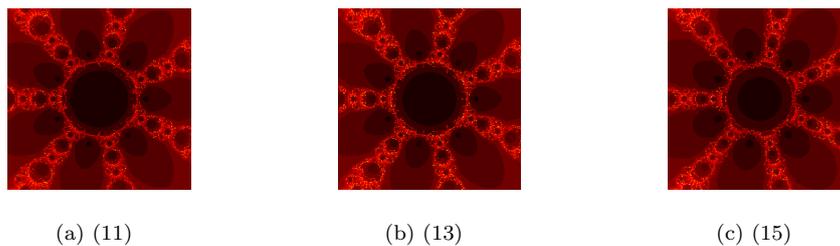
Fig. 8 Basins of attraction of methods (11), (13) and (15) for $f(x) = x^7 - 1$

| (a) (17) | (b) (18) |

Fig. 9 Basins of attraction of methods (17) and (18) for $f(x) = x^7 - 1$

| (a) (19) | (b) (20) |

Fig. 10 Basins of attraction of methods (19) and (20) for $f(x) = x^7 - 1$

Table 4 gives a comparison of μ and N_d of the eighth order methods for the polynomials $f(x) = x^r - 1$, $r = 2, 3, 7, 20$. We also show a column with the percentage of white points for each method. We can observe that all methods (except (18)) are globally convergent for quadratic polynomials, being the mean number of iterations similar in all cases. Proposed methods are also globally convergent for cubic polynomials as well as (19) and (20). Although, the global convergence is missed for higher degree of the polynomials, the number of divergent points is much lower in new methods than in known ones.

(a) (11)　　　　　　　　(b) (13)　　　　　　　　(c) (15)
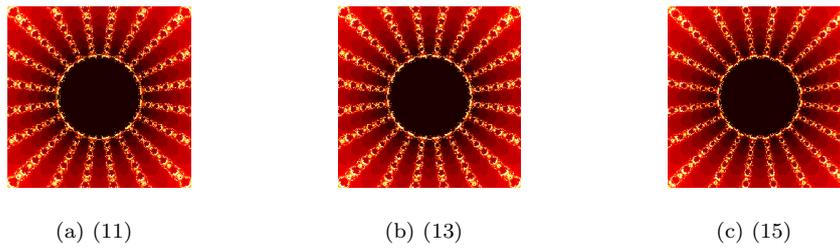
Fig. 11 Basins of attraction of methods (11), (13) and (15) for $f(x) = x^{20} - 1$

(a) (17)　　　　　　　　　　　(b) (18)
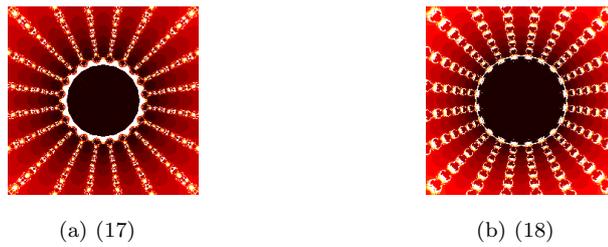
Fig. 12 Basins of attraction of methods (17) and (18) for $f(x) = x^{20} - 1$

(a) (19)　　　　　　　　　　　(b) (20)

Fig. 13 Basins of attraction of methods (19) and (20) for $f(x) = x^{20} - 1$

Figures 2 (a)-(c) show the basins of attraction for our new methods (11), (13) and (15), respectively for the quadratic polynomial. The dark regions show the roots $\alpha_1 = -1$ and $\alpha_2 = 1$. This shows the rapid convergence of eighth order methods when the starting points are chosen close to the root. There are no diverging points for the 3 methods and all methods converge in less than 10 iterations as shown by the dark intensity of the colors. Figures (3) (a)-(b) and (4) (a)-(b) show the basins of attraction for the methods (17)-(20), respectively. We observe that there are some diverging points for method (18) which is based on divided difference methods and is the worst method. We can observe the beautiful pictures produced by each method. The most efficient method with the lowest mean iteration number and greatest basins of attraction for the quadratic polynomial is (17). These figures confirm graphically the numerical results shown in Table 4.

Table 4 Comparison of μ, N_d and P_d of the eighth order methods for the polynomials $f(x) = x^r - 1$, $r = 2, 3, 7, 20$.

Method	r = 2			r = 3			r = 7			r = 20		
	N_d	P_d	μ	N_d	P_d	μ	N_d	P_d	μ	N_d	P_d	μ
(11)	0	0	2.61	0	0	2.63	0	0	3.50	1816	2.8	5.91
(13)	0	0	2.92	0	0	2.88	2	0.003	3.63	2097	3.2	5.92
(15)	0	0	3.09	0	0	2.72	4	0.006	3.51	1630	2.5	5.79
(17)	0	0	2.34	66	0.1	3.12	6382	9.7	4.47	5380	8.2	5.34
(18)	772	1.18	3.22	10534	16.1	4.58	8768	13.4	4.57	7140	10.9	5.98
(19)	0	0	2.58	0	0	3.08	72	0.1	3.86	2792	4.3	4.93
(20)	0	0	2.53	0	0	3.16	160	0.2	4.35	3418	5.2	5.87

Figures 5, 6 and 7 show the basins of attractions for the cubic polynomial. There are some diverging points for method (17) but a lot for method (18). The other methods are globally convergent. The best method in terms of lowest mean iteration number (see Table 4) and largest basins of attraction is method (11).

As the degree of the polynomial increases from 3 to 7, the methods have difficulties and their mean iteration number increases. Our methods have very little diverging points compared to the other ones. Method (11) is globally convergent. Figures 8, 9 and 10 show the basins of attractions for finding the 7th roots of unity. The fractal of the methods (11), (13) and (15) are almost similar. We also observe in Figure 9 methods (17) and (18) have a lot of white (divergent) points which is confirmed by Table 4. The method (11) is again the most efficient method.

For a large $r = 20$, the complexity of the fractals increases for the methods as shown as in Figures 11, 12 and 13. The fractals are almost similar and the number of diverging points are high for all methods. Method (15) has the lowest number of diverging points whereas method (19) has the lowest mean iteration number of the converging points. On the whole, we see that our methods are better than the other methods in the literature.

6 Conclusion

Our main goal and motivation in constructing iterative methods for solving nonlinear equations is to attain as high as possible order of convergence with minimal computational cost. The most efficient existing root solvers are based on multi-point iterations, first studied in the Traub's book [24]; and recently have been analyzed in the Ph.D. Thesis [25].

A novel eighth-order class of derivative-involved methods without memory including three steps and four functional evaluations per full cycle to reach the

optimal convergence order 8 have developed by using the method of Chun and Neta (1), and weight function technique. The contributed class has arrived at the optimal efficiency index $8^{\frac{1}{4}} \approx 1.682$. The convergence analysis of (2) has been made. Finally, numerical examples were employed to support the theory presented in this paper. The basins of attraction of the new methods have also been provided and they have shown that the new methods can compete with other optimal eighth order methods in the literature.

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