SCIENTIFIC REPORT

New trends on the numerical representability of semiordered structures

F.J. Abrísqueta, M.J. Campión, R.G. Catalán, J.R. De Miguel, A. Estevan, E. Induráin, M. Zudaire, Universidad Pública de Navarra
L. Agud, Universidad Politécnica de Valencia
J.C. Candeal, Universidad de Zaragoza
S. Díaz, D. Martinetti, S. Montes, Universidad de Oviedo
J. Gutiérrez García, Universidad del País Vasco UPV/EHU

Abstract

We introduce a survey, including the historical background, on different techniques that have recently been issued in the search for a characterization of the representability of semiordered structures, in the sense of Scott and Suppes, by means of a real-valued function and a strictly positive threshold of discrimination.

1. Introduction

Interval orders are perhaps the best class of ordered structures to build models of uncertainty or to represent and manipulate vague or imperfectly described pieces of knowledge. Dealing with different classes of orderings $\prec$ defined on a nonempty set X, the concept of an interval order was introduced[1] by Peter C. Fishburn (see [36]) in contexts coming from Mathematical Psychology (Measurement Theory) and Economics (Utility Theory), in order to study models of preference or measurement orderings whose associated indifference may fail to be transitive. See also[2] [72, 66, 35, 40, 55, 56], or [14]. As pointed out in [40], intransitive preferences may occur in a wide variety of decisional contexts that include economic consumer theory, multiattribute utility theory, game theory, preference between time streams, and decision making under risk and uncertainty. In Fishburn's own words (see [40]):

> *<<Intransitive preferences have been a topic of curiosity, study, and debate over the past 40 years. Many economists and decision theorists insist on transitivity as the cornerstone of rational choice, and even in behavioral decision theory intransitivities are often attributed to faulty experiments, random or sloppy choices, poor judgment, or unexamined biases. But others see intransitive preferences and potential truths of reasoned comparisons and propose representations of preferences that accommodate intransitivities>>.*

Obviously, a key point in the analysis of the aforementioned problem ("representations of preferences that accommodate intransitivities) consists in the search of numerical representations (see e.g. [41]) to deal with one of the most classical structures that have been built to model nontransitive assessments, namely, that of interval ordered structures.

The representability of an interval order $\mathcal{R}$ defined on a nonempty set X by means of two real valued functions $F, G : X \to \mathbb{R}$ such that $x\mathcal{R}y \iff G(x) < F(y) \quad (x, y \in X)$ leads us in a natural way to use interval methods to deal with this kind of orderings, namely an element $x \in X$ would be assigned an interval $[F(x), G(x)]$ (that may eventually collapse into a point if $F(x) = G(x)$) of real numbers. This use of interval-valued correspondences to represent the imprecise or uncertain has many motivations (see e.g. [9, 26]). An appealing particular case of representable interval orders requires all the image intervals to have the same length, with problems of inexact measurement in mind. This will correspond to semiorders that are representable in the Scott and Suppes sense. This concept of a representable semiorder will be the subject matter of the present paper[3]. In this context, we may observe that one of the main functions of a computer is "data processing. The set of data usually comes from measurement, so that we try to know the characteristics of the physical quantities that characterize our world. But, in many processes, the data we are dealing with are not absolutely precise. This leads to the inaccuracy in the result of data processing.

The concept of a semiorder was introduced in [53] to deal with innacuracies in measurements where a nonnegative threshold of discrimination is involved. The original idea was that of presenting a mathematical model of preferences enable to capture situations of "intransitive indifference with a threshold of discrimination:

> *<<Suppose, for instance, that a man is not able to declare different two quantities of a same thing when such two quantities do not differ more than a threshold of discrimination or perception, α. This threshold is a non-negative real number, and it is supposed to be the same for every individual. That is, if $a \prec b$ means here "a man is able to realize that the quantity a is smaller than b, then we have $a \prec b \iff a + \alpha < b$.*
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> *A classical example, attributed to Armstrong ([6]), considers a man that prefers a cup of coffee with a whole portion of sugar, to a cup of coffee with no sugar at all. If such man is forced to declare his*

[1]Under a different name, the concept of an interval order was already implicit much earlier, in the work of Norbert Wiener. (See e.g. [74]).

[2]All these studies are framed in the crisp setting. Possible extensions to the fuzzy setting have recently been studied in [28, 27, 49].

