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López Pellicer, M.; JERZY KAKOL; O. Okunev (2014). Compact covers andf unction spaces. Journal of Mathematical Analysis and Applications. 411(1):372-380. doi:10.1016/j.jmaa.2013.09.046

http://dx.doi.org/10.1016/j.jmaa.2013.09.046.

The final publication is available at

http://dx.doi.org/10.1016/j.jmaa.2013.09.046

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COMPACT COVERS AND FUNCTION SPACES

J. KĄKOL, M. LOPEZ-PELLICER, O. OKUNEV

Abstract. For a Tychonoff space X, we denote by $C_p(X)$ and $C_c(X)$ the space of continuous real-valued functions on X equipped with the topology of pointwise convergence and the compact-open topology respectively.

Providing a characterization of the Lindelöf Σ-property of X in terms of $C_p(X)$, we extend Okunev's results by showing that if there exists a surjection from $C_p(X)$ onto $C_p(Y)$ (resp. from $L_p(X)$ onto $L_p(Y)$) that takes bounded sequences to bounded sequences, then υY is a Lindelöf Σ-space (respectively K-analytic) if υX has this property. In the second part, applying Christensen's theorem, we extend Pelant's result by proving that if X is a separable completely metrizable space and Y is first countable, and there is a quotient linear map from $C_c(X)$ onto $C_c(Y)$, then Y is a separable completely metrizable space. We study also the non-separable case, and consider a different approach to the result of J. Baars, J. de Groot, J. Pelant and V. Valov, which is based on the combination of two facts: Complete metrizability is preserved by ℓ_p-equivalence in the class of metric spaces (J. Baars, J. de Groot, J. Pelant). If X is completely metrizable and ℓ_p-equivalent to a first countable Y, then Y is metrizable (V. Valov). Some additional results are presented.

1. Introduction

All spaces considered in this article are assumed to be completely regular and Hausdorff. We use terminology and notation as in [14]. We say that a set A in a space X is *functionally bounded in* X if every continuous real-valued function on X is bounded on A. A space X is a *μ-space* if every closed functionally bounded subspace of X is compact. A *Polish space* is a separable completely metrizable space. The symbol ω denotes the smallest infinite ordinal (so ω is the set of all non-negative integers).

We denote by $C_p(X)$ and $C_c(X)$ the spaces of continuous real-valued functions on X endowed with the topology of pointwise convergence and the compact open topology respectively. $L_p(X)$ is the topological dual of $C_p(X)$ endowed with the weak *-topology. We assume that X is a subspace of $L_p(X)$ by virtue of the standard embedding $x \mapsto \hat{x}$ where $\hat{x}(f) = f(x)$ for each $f \in C_p(X)$.

The research was supported for the first named author by National Center of Science, Poland, grant no. N N201 605340, and for the first and second named authors by Generalitat Valenciana, Conselleria d'Educació i Esport, Spain, Grant PROMETEO/2013/058.

Recall that Nagata's theorem states that if the topological rings $C_p(X)$ and $C_p(Y)$ are topologically isomorphic, then X and Y are homeomorphic, see [6]. This suggests the following problem, see [2].

Two spaces X and Y are said to be *t-equivalent* (*ℓ_p-equivalent*) if the spaces $C_p(X)$ and $C_p(Y)$ are homeomorphic (linearly homeomorphic). We say that a topological property $\mathcal{P}$ is preserved by t-equivalence (ℓ_p-equivalence) if whenever two spaces X and Y are t-equivalent (ℓ_p-equivalent) and X has the property $\mathcal{P}$, Y has the property $\mathcal{P}$ too.

What topological properties are preserved by the relations of t-equivalence and ℓ_p-equivalence?

Clearly, a property $\mathcal{P}$ is preserved by the relation of t-equivalence (ℓ_P-equivalence) if and only if there is a *dual* topological (linear topological) property $\mathcal{Q}$ such that X has $\mathcal{P}$ if and only if $C_p(X)$ has $\mathcal{Q}$; in different words, if $\mathcal{P}$ "admits a description in terms of the (linear) topological structure of $C_p(X)$".

We will say that two spaces X and Y are *ℓ_c-equivalent* if the spaces $C_c(X)$ and $C_c(Y)$ are linearly homeomorphic. Note that if X and Y are ℓ_p-equivalent and X is a μ-space, then the spaces X and Y are also ℓ_c-equivalent, see [5]. It is well known (by a combination of Milyutin's and Pestov's results, see [5, Theorem 3]), that $[0,1]$ and $[0,1]\times[0,1]$ are ℓ_c-equivalent but not ℓ_p-equivalent. On the other hand, if X and Y are ℓ_p-equivalent and X is *Dieudonné complete* (in particular, if X is paracompact or realcompact), then X and Y are ℓ_c-equivalent, see [5, Theorem 1].

We say that a space X *ℓ_c-covers* a space Y if there is a continuous open linear mapping from $C_c(X)$ onto $C_c(Y)$. Clearly, if X and Y are ℓ_c-equivalent, then each of the two ℓ_c-covers the other.

There are many known results about preservation and non-preservation of various topological properties by t-equivalence and ℓ_p-equivalence; see, e.g., [5], [6], [8], [24], [32]. For example, metrizability, local compactness, the countability of weight, normality and paracompactness are not ℓ_p-invariant. On the other hand, hemicompactness, the property of being an $\aleph_0$-space, the Lindelöf Σ-property, K-analyticity and analyticity are preserved by ℓ_p-equivalence.

We denote by $\mathbb{P}$ the space ω^ω endowed with the Tychonoff product topology (with all the factors discrete). We equip the space $\mathbb{P}$ with the natural partial order: $p \le q$ if and only if $p(n) \le q(n)$ for all $n \in \omega$. For an element p of $\mathbb{P}$ and a natural number k we denote by $p|k$ the finite sequence $(p(1),\dots,p(k))$. Given a finite sequence $\sigma = (\sigma_1,\dots,\sigma_n)$ of natural numbers, we denote by $W(\sigma)$ the set $\{p \in \mathbb{P} : p|n = \sigma\}$. Clearly, for every $p \in \mathbb{P}$, the family of sets $\{W(p|n) : n \in \omega\}$ is a base of open neighborhoods of p in $\mathbb{P}$.

A family of subspaces $\mathcal{R} = \{A_p : p \in \mathbb{P}\}$ of a space X is called a *resolution* of X if it covers X and $A_p \subset A_q$ whenever $p \le q$. We say that a resolution $\mathcal{R}$ is *compact* if each element A_p of $\mathcal{R}$ is compact. If X is a topological vector space, we say that a resolution $\mathcal{R}$ of X is *bounded* if each element of $\mathcal{R}$ is bounded in X (that is, absorbed by any neighbourhood of zero). A resolution $\mathcal{R}$ *swallows compact sets* if every compact subspace of X is contained in some element of $\mathcal{R}$.

