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CARISTI'S TYPE MAPPINGS ON COMPLETE PARTIAL METRIC SPACES

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Abstract. We introduce a new type of Caristi's mapping on partial metric spaces and show that a partial metric space is complete if and only if every Caristi mapping has a fixed point. From this result we deduce a characterization of bicomplete weightable quasi-metric spaces. Several illustrative examples are given.
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1. Introduction and preliminaries

The notion of a partial metric space was introduced by Matthews [13] as a part of the study of denotational semantics of data flow networks. In particular, he obtained, among other results, a partial metric version of the Banach fixed point theorem ([13, Theorem 5.3]). Later on, Valero [19], Oltra and Valero [15], Altun et al [2], [3], and Ilic et al [10], gave some generalizations of the result of Matthews.

Let us recall that a partial metric on a set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

(p_1) $x = y \Longleftrightarrow p(x, x) = p(x, y) = p(y, y)$ (T_0-separation axiom),
(p_2) $p(x, x) \le p(x, y)$ (small self-distance axiom),
(p_3) $p(x, y) = p(y, x)$ (symmetry),
(p_4) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$ (modified triangular inequality).

A partial metric space (for short PMS) is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

It is clear that, if $p(x, y) = 0$, then, from (p_1) and (p_2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

At this point it seems interesting to remark the fact that partial metric spaces play an important role in constructing models in the theory of computation (see for instance [1], [8], [17], [18], [20], etc).

A basic example of a PMS is the pair $([0,\infty),p)$, where $p(x,y)=\max\{x,y\}$ for all $x,y\in[0,\infty)$.

For another example, let I denote the set of all intervals $[a,b]$ for any real numbers $a\leq b$. Let $p:I\times I\to[0,\infty)$ be the function such that $p([a,b],[c,d])=\max\{b,d\}-\min\{a,c\}$. Then (I,p) is a PMS.

Other examples of partial metric spaces which are interesting from a computational point of view may be found in [7], [13], etc.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p-balls

$$\{B_p(x,\varepsilon):x\in X,\varepsilon>0\},$$

where

$$B_p(x,\varepsilon)=\{y\in X:p(x,y)<p(x,x)+\varepsilon\},$$

for all $x\in X$ and $\varepsilon>0$.

Observe that a sequence $\{x_n\}_{n\in\omega}$ (by ω we denote the set of all non-negative integer numbers) in a PMS (X,p) converges to a point $x\in X$, with respect to τ_p, if and only if $p(x,x)=\lim_{n\to\infty}p(x,x_n)$.

If p is a partial metric on X, then the function $p^s:X\times X\to[0,\infty)$ given by

$$p^s(x,y)=2p(x,y)-p(x,x)-p(y,y), \tag{1.1}$$

is a metric on X.

Definition 1.1. Let (X,p) be a PMS.

(i) A sequence $\{x_n\}_{n\in\omega}$ in (X,p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty}p(x_n,x_m)$.

(ii) (X,p) is called complete if every Cauchy sequence $\{x_n\}_{n\in\omega}$ in X converges, with respect to τ_p, to a point $x\in X$ such that $p(x,x)=\lim_{n,m\to\infty}p(x_n,x_m)$.

The following lemma plays an important role in obtaining fixed point results on a PMS.

Lemma 1.2. (Matthews [13], Oltra and Valero [15]). *Let (X,p) be a PMS. Then:*

(a) $\{x_n\}_{n\in\omega}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(b) (X,p) is complete if and only if (X,p^s) is complete.

The following fact will be also useful.

Lemma 1.3. (Romaguera [16]). *Let (X,p) be a PMS. Then, for each $x\in X$, the function $p_x:X\to[0,\infty)$ given by $p_x(y)=p(x,y)$ is lower semicontinuous for (X,p^s).*

Caristi proved in [4] that if T is a self mapping of a complete metric space (X,d) such that there is a lower semicontinuous function $\phi:X\to[0,\infty)$ satisfying

$$d(x,Tx)\leq\phi(x)-\phi(Tx), \tag{1.2}$$

for all $x\in X$, then T has a fixed point.

If there exists a a lower semicontinuous function $\phi:X\to[0,\infty)$ satisfying (1.2), then T is called Caristi mapping for (X,d).