[3]This paper is an extended version, now incorporating our most recent results (see [17, 21, 2, 13, 22, 27, 49, 1, 32, 20]), of a previous work (namely, [48]) that was communicated at the congress FLINS 2008 held in Madrid, Spain. A former, more elementary version, was presented at the congress on Ordinal and Symbolic Data Analysis (OSDA 2007), held in Ghent, Belgium.

preference between a cup with no sugar at all and a cup with only one molecule of sugar, he will declare them indifferent. The same will occur if he compares a cup with n molecules and a cup with $n+1$ molecules of sugar. However, after a very large number of intermediate comparisons we would finally confront him with a cup that has a whole portion of sugar that he is able to discriminate from the cup with no sugar at all. Here, we observe a clear intransitivity of indifferences.>>

In this model, the threshold is constant (the same for all agents or ways of measure). Consequently, different features can only be distinguished if they differ in a quantity bigger than the corresponding threshold. Otherwise they are declared indifferent, so giving rise to intransitivity of this associated indifference as in the previous classical example. Classical studies on semiorders appeared in [34, 35, 37, 38, 39, 54, 42, 43, 44, 62, 63, 8, 14, 45], or [64]. In some of those studies were obtained either necessary or else sufficient conditions for the existence of a numerical representation of a semiorder $\prec$ defined on a nonempty set X by means of a real-valued function $f : X \to \mathbb{R}$ and a non-negative constant or discrimination threshold $\lambda \geq 0$ such that $a \prec b \iff f(a) + \lambda < f(b) \quad (a, b \in X)$. Some characterizations of the existence of those representations appeared in [24]. This kind of numerical representations of semiordered structures is also encountered in a wide range of applications, not only in data processing coming mainly from Computer Theory or Artificial Intelligence, but also in extensive measurement in Mathematical Psychology (see [52]), choice theory under risk (see [33]), decision-making under risk (see [67]), modellization of choice with errors (see [4]), social welfare theory (see [59]), general equilibrium theory in Economics (see [50]), and expected utility theory in mixture spaces (see [73]).

The structure of the paper goes as follows:

After the Introduction (Section 1), the necessary notations and background definitions are introduced in Section 2 (preliminaries). Section 3 is a guided tour to the history of the numerical representability of semiorders, in which we revise the most important achievements arising in this literature. Section 4 is devoted to show the main results about the characterization of the numerical representability of semiorders by means of a real-valued utility function and a strictly positive threshold of discrimination, in the sense of Scott and Suppes. An account of the main techniques that have been introduced to get the key results is presented and discussed in Section 5. The relationship between the numerical representability of semiorders and the solution of certain functional equations is analyzed in Section 6. Semiorders with special properties of algebraic or topological nature are considered in Section 7. A relationship between representable semiorders and a particular case of fuzzy numbers is shown in Section 8. The possibility of extending the concept of a semiorder to the fuzzy setting is discussed in Section 9. A short list of open questions (Section 10) closes the paper.

2. Preliminaries

Let X be a nonempty set. Let $\prec$ be an asymmetric binary relation defined on X. Associated to $\prec$ we define the reflexive and total binary relation $\precsim$ given by $x \precsim y \iff \neg(y \prec x) \quad (x, y \in X)$, the symmetric binary relation $\sim$, called *indifference*, given by $x \sim y \iff [(\neg(x \prec y)) \wedge (\neg(y \prec x))] \; (x, y \in X)$, and the *dual* relation $\prec_d$ defined by $x \prec_d y \iff y \prec x$ for all $x, y \in X$.