As usual, a set-valued mapping $T\colon X \to Y$ is called *compact valued* if the set $T(x)$ is compact for every $x \in X$, and is *upper-semicontinuous* if for every open set V in Y, the set $\{x \in X : T(x) \subset V\}$ is open. We abbreviate "compact-valued upper semicontinuous" as "usco". For a set $A \subset X$ we denote $T(A) = \bigcup\{T(x) : x \in A\}$, and we say that T is *onto* Y if $T(X) = Y$. We denote the family of all compact subspaces of a space X by $\mathcal{K}(X)$ (so compact-valued mappings to X are the same as functions to $\mathcal{K}(X)$).

In Section 2 we find a characterization of Lindelöf Σ-property of υX in terms of the linear topological structure of $C_p(X)$ and use it to show that if υX is a Lindelöf Σ-space or a K-analytic space and there exists a surjection from $L_p(X)$ onto $L_p(Y)$ that takes bounded sequences to bounded sequences, then υY is a Lindelöf Σ-space (respectively, K-analytic); we prove a similar statement for the Lindelöf Σ-property of υX and mappings between $C_p(X)$ and $C_p(Y)$. This supplements some earlier results of Okunev [22].

A. Arhangel'skii asks in [4, Problem 20] if *a first countable space Y which is ℓ_p-equivalent to a metrizable space X must also be metrizable.* In [8, Theorem 3.3] J. Baars, J. de Groot and J. Pelant proved that *complete metrizability is preserved by ℓ_p-equivalence in the class of metrizable spaces.* They also gave an alternative proof for separable metrizable spaces by using Christensen's Theorem (1 below), [8, Theorem 5.1, Theorem 5.3]. Later on, Valov proved, using the results in [30], that the answer to the Arhangel'skii's problem is positive for Čech-complete spaces Y, see [32, Corollary 4.6]. The combination of the above facts yields: *The property of being a completely metrizable space is preserved by the ℓ_p-equivalence for spaces satisfying the first axiom of countability.*

In Section 3 of this article we discuss the preservation of complete metrizability by ℓ_c-equivalence. We give a different proof of the preservation of complete metrizability in the case of ℓ_c-equivalent spaces X and Y where X is Polish and Y is first-countable, and discuss the non-separable case. Note however that from Valov's quite technical [32, Corollary 4.6] it follows that if X is completely metrizable, Y is paracompact first countable, and there is a continuous surjection from $C_c(X)$ onto $C_c(Y)$, then Y is completely metrizable. Our approach is different and uses a property of $C_c(X)$ which is preserved by linear open maps and characterizes spaces X with a compact resolution swallowing compact sets (Theorem 14, Corollary 15). The importance of this concept stems from the following deep result of J. P. R. Christensen [12, Theorem 3.3].

Theorem 1. *A metrizable space X is Polish if and only if X admits a compact resolution swallowing compact sets.*

We partially extend Theorem 1 to the non-separable case, see Proposition 16 below, and we apply this extension to show that if a space Y is ℓ_c-covered by a completely metrizable space of weight κ, then Y admits a compact cover swallowing compact sets similar to a resolution. In particular, if Y is of pointwise countable type and X is Polish, then Y is separable and completely metrizable (Corollary 24). The separable case will be deduced from Theorem 21 showing that if X is an $\aleph_0$-space and there is an open continuous linear mapping from $C_c(X)$ onto $C_c(Y)$, then Y is an $\aleph_0$-space. Indeed, the latter fact and Theorem 1 apply to prove that if for a separable completely metrizable space X there exists a quotient linear map from $C_c(X)$ onto $C_c(Y)$ and Y is first-countable, then Y is separable and completely metrizable.

Motivated by the argument providing Corollary 23, we present another proof (quite different from the one in [32, Proposition 4.8]) showing that if Y is a wq-space ℓ_c-covered by a locally compact μ-space X, then Y is a locally compact μ-space. Our approach uses the concept of Baire-likeness of topological vector spaces. This extends a result of [4] or [5, Theorem 12] with the same conclusion for ℓ_p-equivalent spaces X and Y. McCoy and Ntantu [19] proved a similar result for ℓ_p-equivalent spaces X and Y where X is separable, metrizable and locally compact, and Y is first-countable. The same conclusion was obtained in [16] for locally compact paracompact spaces X.

Some interesting part of the paper deals with compact $\mathbb{P}(\kappa)$ resolutions swallowing compact sets (Theorem 14) to provide a nice characterization for a large class of locally convex spaces $C_c(X)$ to have a $\mathfrak{G}_\kappa$-basis. This leads to Proposition 16 stating that if X is a completely metrizable space of weight κ, then $C_c(X)$ has a $\mathfrak{G}_\kappa$-basis. The concept of a $\mathfrak{G}_\kappa$-basis seems to be a good tool to study a non-separable version (still an open question) of a remarkable Christensen theorem several times mentioned in the paper, see Theorem 17.

Recall that a space Y is *of pointwise countable type* [6] if for every $y \in Y$ there exists a compact set K that contains y and has a countable base of neighbourhoods in Y. The class of spaces of pointwise countable type contains, in particular, all first countable spaces and all Čech-complete spaces.

A space Y is a *wq-space* [32] if for each $y \in Y$ there is a sequence $\{U_n : n \in \omega\}$ of open neighbourhoods of y such that if $y_n \in U_n$ for each $n \in \omega$ then $\{y_n : n \in \omega\}$ is functionally bounded in Y. It is well known (and is easy to verify) that every space of pointwise countable type is a wq-space; moreover, every wq- and μ-space is of pointwise countable type.

2. ℓ_p-EQUIVALENCE AND THE LINDELÖF Σ-PROPERTY OF υX

A space X is a *Lindelöf Σ-space* if there is a compact-valued upper semi-continuous mapping from a separable metrizable space onto X; since every separable metrizable space is a continuous image of a subspace of $\mathbb{P}$, we can characterize Lindelöf Σ-spaces as images under usco mappings of subspaces of $\mathbb{P}$.

If there is a compact-valued upper semicontinuous mapping from $\mathbb{P}$ onto X, then X is called *K-analytic*, see [26]. A space X is called *quasi-Souslin* if there exists a set-valued mapping T from $\mathbb{P}$ onto X such that if a sequence p_n converges to a point p in $\mathbb{P}$ and for each $n \in \omega$, $x_n \in T(p_n)$, then the sequence $(x_n : n \in \omega)$ has a limit point in $T(p)$, see [31, I.4.2]. It is easy to verify that a space X is quasi-Souslin if and only if there is a countably compact-valued upper semicontinuous mapping from $\mathbb{P}$ onto X.