There are a lot of generalizations of Caristi's fixed point theorem in the literature. Furthermore, Kirk proved in [12] that a metric space (X, d) is complete if and only if every Caristi mapping for (X, d) has a fixed point.

In [16], Romaguera discussed the extension of Kirk's theorem to a PMS. To this end, he proposed the following two notions of a Caristi mapping in this context.

(i) A self mapping T of a PMS (X, p) is called a p-Caristi mapping on X if there is a function $\phi : X \to [0, \infty)$ which is lower semicontinuous for (X, p) and satisfies

$$p(x, Tx) \leq \phi(x) - \phi(Tx) \tag{1.3}$$

for all $x \in X$.

(ii) A self mapping T of a PMS (X, p) is called a p^s-Caristi mapping on X if there is a function $\phi : X \to [0, \infty)$ which is lower semicontinuous for (X, p^s) and satisfies (1.3).

Clearly, every p-Caristi mapping is a p^s-Caristi mapping, but the converse is not true in general.

Also in the same paper, Romaguera defined a 0-complete PMS as (X, p) in which every 0-Cauchy sequence, converges with respect to τ_p, to a point z such that $p(z, z) = 0$, where a sequence $\{x_n\}_{n\in\omega}$ in a PMS is called 0-Cauchy if $\lim_{m,n\to\infty} p(x_n, x_m) = 0$.

It is clear that every complete PMS is 0-complete. However, the converse is not true in general (see [16, p. 3]).

The following result provides the extension of Kirk's theorem to PMS obtained in [16, Theorem 2.3].

Theorem 1.4. *A PMS (X, p) is 0-complete if and only if every p^s-Caristi mapping on X has a fixed point.*

Very recently, Karapinar [11] obtained the following generalization of Caristi's fixed point theorem.

Theorem 1.5. *Let (X, p) be a complete PMS, then every p-Caristi mapping on X has a fixed point.*

Remark 1.6. Observe that Theorem 1.5 is an immediate consequence of Theorem 1.4 because every complete PMS is 0-complete and every p-Caristi mapping is a p^s-Caristi mapping.

Since the identity mapping on a PMS (X, p) is a p^s-Caristi mapping only in the case that p is a metric, we shall propose in the next section a new notion of Caristi mapping which avoids this disadvantage. For our surprise, complete partial metric spaces can be characterized as those partial metric spaces for which every Caristi mapping (in this new sense) has a fixed point. From this result we shall deduce a characterization of bicomplete weightable quasi-metric spaces. Several illustrative examples are also given.

2. The results

Definition 2.1. A self mapping T of a PMS (X, p) is called a Caristi mapping on X if there is a function $\phi : X \to [0, \infty)$ which is lower semicontinuous for (X, p^s) and satisfies

$$p(x, Tx) \leq p(x, x) + \phi(x) - \phi(Tx)$$

for all $x \in X$.

Obviously, the identity mapping on a PMS (X, p) is a Caristi mapping (in the sense of the above definition).

Moreover, it is clear that every p^s-Caristi mapping on (X, p) is a Caristi mapping. The next example shows that the converse does not hold, in general.

Example 2.2. Let $X = [1, \infty)$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a complete PMS (and so it is 0-complete). Suppose that $T : X \to X$ is a p^s-Caristi mapping. Then, by Theorem 1.4, T must have a fixed point in X, say z. Since T is p^s-Caristi, there is a function $\varphi : X \to [0, \infty)$ which is lower semicontinuous for (X, p^s) and such that for each $x \in X$,

$$p(x, Tx) \leq \varphi(x) - \varphi(Tx).$$

Since z is a fixed point of T, we have

$$p(z, z) = p(z, Tz) \leq \varphi(z) - \varphi(Tz) = \varphi(z) - \varphi(z) = 0.$$

Therefore, the definition of p implies $z = 0$, which is a contradiction since $0 \notin X$. Thus there is no p^s-Caristi mapping on X.

Now let $T : X \to X$ defined by $Tx = (x + 1)/2$ and $\phi : X \to [0, \infty)$ defined by $\phi(x) = 1$ for all $x \in X$. Clearly, ϕ is lower semicontinuous for (X, p^s). Moreover, we have

$$p(x, Tx) = \max\{x, \frac{x+1}{2}\} = x = p(x, x) + \phi(x) - \phi(Tx),$$

for all $x \in X$, and thus, T is a Caristi mapping on X.