An *interval order* $\prec$ is an asymmetric binary relation such that $[(x \prec y)$ and $(z \prec t)] \Rightarrow [(x \prec t)$ or $(z \prec y)] \; (x, y, z, t \in X)$. An interval order $\prec$ is said to be a *semiorder* if $[(x \prec y)$ and $(y \prec z)] \Rightarrow [(x \prec w)$ or $(w \prec z)]$ for every $x, y, z, w \in X$. An interval order $\prec$ defined on X is said to be *representable* (as an interval order) if there exist two real valued functions $u, v : X \longrightarrow \mathbb{R}$ such that $x \prec y \iff v(x) < u(y) \quad (x, y \in X)$. Also, a semiorder $\prec$ defined on X is said to be *representable* (now, as a semiorder!) *in the sense of Scott and Suppes* (see [68]) if there exist a real-valued function $u : X \to \mathbb{R}$ and a nonnegative constant or "discrimination threshold $\lambda \geq 0$ such that $x \prec y \iff \lambda < u(y) - u(x) \quad (x, y \in X)$. (Obviously, if such a representation exists with $\lambda = 0$, the associated indifference $\sim$ is transitive). There exist interval orders that fail to be representable (as interval orders). Also, there exist semiorders that are not representable in the sense of Scott and Suppes. (See [61, 24] for further details). Let us recall that a *preorder* $\precsim$ on an arbitrary nonempty set X is a binary relation on X which is reflexive and transitive. If $\precsim$ is a preorder on X, then the pair $(X, \precsim)$ is said to be a preordered set. An antisymmetric preorder is said to be an *order*. A *total preorder* $\precsim$ on a set X is a preorder such that if $x, y \in X$ then $[x \precsim y]$ or $[y \precsim x]$. If $(X, \precsim)$ is a preordered set then a real-valued function $u : X \to \mathbb{R}$: is said to be an *order-monomorphism* for $\precsim$ if, for every $x, y \in X$, it holds that $[x \precsim y \iff u(x) \leq u(y)]$. A total preorder $\precsim$ on X is called *representable* if there is an order-monomorphism for $\precsim$. Following [34, 36], associated with an interval order $\prec$ defined on a nonempty set X, we shall consider two new binary relations $\prec^*$ and $\prec^{**}$ given by $x \prec^* y \iff x \prec z \precsim y$ for some $z \in X \quad (x, y \in X)$, and, similarly, $x \prec^{**} y \iff x \precsim z \prec y$ for some $z \in X \quad (x, y \in X)$. We denote $x \precsim^* y \iff \neg(y \prec^* x)$, $x \sim^* y \iff x \precsim^* y \precsim^* x$, $x \precsim^{**} y \iff \neg(y \prec^{**} x)$ and $x \sim^{**} y \iff x \precsim^{**} y \precsim^{**} x \quad (x, y \in X)$. As a matter of fact, both the binary relations $\precsim^*$ and $\precsim^{**}$ are total preorders on X. Moreover, the indifference relation $\sim$ associated with the interval order $\prec$ is transitive if and only if $\precsim^*$, $\precsim^{**}$ and the binary relation $\precsim$ associated with the interval order $\prec$ coincide. In particular, in this case $\precsim$ is also a total preorder on X. (See [58] for more details). Moreover, a new binary relation $\precsim^0$ is defined on X with the help of $\prec^*$ and $\prec^{**}$, by declaring that $x \precsim^0 y \iff [(x \precsim^* y) \wedge (x \precsim^{**} y)] \quad (x, y \in X)$. This new binary relation $\precsim^0$ allows us to characterize semiorders among interval orders, as proved in [34, 35], namely, if X is a nonempty set and $\prec$ an interval order on X, then $\prec$ is indeed a semiorder if and only if $\precsim^0$ is a total preorder on X. To conclude this section, we point out that certain sets of fuzzy numbers can actually be considered as alternative codomains (instead of the almost universal use of real numbers) to represent interval orders and semiorders defined on a set. This easy fact consists in a natural translation of real intervals as symmetric triangular fuzzy numbers, so that intervals of the same length correspond to symmetric triangular fuzzy numbers of unitary base. A suitable lexicographic ordering is defined on the set of symmetric triangular fuzzy numbers, so that representable interval orders (respectively, semiorders that are representable in the sense of Scott and Suppes) can be identified to a subset of symmetric triangular fuzzy numbers (respectively, symmetric triangular fuzzy numbers of unitary base) ordered in that way. (See [16, 9, 21, 13, 32] for a further account).

3. Historical background: a revision of the literature on semiorders

In this section we provide a revision of the key papers in the literature of semiorders, with an special interest in the problem of the numerical representation.