A topological vector space E is called *web-bounded* if there exist an $S \subset \mathbb{P}$ and a set-valued mapping A from S onto E such that whenever $p \in S$ and $x_k \in A(W(p|k))$, the sequence $\{x_k : k \in \omega\}$ is bounded in E (that is, is absorbed by any neighbourhood of zero in E). Clearly, every topological vector space with a bounded resolution, in particular, every vector quasi-Souslin and every vector Lindelöf Σ-space, is web-bounded. Note that every linear subspace of a web-bounded space is web-bounded, and every image of a web-bounded space under a continuous linear mapping is web-bounded.

The next statement is Theorem 3.5 in [23].

Proposition 2. *The space υX is a Lindelöf Σ-space if and only if there exists a Lindelöf Σ-space Z such that $C_p(X) \subset Z \subset \mathbb{R}^X$.*

As usual, we denote by E' the weak dual space of E. Clearly, E' is a subspace of $\mathbb{R}^E$.

Theorem 3. *Let E be a locally convex space. If E is web-bounded, then there is a linear Lindelöf Σ-space Z such that $E' \subset Z \subset \mathbb{R}^E$. In particular, E' is web-bounded.*

Proof. Let $S \subset \mathbb{P}$ and $A \colon S \to E$ be a set-valued mapping as in the definition of a web-bounded space. Put

$$Z = \{\phi \in \mathbb{R}^E : \text{for each } p \in S \text{ there is an } n \in \omega \text{ such that } \phi \text{ is bounded on } A(W(p|n))\}.$$

Let us verify that $E' \subset Z$. Let $\phi \in E'$, and suppose $\phi \notin Z$. Then for some $p \in S$ and every $n \in \omega$ there is a point $x_n \in A(W(p|n))$ such that $|\phi(x_n)| > n$. Then the sequence $\{x_n : n \in \omega\}$ is unbounded in E, in contradiction with the definition of a web-bounded space.

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ be the natural two-point compactification of $\mathbb{R}$. For every finite sequence σ of naturals of length n put

$$Z_\sigma = \{\phi \in \bar{\mathbb{R}}^E : \phi(A(W(\sigma))) \subset [-n, n]\}.$$

The set Z_σ is closed in the compact space $\bar{\mathbb{R}}^E$, hence is compact. Furthermore, the family $\{Z_\sigma : \sigma \text{ is a finite sequence of naturals}\}$ is countable, and $Z = \bigcap_{p\in S}\bigcup_{n\in\omega} Z_{p|n}$. It follows that for every $\phi \in Z$ and $\psi \in \bar{\mathbb{R}}^E \setminus Z$ there are $p \in S$ and $n \in \omega$ such that $\phi \in Z_{p|n}$ and $\psi \notin Z_{p|n}$. By Proposition IV.9.2 in [6], Z is a Lindelöf Σ-space.

Obviously, Z is a linear subspace of $\mathbb{R}^E$, so Z is web-bounded. Since E' is a linear subspace of Z, E' is web-bounded. □

We say that a locally convex space E is *weak* if its topology coincides with the $*$-weak topology. It is well-known that if a space E is weak, then $(E')'$ is linearly homeomorphic to E (see, e.g., IV.1.2 in [27]).

Corollary 4. *Let E be a weak space. Then E is web-bounded if and only if E' is web-bounded.*

In particular, $C_p(X)$ is a weak space.

Corollary 5. *The space $C_p(X)$ is web-bounded if and only if $L_p(X)$ is web-bounded.*

The following lemma is a part of [18, Theorem 9.15]; for the sake of completeness we give the proof here.

Lemma 6. *The space $C_p(X)$ is web-bounded if and only if υX is a Lindelöf Σ-space.*

Proof. If υX is a Lindelöf Σ-space, then $L_p(\upsilon X)$ is a Lindelöf Σ-space (because υX is a Hamel base for $L_p(\upsilon X)$, so $L_p(\upsilon X)$ is a countable union of continuous images of products of finite powers of υX with compact spaces; see [6, Proposition 0.5.13]), hence $L_p(\upsilon X)$ is web-bounded. It follows by Corollary 5 that the space $C_p(\upsilon X)$ is web-bounded. The restriction mapping $r_X\colon C_p(\upsilon X) \to C_p(X)$ (defined by the rule $r_X(f) = f|X$ for all $f \in C_p(\upsilon X)$) is linear, continuous and onto, so the space $C_p(X)$ is web-bounded.

Conversely, assume that $C_p(X)$ is web-bounded. By Theorem 3, there is a Lindelöf Σ-space Z such that $L_p(X) \subset Z \subset \mathbb{R}^{C(X)}$.

From the fact that X is C-embedded in $\mathbb{R}^{C(X)}$ it follows that υX is homeomorphic to the closure of of X in $\mathbb{R}^{C(X)}$. Since $X \subset Z \subset \mathbb{R}^{C(X)}$, and Z is Lindelöf, υX is homeomorphic to a closed subspace of Z, and therefore is a Lindelöf Σ-space. □

Corollary 7. *The space $L_p(X)$ is web-bounded if and only if υX is a Lindelöf Σ-space.*

Indeed, it is clear from the definition that web-boundedness is preserved under mappings that take bounded sequences to bounded sequences.

A set B in a topological vector space is bounded if and only if every sequence in B is bounded. It follows that the image of a space with a bounded resolution under a mapping that takes bounded sequences to bounded sequences has a bounded resolution. Now from Lemma 6 we arrive at the following.

Theorem 8. *Let X and Y be spaces, and assume that there exists a surjective mapping $h\colon C_p(X) \to C_p(Y)$ that takes bounded sequences to bounded sequences. If υX is a Lindelöf Σ-space, then υY is a Lindelöf Σ-space.*

Theorem 9. *Let X and Y be spaces, and assume that there exists a surjective mapping $h\colon L_p(X) \to L_p(Y)$ that takes bounded sequences to bounded sequences. If υX is a Lindelöf Σ-space, then υY is a Lindelöf Σ-space.*

Theorem 10. *$L_p(X)$ has a bounded resolution if and only if υX is a K-analytic space. In particular, if X is quasi-Souslin, then $L_p(X)$ has a bounded resolution*

Proof. Let $\{A_p : p \in \mathbb{P}\}$ be a bounded resolution in $L_p(X)$. The family $\{A_p \cap X : p \in \mathbb{P}\}$ is a resolution in X consisting of functionally bounded sets. For each $p \in \mathbb{P}$ let B_p be the closure of $A_p \cap X$ in the space υX. Then the subspace $Y = \bigcup\{B_p : p \in \mathbb{P}\}$ is quasi-Souslin. It follows that υY is K-analytic. Indeed, let T be a set-valued mapping that witnesses Y being quasi-Souslin. Each $T(p)$ is countably compact, so its closure $\overline{T(p)}$ in υY is compact. The mapping $\bar{T}\colon p \mapsto \overline{T(p)}$ is upper semicontinuous, so $Z = \bigcup_{p\in\mathbb{P}} \overline{T(p)}$ is K-analytic.