Note that T has fixed point (indeed, $z = 1$ is its unique fixed point). This fact is not casual as Theorem 2.3 below, shows.

Theorem 2.3. *A PMS (X, p) is complete if and only if every Caristi mapping on X has a fixed point.*

Proof. Suppose that (X, p) is complete and let T be a Caristi mapping on X. Then there exists a function $\phi : X \to [0, \infty)$ which is a lower semicontinuous for (X, p^s) and satisfies

$$p(x, Tx) \leq p(x, x) + \phi(x) - \phi(Tx),$$

for all $x \in X$. Hence

$$2p(x, Tx) - p(Tx, Tx) \leq 2p(x, x) + 2\phi(x) - 2\phi(Tx) - p(Tx, Tx). \tag{2.1}$$

Now let $\beta : X \to [0, \infty)$ given by $\beta(x) = p(x, x)$ for all $x \in X$. Then β is lower semicontinuous for (X, p^s), so the function $\psi := \beta + 2\phi$ is also lower semicontinuous for (X, p^s). So the inequality (2.1) can be written as

$$2p(x, Tx) - p(x, x) - p(Tx, Tx) \leq \psi(x) - \psi(Tx),$$

i.e.,

$$p^s(x, Tx) \leq \psi(x) - \psi(Tx),$$

for all $x \in X$. Since, by Lemma 1.2, (X, p^s) is a complete metric space, and ψ is lower semicontinuous for (X, p^s), we can apply Caristi's fixed point theorem, and thus T has a fixed point.

The converse follows from a slight modification of the proof of the 'if' part of Theorem 1.4 above (Theorem 2.3 in [16]). Indeed, suppose that there is a Cauchy sequence $\{x_n\}_{n\in\omega}$ of distinct points in (X, p) which is not convergent in (X, p^s). Construct a subsequence $\{y_n\}_{n\in\omega}$ of $\{x_n\}_{n\in\omega}$ such that

$$p(y_n, y_{n+1}) - p(y_n, y_n) < 2^{-(n+1)},$$

for all $n \in \omega$.

Put $A = \{y_n : n \in \omega\}$. Clearly A is closed in (X, p^s).

Now define $T : X \to X$ by $Tx = y_0$, if $x \in X\backslash A$ and $Ty_n = y_{n+1}$ for all $n \in \omega$, and define $\phi : X \to [0, \infty)$ by $\phi(x) = p(y_0, x) + 1$ if $x \in X\backslash A$ and $\phi(y_n) = 2^{-n}$ for all $n \in \omega$. Note that $\phi(y_{n+1}) < \phi(y_n)$ for all $n \in \omega$ and that $\phi(y_0) \le \phi(x)$ for all $x \in X\backslash A$. By using Lemma 1.3 we deduce that ϕ is lower semicontinuous on (X, p^s).

Moreover, for each $x \in X\backslash A$ we have

$$\begin{aligned} p(x, Tx) &= p(x, y_0) \\ &= \phi(x) - \phi(y_0) \\ &= \phi(x) - \phi(Tx) \\ &\le p(x, x) + \phi(x) - \phi(Tx), \end{aligned}$$

and for each $y_n \in A$ we have

$$\begin{aligned} p(y_n, Ty_n) &= p(y_n, y_{n+1}) \\ &\le p(y_n, y_n) + 2^{-(n+1)} \\ &= p(y_n, y_n) + \phi(y_n) - \phi(Ty_n). \end{aligned}$$

Therefore T is a Caristi mapping on X without fixed point, a contradiction. This concludes the proof.

The following example illustrates Theorem 2.3.

Example 2.4. Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a complete PMS (and so it is 0-complete). Define $T : X \to X$ by $Tx = \sqrt{x}$ and define $\phi : X \to [0, \infty)$ by $\phi(x) = 1 - x$. Then ϕ is continuous and so it is lower semicontinuous function for (X, p^s). Also we have

$$p(x, Tx) = \sqrt{x} \le p(x, x) + \phi(x) - \phi(Tx)$$

for all $x \in X$. Therefore, T is a Caristi mapping on X, so by Theorem 2.3, T has a fixed point. Note that, nevertheless, Theorem 1.4 can not be applied to this example, because

$$p(1, T1) = 1 \nleq 0 = \phi(1) - \phi(T1),$$

for any function $\phi : X \to [0, \infty)$.