Since $Y \subset Z \subset \upsilon Y$, we have $Z = \upsilon Z = \upsilon Y$, so υY is K-analytic. Since $X \subset Y \subset \upsilon X$, we conclude that $\upsilon X = \upsilon Y$ is K-analytic.

Conversely, assume that υX is K-analytic. Then by [6, Proposition 0.5.13], the space $L_p(\upsilon X)$ is K-analytic; therefore, it has a compact resolution $\{A_p : p \in \mathbb{P}\}$. As $L_p(X)$ is embedded in $L_p(\upsilon X)$, the family $\{A_p \cap L_p(X) : \alpha \in \mathbb{P}\}$ is a bounded resolution of $L_p(X)$. □

Theorem 11. *If there exists a mapping from $L_p(X)$ onto $L_p(Y)$ that takes bounded sequences to bounded sequences, and υX is K-analytic, then υY is K-analytic.*

Problem 12. *Suppose the space υX is K-analytic and there is a mapping from $C_p(X)$ onto $C_p(Y)$ that takes bounded sequences to bounded sequences. Must the space υY be K-analytic?*

3. Resolutions and completeness

Let κ be an infinite cardinal. We denote $\mathbb{P}(\kappa) = ([\kappa]^{<\omega})^\omega$ where $[\kappa]^{<\omega}$ is the familiy of all finite subsets of κ; thus, for every $p \in \mathbb{P}(\kappa)$ and $n \in \omega$, $p(n)$ is a finite subset of κ. We endow the set $\mathbb{P}(\kappa)$ with the partial order by putting $p \le q$ if for every $n \in \omega$, $p(n) \subset q(n)$. We denote by $c(p)$ the cardinality of the set $p(0)$.

We say that a compact-valued mapping $\phi\colon \mathbb{P}(\kappa) \to X$ is a *compact* $\mathbb{P}(\kappa)$*-resolution for* X if $X = \phi(\mathbb{P}(\kappa))$ and $\phi(p) \subset \phi(q)$ whenever $p \leq q$. If, moreover, every compact subset of X is contained in $\phi(p)$ for some $p \in \mathbb{P}(\kappa)$, then we say that ϕ is a *compact* $\mathbb{P}(\kappa)$*-resolution swallowing compact sets.*

Let $i\colon \mathbb{P} \to \mathbb{P}(\omega)$ be the function such that $i(p)(n) = \{k \in \omega : k \leq p(n)\}$ for all $p \in \mathbb{P}$ and $n \in \omega$; put $\mathbb{P}_0 = i(\mathbb{P})$. Obviously, $\mathbb{P}_0$ is a cofinal subset of $\mathbb{P}(\omega)$ order-isomorphic to $\mathbb{P}$; furthermore, for every $s \in \mathbb{P}(\omega)$ there is a unique minimal element s_0 in the set $\{t \in \mathbb{P}_0 : s \leq t\}$.

Proposition 13. *A space X has a compact resolution if and only if X has a compact $\mathbb{P}(\omega)$-resolution. Moreover, X has a compact resolution swallowing compact sets if and only if X has a compact $\mathbb{P}(\omega)$-resolution swallowing compact sets.*

Proof. This proposition follows from [11, Proposition 3.3(a)]. For the sake of completeness we give the following direct proof. Suppose X has a compact resolution $K\colon \mathbb{P} \to \mathcal{K}(X)$. Define $\phi\colon \mathbb{P}(\omega) \to \mathcal{K}(X)$ by the rule: $\phi(p) = K(i^{-1}(p_0))$ where p_0 is the minimum element of $\mathbb{P}_0$ greater or equal to p. Then ϕ is a compact $\mathbb{P}(\omega)$-resolution for X. Clearly, $\phi(\mathbb{P}(\omega)) = K(\mathbb{P})$, so if K is compact swallowing, then so is ϕ.

Now assume that ϕ is a compact $\mathbb{P}(\omega)$-resolution for X. For each $s \in \mathbb{P}$ put $K(s) = \phi(i(s))$. Since $\mathbb{P}_0$ is cofinal in $\mathbb{P}(\omega)$, the function $K\colon \mathbb{P} \to \mathcal{K}(X)$ is a compact resolution for X. Obviously, if ϕ is compact-swallowing, then so is K. ☐

Following [15], we will say that a base $\{U_p : p \in \mathbb{P}\}$ of neighborhoods of zero of a locally convex space E is a $\mathfrak{G}$*-base* if $U_q \subset U_p$ whenever $p \leq q$.

We will say that a base $\{U_p : p \in \mathbb{P}(\kappa)\}$ of neighborhoods of zero of E is a $\mathfrak{G}_\kappa$*-base* if $U_q \subset U_p$ whenever $p \leq q$. By an argument similar to that in the proof of Proposition 13, a space E has a $\mathfrak{G}$-base if and only if it has a $\mathfrak{G}_\omega$-base.

Theorem 14. *Let κ be an infinite cardinal. A space X has a compact $\mathbb{P}(\kappa)$-resolution swallowing compact sets if and only if $C_c(X)$ has a $\mathfrak{G}_\kappa$-base.*

Proof. Let $\phi\colon \mathbb{P}(\kappa) \to \mathcal{K}(X)$ be a compact $\mathbb{P}(\kappa)$-resolution swallowing compact sets. Assign to each $p \in \mathbb{P}(\kappa)$ the set $U_p = \{f \in C_c(X) : |f(x)| \leq \frac{1}{c(p)+1}$ for all $x \in \phi(p)\}$. The compactness of each $\phi(p)$, $p \in \mathbb{P}(\kappa)$ and the compact-swallowing property imply that the family $\mathcal{U} = \{U_p : p \in \mathbb{P}(\kappa)\}$ is a base of neighborhoods of zero in $C_c(X)$. If $p \leq q$, then $\phi(p) \subset \phi(q)$ and $p(0) \subset q(0)$, so $U_q \subset U_p$, and $\mathcal{U}$ is a $\mathfrak{G}_\kappa$-base.

Conversely, let $\{U_p : p \in \mathbb{P}(\kappa)\}$ be a $\mathfrak{G}_\kappa$-base of $C_c(X)$. For each $p \in \mathbb{P}(\kappa)$ put

$$K_p = \{x \in X : |f(x)| \leq c(p) \text{ for all } f \in U_p\}.$$

Clearly, the sets K_p are closed in X, and $K_p \subset K_q$ whenever $p \leq q$. Let us verify that these sets are compact.

Let $p \in \mathbb{P}(\kappa)$. Then the set $\{f \in C_c(X) : f(K_p) \subset (-1,1)\}$ contains the set $\frac{1}{c(p)+1}U_p$, so it is a neighnorhood of 0 in $C_c(X)$. It follows that K_p is contained in a compact subset of X; since K_p is closed in X, it is compact.

Let us now verify that every compact set C in X is contained in K_p for some $p \in \mathbb{P}(\kappa)$. Indeed, since the family $\{U_p : p \in \mathbb{P}(\kappa)\}$ is a base at 0 of $C_c(X)$, we have $U_q \subset \{ f \in C_c(X) : f(C) \subset (-1,1)\}$ for some $q \in \mathbb{P}(\kappa)$. Find a $p \in \mathbb{P}(\kappa)$ so that $q \leq p$ and $p(0) \neq \emptyset$. Then for every $x \in C$ and $f \in U_p$ we have $|f(x)| < 1 \leq c(p)$, so $x \in K_p$. We have proved that $C \subset K_p$.

Thus, the mapping $\phi\colon \mathbb{P}(\kappa) \to \mathcal{K}(X)$ such that $\phi(p) = K_p$ for all $p \in \mathbb{P}$, is a compact-swallowing compact $\mathbb{P}(\kappa)$-resolution for X. □

Corollary 15. *If a space X has a compact $\mathbb{P}(\kappa)$-resolution swallowing compact sets, and X ℓ_c-covers Y, then Y has a compact $\mathbb{P}(\kappa)$-resolution swallowing compact sets.*

If X and Y are ℓ_p-equivalent, the conclusion of the last theorem holds. Indeed, if $T\colon C_p(X) \to C_p(Y)$ is a linear homeomorphismn, then T is also a linear homeomorphism between $C_c(X)$ and $C_c(Y)$ by [5, Propositions II.1.1, II.1.4].

Note also that Corollary 15 fails if we only assume t-equivalence of the spaces X and Y (see [29, Example 3.4]): the spaces $C_p(\mathbb{Q})$ and $C_p(\omega+1)$ are homeomorphic by [13], but the the space of rationals $\mathbb{Q}$ does not admit a compact resolution swallowing compact sets, because otherwise by Theorem 1 it would have to be Čech-complete. Corollary 15 also shows that $C_p(\mathbb{Q})$ and $C_p(\omega+1)$ are not linearly homeomorphic. By [29, Corollary 2.2], the property of having a compact resolution is preserved by t-equivalence.

Proposition 16. *Every Čech-complete space X of weight κ has a $\mathbb{P}(\kappa)$-compact resolution swallowing compact sets. Consequently, if X is a completely metrizable space of weight κ, then $C_c(X)$ admits a $\mathfrak{G}_\kappa$-base.*

Conversely, if X is a metrizable space and $C_c(X)$ has a $\mathfrak{G}$-base, then X is Polish.

Proof. Let $\mathfrak{B}$ be a base of cardinality κ for X. Fix a countable family $\{W_n : n \in \omega\}$ of open sets in βX so that $X = \bigcap_{n\in\omega} W_n$. Let $\mathfrak{B}_n$ be the family of all elements of $\mathfrak{B}$ whose closures in βX are contained in the set W_n. Clearly, $\mathfrak{B}_n$ is a base of X for each $n \in \omega$. Enumerate each $\mathfrak{B}_n$ in type κ: $\mathfrak{B}_n = \{B_{n\alpha} : \alpha \in \kappa\}$. For every finite set $A \subset \kappa$ put $F_n(A) = \bigcup\{\overline{B_{n\alpha}} : \alpha \in A\}$ where the closures are taken in βX. Then $F_n(A)$ is a compact subset of βX contained in W_n. Define $\phi\colon \mathbb{P}(\kappa) \to \mathcal{K}(X)$ by the rule:

$$\phi(p) = \bigcap\{F_n(p(n)) : n \in \omega\}.$$

The function ϕ is well defined, since the intersection of compact sets $F_n(p(n))$ is compact, and is contained in X, because $F_n(p(n)) \subset W_n$ and $\bigcap_{n\in\omega} W_n = X$. It is obvious from the construction that $\phi(p) \subset \phi(q)$ whenever $p \leq q$. Let us verify that for every compact K in X there exists $p \in \mathbb{P}(\kappa)$ such that $K \subset \phi(p)$. Indeed, for each $n \in \omega$ there exists a finite $A_n \subset \kappa$ such that $K \subset \bigcup\{B_{n\alpha} : \alpha \in A_n\}$. Then the element p of $\mathbb{P}(\kappa)$ such that $p(n) = A_n$ for all $n \in \omega$ is as required.

If X is metrizable and $C_c(X)$ has a $\mathfrak{G}$-base, then by Theorem 14, X has a compact resolution swallowing compact sets, so X is Polish by Theorem 1. □

It is natural to ask whether every metrizable space of weight κ such that $C_c(X)$ has a $\mathfrak{G}_\kappa$-base must be completely metrizable. The next statement, proved by David Guerrero Sánchez and presented here with his kind permission, shows that the answer generally is negative.

Theorem 17. *Let X be a metrizable space of weight $\leq \kappa$. Then X has a compact-swallowing compact $\mathbb{P}(\kappa^\omega)$-resolution.*

Proof. Let X be a metric space of weight $\leq \kappa$. Then X has at most $\lambda = k^\omega$ compact subsets.

Therefore, we can write $\mathcal{K}(X) = \{K_\alpha : \alpha \in \lambda\}$ (we do not assume that the enumeration is injective).

Define $\phi(p) = \bigcup\{K_\alpha : \alpha \in p(0)\}$ for each $p \in \mathbb{P}(\lambda)$. Clearly, ϕ is a compact-swallowing compact $\mathbb{P}(\lambda)$-resolution for X. □

In particular, every metric space X of weight $\mathfrak{c}$ has a compact-swallowing compact $\mathbb{P}(\mathfrak{c})$-resolution, so by Theorem 14, $C_c(X)$ has a $\mathfrak{G}_\mathfrak{c}$-base. Of course, a metrizable space of weight $\mathfrak{c}$ need not be completely metrizable.

4. METRIZABILITY AND LOCAL COMPACTNESS

It is well known that metrizability and local compactness are not preserved by the relation of ℓ_p-equivalence: this is shown by the very first known example of non-homeomorphic ℓ_p-equivalent spaces, the countable sum of convergent sequences and the countable Fréchet fan (see, e.g., [6]).

On the other hand, there are several results that show that a space ℓ_p-equivalent or ℓ_c-equivalent to a metrizable or a locally compact space must be metrizable or locally compact if we assume that it has some additional property, such as first countability or pointwise countable type.

In this section we obtain some extensions and versions of results of this type.

We need the following few auxiliary results. Recall that a space X is *submetrizable* if there is a continuous bijection from X onto a metrizable space.