In his paper [13], Matthews also introduced and studied the notion of a weightable quasi-metric space; in particular, he established (see Theorem 2.5 below) the precise relationship between partial metric spaces and weightable quasi-metric spaces. We conclude the paper by establishing the weightable quasi-metric counterpart of Theorem 2.3. To this end, we recall the following pertinent concepts and facts.

A quasi-metric on a set X is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$: (i) $x = y \Leftrightarrow d(x, y) = d(y, x) = 0$; (ii) $d(x, z) \le d(x, y) + d(y, z)$.

A quasi-metric space is a pair (X, d) such that X is a set and d is a quasi-metric on X.

If d is a quasi-metric on X, then the function $d^s : X \times X \to [0, \infty)$ defined by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$, is a metric on X.

A quasi-metric space (X, d) is said to be bicomplete if the metric space (X, d^s) is complete.

According to [13], a quasi-metric space (X, d) is weightable provided that there exists a function $w : X \to [0, \infty)$ such that for all $x, y \in X$, $d(x, y) + w(x) = d(y, x) + w(y)$. The function w is said to be a weight function for (X, d) and the quasi-metric d is weightable by the function w.

The following result was proved by Matthews [13].

Theorem 2.5. *(a) Let (X, p) be a partial metric space. Then, the function $d_p : X \times X \to [0, \infty)$ defined by $d_p(x, y) = p(x, y) - p(x, x)$ for all $x, y \in X$, is a weightable quasi-metric on X with weight function w given by $w(x) = p(x, x)$ for all $x \in X$.*

(b) Conversely, if (X, d) is a weightable quasi-metric space with weight function w, then the function $p_d : X \times X \to [0, \infty)$ defined by $p_d(x, y) = d(x, y) + w(x)$ for all $x, y \in X$, is a partial metric on X.

The following fact is well-known (see for instance [14]) and easy to verify.

Lemma 2.6. *A weightable quasi-metric space (X, d) is bicomplete if and only if (X, p_d) is a complete PMS.*

For quasi-metric spaces we propose the following natural notion of a Caristi mapping.

Definition 2.7. A self mapping T of a quasi-metric space (X, d) is called a Caristi mapping on X if there is a function $\phi : X \to [0, \infty)$ which is lower semicontinuous for (X, d^s) and satisfies $d(x, Tx) \leq \phi(x) - \phi(Tx)$, for all $x \in X$.

Remark 2.8. Hicks [9], Ćirić [5] and Cobzaş [6], among others, have obtained quasi-metric versions of Cariti's fixed point theorem with different approaches to the one given here.

Remark 2.9. It follows from Theorem 2.5 that a self mapping T of a weightable quasi-metric space (X, d) is a Caristi mapping, in the sense of Definition 2.7, if and only if T is a Caristi mapping, for the PMS (X, p_d).

Combining Lemma 2.6, Theorem 2.3 and Remark 2.9, we obtain the following.

Theorem 2.10. *A weightable quasi-metric space (X, d) is bicomplete if and only if every Caristi mapping on X has a fixed point.*

Finally, we present an example which shows that "weightable" can not be omitted in the preceding theorem.

Example 2.11. Let $X = (0, \infty)$ and let $d : X \times X \to [0, \infty)$ defined by $d(x, y) = x - y$ if $x \geq y$, and $d(x, y) = 1$ otherwise. Then d is a quasi-metric on X and $d^s(x, y) = \max\{1, |x - y|\}$ for all $x \neq y$. Hence (X, d^s) is a complete metric space, so (X, d) is bicomplete. Now define $T : X \to X$ by $Tx = x/2$, and $\phi : X \to [0, \infty)$ by $\phi(x) = x$ for all $x \in X$. Note that d^s generates the discrete topology on X, so ϕ is obviously lower semicontinuous for (X, d^s). We also have

$$d(x, Tx) = d(x, \frac{x}{2}) = \frac{x}{2} = \phi(x) - \phi(Tx),$$

for all $x \in X$. Consequently T is a Caristi mapping on X without fixed point.

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