Proposition 18. *Let X be a submetrizable space. If there exists a continuous map from $C_c(X)$ onto $C_c(Y)$, then $C_c(C_c(Y))$ is submetrizable. In particular, if Y is of pointwise countable type, then Y is submetrizable and first countable.*

Proof. If X is submetrizable, then by [19, Theorem 5.6.2], $C_c(X)$ has a dense σ-compact subspace. Hence, $C_c(Y)$ has a dense σ-compact subspace, and by [19, Corollary 4.3.2], $C_c(C_c(Y))$ is submetrizable.

Recall that every space of pointwise countable type is a k-space, see [14, Chapter 2.3], so Y is homeomorphic to a subspace of $C_c(C_c(Y))$. Hence, Y is submetrizable, so all compact sets in Y are metrizable. To see that Y is first countable we apply the *transitivity of character for compact sets* (the fact that if $F_1 \subset F_2$ are compact sets in a space Y, F_1 has countable character in F_2, and F_2 has countable character in Y, then F_1 has countable character in Y, see [1, Proposition 3.3]); we apply this to any singleton $F_1 = \{y_0\}$ and a compact set F_2 of countable character in Y that contains y_0 to prove the countability of character of Y at y_0. Thus, Y is first countable. □

Recall that a space X is an $\aleph_0$-*space* if X has a countable *k-network*, that is, a countable family of sets $\mathcal{N}$ such that for every compact set K and every open set U in X such that $K \subset U$, there is an $N \in \mathcal{N}$ with $K \subset N \subset U$, see [20]. It is well known [20] that every image of an $\aleph_0$-space under a perfect mapping is an $\aleph_0$-space and every closed subspace of an $\aleph_0$-space is an $\aleph_0$-space.

A function $f\colon X \to Y$ is called *k-continuous* if its restriction to every compact subspace of X is continuous. For a space X, let $\Delta_X\colon X \to C_c(C_c(X))$ be the map such that $\Delta_X(x)(f) = f(x)$ for all $x \in X$ and $f \in C_c(X)$. It is well-known that the mapping Δ_X is injective, and that it is an embedding if and only if it is continuous.

Lemma 19. *The mapping Δ_X is k-continuous.*

Proof. Let K be a compact subset of X and $F = C_c(C_c(X)) \setminus [C, V]$ where C is a compact subset of $C_c(X)$ and V is an open subset of $\mathbb{R}$. Clearly, $\Delta_X^{-1}(F) = \{x \in X : f(x) \notin V$ for some $f \in C\}$.

Let x be the limit of a convergent net $(x_i : i \in I)$, with each $x_i \in K \cap (\Delta_X)^{-1}(F)$. To prove the lemma it is enough to show that $x \in \Delta_X^{-1}(F)$.

For each $i \in I$ fix an $f_i \in C$ with $f_i(x_i) \notin V$. The compactness of C implies that the net $(f_i : i \in I)$ has a subnet that converges uniformly on K to some $f \in C$. Then from $f_i(x_i) \notin V$, for each $i \in I$, follows that $f(x) \notin V$, implying that $x \in \Delta_X^{-1}(F)$. □

Denote by $\tilde{X}$ the image of X under the mapping $\Delta_X : X \to C_c(C_c(X))$ and by $\hat{X}$ the image of X under the mapping $\Delta_X : X \to C_p(C_c(X))$. It is well known (see e.g [19]) that Δ_X is a homeomorphism from X to $\hat{X}$, and $\hat{X}$ is closed in $C_p(C_c(X))$; it follows that $\tilde{X}$

is a closed subspace of $C_c(C_c(X))$ and that the restriction to $\tilde{X}$ of the identity mapping $C_c(C_c(X)) \to C_p(C_c(X))$ is a continuous bijection onto $\hat{X}$ whose inverse is k-continuous.

Lemma 20. *Let X be a space and $f\colon X \to Y$ a continuous bijection whose inverse is k-continuous. If X is an $\aleph_0$-space, then so is Y.*

Proof. Note that a set K in Y is compact if and only if $f^{-1}(K)$ is compact. Now it is immediate that if $\mathcal{N}$ is a countable k-network in X, then $\{f(N) : N \in \mathcal{N}\}$ is a countable k-network in Y. □

The following theorem extends Arhangel'skii's result from [4] with the same conclusion as below but for ℓ_p-equivalent spaces X and Y, see also [5, Theorem 12]. Note that in general the $\aleph_0$-space property in not preserved by open maps.

Theorem 21. *If X is an $\aleph_0$-space and there is an open continuous linear mapping from $C_c(X)$ onto $C_c(Y)$, then Y is an $\aleph_0$-space.*

Proof. Let $h\colon C_c(X) \to C_c(Y)$ be an open continuous linear mapping. Let $h_c^*\colon C_c(C_c(Y)) \to C_c(C_c(X))$ and $h_p^*\colon C_p(C_c(Y)) \to C_p(C_c(X))$ be the dual mappings; the mapping h_c^* is continuous, and the mapping h_p^* is an embedding, see e.g. [19, Corollary 2.2.8].

The mapping h_p^* is a closed embedding, because h is quotient. Denote

$$Z_p = h_p^*(\hat{Y}) \text{ and } Z_c = h_c^*(\tilde{Y}).$$

It follows that Z_p is closed in $C_p(C_c(X))$, and hence Z_c is closed in $C_c(C_c(X))$. Moreover, h_p^* maps homeomorphically $\hat{Y}$ onto Z_p, because h_p^* is an embedding.

Let $j\colon Z_c \to Z_p$ be the restriction to Z_c of the identity mapping $C_c(C_c(X)) \to C_p(C_c(X))$, and let $i\colon \tilde{Y} \to \hat{Y}$ be the restriction to $\tilde{Y}$ of the identity mapping $C_c(C_c(Y)) \to C_p(C_c(Y))$. Clearly, i and j are continuous bijections, and i^{-1} is k-continuous by Lemma 19.

We have $j \circ (h_c^*|\tilde{Y}) = (h_p^*|\hat{Y}) \circ i$, so $j^{-1} = (h_c^*|\tilde{Y}) \circ i^{-1} \circ (h_p^*|\hat{Y})^{-1}$, whence j^{-1} is k-continuous.

By [20] the space $C_c(C_c(X))$ is an $\aleph_0$-space, so its closed subspace Z_c is an $\aleph_0$-space. From Lemma 20 now follows that Z_p is an $\aleph_0$-space. Since Z_p is homeomorphic to Y, Y is an $\aleph_0$ space. □

Since every first countable $\aleph_0$-space is second-countable, see [20], [21], Theorem 21 applies to prove the following extension of Arhangel'skii's [4, Theorem 16] (see also [5]).

Corollary 22. *Let X be a second-countable space and Y be first-countable. If there is an open continuous linear mapping from $C_c(X)$ onto $C_c(Y)$, then Y is second-countable.*

The next corollary extends Pelant's result [5, Theorem 3.27].

Corollary 23. *Let X be a separable completely metrizable space and Y a first-countable space. If there is an open continuous linear mapping from $C_c(X)$ onto $C_c(Y)$, then Y is a separable completely metrizable space.*

Proof. By Theorem 21, Y is second-countable. Since X is Polish, it has a compact swallowing compact resolution, so $C_c(X)$ has a $\mathfrak{G}$-base. It follows that $C_c(Y)$ has a $\mathfrak{G}$-base, so Y has a compact-swallowing compact resolution, and hence is completely metrizable by Christensen's theorem. $\square$

Proposition 18 and Corollary 23 give a relatively simple proof for the next statement, which can also be obtained from a combination of results of V. Valov, J. de Groot and J. Pelant.

Corollary 24. *If X is a Polish space, Y is a space of pointwise countable type, and X ℓ_c-covers Y, then Y is Polish.*

It is well-known that $C_c(X)$ is metrizable if and only if X is *hemicompact*, i.e. has a compact-swallowing sequence of compact sets, see [19], Theorem 4.4.2.

Proposition 25. *Let X be a locally compact space. Then X is hemicompact if and only if X is a μ-space with a compact resolution swallowing compact sets.*

Proof. Clearly, every hemicompact space is a μ-space with a compact resolution swallowing compact sets.

For the converse assume that X is a μ-space admitting a compact resolution swallowing compact sets. The μ-property of X implies that $C_c(X)$ is barrelled (see Theorem 10.1.20 in [25]). Since X is locally compact, the space $C_c(X)$ it is *Baire-like*, that is, for every increasing countable cover $\{A_n : n \in \omega\}$ of $C_c(X)$ by symmetric convex closed sets there exists an $m \in \omega$ such that A_m is a neighbourhood of zero, see [17, Lemma 2.1] or the proof of Theorem 27 below. Next, by Theorem 14 (with $\kappa = \omega$), the space $C_c(X)$ has a $\mathfrak{G}$-base, say, $\{U_p : p \in \mathbb{P}\}$. We may assume without loss of generality that all sets U_p are convex, symmetric and closed in $C_c(X)$. For each finite sequence of naturals σ of length n, put $C_\sigma = \bigcap\{U_p : p|n = \sigma\}$. Then $C_{p|n} \subset U_p$. Moreover, the sequence $\{C_{p|n} : n \in \omega\}$ is increasing, and for each bounded set $B \subset C_c(X)$, $B \subset kC_{p|k}$ for some $k \in \omega$. Indeed, assume that there is a bounded set B in $C_c(X)$ such that $B \not\subset kC_{p|k}$ for all $k \in \omega$. For each $k \in \omega$ choose $x_k \in B \setminus kC_{p|k}$. Choose $p_k \in \mathbb{P}$ so that $p_k|k = p|k$ and $x_k \notin kU_{p_k}$. Let $q(n) = \max\{p_k(n) : k \in \omega\}$; then $q \in \mathbb{P}$ (note that $p_k(n) = p(n)$ for $k \geq n$, so q is well-defined). Clearly, $p_k \leq q$ for all $k \in \omega$. Therefore, $U_q \subset U_{p_k}$ for all $k \in \omega$, and $x_k \notin kU_q$. Thus, $\{x_k : k \in \omega\}$ is an unbounded sequence in B, a contradiction with B being bounded.

The Baire-likeness of $C_c(X)$ now implies that there exists a $k \in \omega$ such hat $C_{p|k}$ is a neighbourhood of zero in $C_c(X)$. Put

$$\mathfrak{D} = \{C_\sigma : \ \sigma \text{ is a finite sequence of naturals and } C_\sigma \text{ is a neighborhood of zero in } C_c(X)\}.$$

Then $\mathcal{D}$ is a countable family of neighborhoods of zero in $C_c(X)$. By the above argument, for any neighborhood of zero U in $C_c(X)$ there are a $p \in \mathbb{P}$ and $k \in \omega$ such that $C_{p|k} \in \mathfrak{D}$ and $C_{p|k} \subset U$. Thus, $\mathfrak{D}$ is a countable base of neighborhoods of zero in $C_c(X)$. It follows that $C_c(X)$ is metrizable. Hence, Y is hemicompact. □

Example 26. *The assumption of local compactness of X in Proposition 25 cannot be omitted: there exists a σ-compact Čech-complete not hemicompact space that has a compact resolution swallowing compact sets.*

Put $X = [0,1] \setminus \{\frac{1}{n+1} : n \in \omega\}$. Then X is a G_δ-set in $[0,1]$, so X is a Polish space; therefore X has a compact resolution swallowing compact sets. Clearly, X is σ-compact and not locally compact; since all hemicompact first countable spaces are locally compact, X is not hemicompact.

In [32, Proposition 4.8] Valov proved that if X and Y are μ-spaces, X is locally compact, Y is a wq-space, and there exists a continuous linear surjection from $C_c(X)$ onto $C_c(Y)$, then Y is locally compact. The above proof of Proposition 25 motivated us to present an apparently simpler approach to Valov's [32, Proposition 4.8]. We use an argument similar to one in [17].

Theorem 27. *Let X be a locally compact μ-space and let Y be a wq-space which is ℓ_c-equivalent to X. Then Y is a locally compact μ-space.*

Proof. Let us first verify that if X is a locally compact μ-space, then $C_c(X)$ is barrelled and Baire-like. To prove the claim it is sufficient to prove that for any decreasing sequence $\{A_n : n \in \omega\}$ of closed non-compact subsets of X there is a function $f \in C(X)$ that is unbounded on each A_n, see [17, Proposition 1.2].

Let $\{A_n : n \in \omega\}$ be a sequence as above. For each $n \in \omega$ choose an $f_n \in C(X)$ unbounded on A_n. If there exists a number $m \in \omega$ such that f_m is unbounded on each A_n, we are done. Therefore, assume that for each $n \in \omega$ there is a $k_n > n$ such that f_n is bounded on A_{k_n}. Since the sequence $\{A_n : n \in \omega\}$ is decreasing, we may assume (taking a suitable subsequence if necessary) that f_n is bounded on A_{n+1} for each $n \in \omega$. If $A = \bigcap_n A_n$ is non-compact, the proof is complete. So assume that A is compact. Then there exists an open neighbourhood H_0 of A whose closure W_0 is compact. Since f_1 is bounded on A_2 and W_0 and unbounded on A_1, there exists $x_1 \in A_1 \setminus (A_2 \cup W_0)$. Then there exists an open neighbourhood H_1 of x_1 with compact closure W_1 and $W_1 \subset X \setminus (A_2 \cup W_0)$. This procedure yields a sequence $D = \{x_n : n \in \omega\}$ in X with a pairwise disjoint sequence

$\{H_n : n \in \omega\}$ of its open neighbourhoods whose closures are compact. It is easy to see that the set $Z = D \cup A$ is closed and non-compact. Since X is a μ-space, there exists a function $f \in C(X)$ unbounded on Z. Then f is unbounded on each A_n. This proves that $C_c(X)$ is Baire-like. By the assumption, $C_c(Y)$ is Baire-like (because the Baire-likeness is inherited by Hausdorff locally convex quotients).

Finally, assume that Y is a wq-space. We already know that $C_c(Y)$ is Baire-like. Then Y is locally compact. Indeed, given a point $y_0 \in Y$ fix a decreasing sequence $\{U_n : n \in \omega\}$ of its neighborhoods as in the definition of a wq-space. Put $W_n = \{f \in C(Y) : \sup_{y \in U_n} |f(y)| \leqslant n\}$ for each $n \in \omega$. Then each W_n is a closed absolutely convex subset of $C_c(Y)$. Note that the sequence $\{W_n : n \in \omega\}$ covers $C_c(Y)$. Indeed, if some function f_0 is not covered, then for each $n \in \omega$ there is a point $z_n \in U_n$ such that $|f_0(z_n)| > n$. It follows that the sequence $\{z_n : n \in \omega\}$ is not functionally bounded, a contradiction with the choice of the sets U_n.

By the Baire-likeness of $C_c(Y)$, there exist $n \in \omega$, $\epsilon > 0$, and a compact subset S of Y such that $\{f \in C(Y) : \sup_{y \in S} |f(y)| < \epsilon\} \subset W_n$. Then $U_n \subset S$, and S is a compact neighborhood of y_0 in Y. □

ACKNOWLEDGEMENT

The authors are grateful to the referee for his comments that have improved the paper.

REFERENCES

[1] A. V. Arhangel'skii, *Bicompact sets and the topology of spaces*, Trudy Moskov. Mat. Obšč. **13** (1965) 3–55 (in Russian).

[2] A. V. Arkhangel'skii, *On linear homomorphisms of function spaces*, Dokl. Akad. Nauk SSSR **264** (1982) 1289-1292.

[3] A. V. Arhangel'skii, V. V. Tkachuk, *Function spaces and topological invariants*, Izdat. Mosk. Gos. Univ., Moscow, 1985.

[4] A. V. Arhangel'skii, *A survey of C_p-theory* Questions & Answers in General Topology **5** (1987) 1–109.

[5] A. V. Arkhangel'skii, *General Topology III*, Encyclopaedia of Mathematical Sciences **51**, Springer, 1991.

[6] A. V. Arkhangel'skii, *Topological function spaces*, Mathematics and its Applications **78**. Kluwer Academic Publishers. Dordrecht Boston London, 1992.

[7] J. Baars, J. de Groot, *On topological linear equivalence of certain function spaces*, CWI Traks **86**, Centrum in Wisk. Inform. Amsterdam, 1982.

[8] J. Baars, J. de Groot, J. Pelant, *Function spaces of completely metrizable spaces*, Trans. Amer.Math.Soc. **340** (1993) 871–883.

[9] B. Cascales, L. Oncina, *Compactoid filters and USCO maps*, J. Math. Anal. Appl. **282** (2003) 826–843.

[10] B. Cascales, J. Orihuela, *On Compactness in Locally Convex Spaces*, Math. Z. **195** (1987) 365–381.

[11] B. Cascales, J. Orihuela, V. V. Tkachuk, *Domination by second countable spaces and Lindelöf Σ-property*, Topology Appl. **158** (2011) 204–214.

[12] J. P. R. Christensen, *Topology and Borel Structure*, North-Holland Mathematics Studies **10**, North-Holland, Amsterdam, 1974.

[13] T. Dobrowolski, S. P. Gulko, J. Mogilski, *Function spaces homeomorphic to the countable product of ℓ_2*, Topology Appl., **34** (1990) 153–160.

[14] R. Engelking, *General Topology*, Monografie Matematiczne **60**.PWN. Warszawa, 1975.

[15] J. C. Ferrando, J. Kąkol, M. López-Pellicer, S. A. Saxon, *Tightness and distinguished Fréchet spaces*, J. Math. Anal. Appl. **324** (2006) 862–881.

[16] S. Gu'lko, O. Okunev, *Local compactness and M-equivalence*, in A.V. Ivanov at al, eds. "Questions of Geometry and Topology" (1986) 14-23 (in Russian).

[17] J. Kąkol, *Remarks on linear homeomorphisms of function spaces*, Acta Math. Hungar. **89** (2000) 315–319.

[18] J. Kąkol, W. Kubiś, M. López Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis*, Developments in Mathematics **24**, Springer, New York Dordrecht Heidelberg London, 2011.

[19] R. A. McCoy, I. Ntantu, *Topological Properties of Spaces of Continuous Functions* Lecture Notes in Mathematics **1315**, Springer, 1988.

[20] E. Michael, *$\aleph_0$-spaces*, J. Math. Mech. **15** (1966) 983–1002.

[21] E. Michael, *A quintuple quotient quest.* General Topology and its Applications **2** (1972) 91–138.

[22] O. Okunev, *Weak topology of an associeted space and t-equivalence*, Math. Notes **46** (1-2) (1990) 534-538.

[23] O. G. Okunev, *On Lindelöf Σ-spaces of continuous functions in the pointwise topology*, Topology Appl. **49** (1993) 149-166.

[24] O. Okunev, *A relation between spaces implied by their t-equivalence*, Topology Appl. **158** (2011) 2158-2164.

[25] P. Pérez Carreras, J. Bonet, *Barrelled Locally Convex Spaces*, North-Holland Mathematics Studies **131**, North-Holland, Amsterdam, 1987.

[26] C. A. Rogers, J. E. Jayne, C. Dellacherie, F. Topsøe. J. Hoffman-Jørgensen, D. A. Martin, A. S. Kechris, A. H. Stone, *Analytic Sets*, Academic Press, 1980.

[27] H. H. Schaefer, Topological Vector Spaces, Springer, 1971.

[28] J. Schmets, *Spaces of Vector-Valued Continuous Functions*, Lecture Notes in Mathematics **1003**, Springer, 1983.

[29] V. V. Tkachuk, *A space $C_p(X)$ is dominated by irrationals if and only if it is K-analytic*, Acta Math. Hungar. **107** (2005) 253-265.

[30] V. Uspenski, *On the topology of a free locally convex space*, Soviet Math. Dokl. **27** (1983) 781-785.

[31] M. Valdivia, *Topics in Locally Convex Spaces*, North-Holland Mathematics Studies **67**, North-Holland, Amsterdam, 1982.

[32] V. Valov, *Function spaces*, Topology Appl. **81** (1997) 1-22.

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