The pioneer work that introduced the concept of a semiorder is [53]. As mentioned in the introduction in that paper the main idea was to introduce a model to cope with intransitivities in preferences that depend on a constant threshold of discrimination. In [68] the problem of the numerical representability of a semiorder is considered from a theoretical point of view. It is shown that any semiorder defined on a finite nonempty set admits a representation through a real-valued function and a nonnegative constant threshold of perception or discrimination. In [37] they appear important results that study semiorders as a particular class of interval orders. General representation theorems for interval orders appear there for the first time. In [58] semiorders and interval orders are considered as particular classes or subclasses of *Ferrers relations*, where a Ferrers relation $\mathcal{Q}$ on a nonempty set X accomplishes that $(x\mathcal{Q}y \wedge z\mathcal{Q}t) \Rightarrow (x\mathcal{Q}t \vee z\mathcal{Q}y)$, for every $x, y, z, t \in X$. They were introduced in [65]. (With our notation, an interval order is an asymmetric Ferrers relation). An axiomatic study is made for this kind of binary relations when they are defined on a nonempty finite set. In [54] it is characterized the numerical representability of semiorders (in the sense of Scott and Suppes) for the case of semiorders defined on infinite, but countable sets. In [37] further results on semiorders considered as a particular class of interval orders are stated for the case of binary relations defined on a nonempty finite set. In [29], it is introduced the key concept of a biorder[4] understood as a correspondence $\mathcal{R}$ from a nonempty set A to a nonempty set B such that A and B are disjoint ($A \cap B = \emptyset$) and $(a\mathcal{R}b \wedge c\mathcal{R}d) \Rightarrow (a\mathcal{R}d \vee c\mathcal{R}b)$, for every $a, c \in A$ and $b, d \in B$. It is shown that an interval order $\prec$ defined on a set X can be interpreted as a suitable biorder defined from X to a copy X^+ of X such that $X^+ \cap X = \emptyset$. Using biorders, new techniques are introduced that allow us to find a new general theorem on the representability of interval orders. (See also [10]). In [42, 43, 44] the problem of the continuous representability of interval orders and semiorders defined on a nonempty set X (finite or not!) endowed with a topology τ is studied in detail, and new techniques to represent semiorders are introduced. Basically, the idea is to modify representations, but just as an interval order, of a given semiorder $\prec$ defined on a nonempty set X, in order to find new representations but now as a semiorder, in the sense of Scott and Suppes, of the binary relation $\prec$. The use of some classical functional equation is implicit. In [67] we find some results that can be used to analyze the Scott-Suppes representability of semiorders in particular cases. The concept of predecessor and successor[5] elements is implicit. These kind of questions concerning continuity of the representations have also been studied more recently in [17], using different techniques. In [62, 63] and [64] they appear several classical studies on semiorders, but mainly devoted to the finite case, using techniques based on Combinatorics and Discrete Mathematics. In [8] the concept of a generalized numerical representation is introduced. A generalized numerical representation consists of a real-valued function and an associated subset of the real plane of a certain type. Some applications to the Scott-Suppes representability of semiorders can be obtained in terms of these generalized numerical representations. A new characterization of the Scott-Suppes separability of semiorders defined on countable infinite sets, alternative to that in [54] is also included. In [24] some characterizations of the Scott-Suppes representability of semiorders are obtained for the general case in terms of suitable extensions of the given semiordered structure. In [2] there is an attempt to find new characterizations of the Scott-Suppes representability of semiorders based only on internal conditions, that is, avoiding the use of extensions of the given structure. Some results in this direction are obtained for particular cases.

Finally, the main problem of finding an internal characterization of the Scott-Suppes representability of semiorders valid for the general case was solved in [22]. (See also [20]).

4. Characterizing the Scott-Suppes representability of semiorders

Any attempt to characterize the Scott-Suppes representability of a semiorder would obviously compare this structure to other well-known structures where a characterization of the representability is already known, as it is the case of total preorders and interval orders. We recall that a total preorder $\mathcal{R}$ defined on a nonempty set X is representable if and only if X is *perfectly separable*, that is, there exists a countable subset $D \subseteq X$ such that for given $x, y \in X$ for which $\neg(y\mathcal{R}x)$ holds, there exists $d \in D$ such that $(x\mathcal{R}d) \wedge (d\mathcal{R}y)$. (See the first chapters in [15] for details). Moreover, an interval order $\prec$ defined on a nonempty set X is representable if and only if it is *interval-order separable*, that is, there exists a countable subset $D \subseteq X$ such that for given $x, y \in X$ with $x \prec y$, there exists $d \in D$ such that $x \prec d \precsim^{**} y$. (See e.g. [61] for further details).

About semiorders, until the recent solution obtained in [22] was at hand, no "internal characterization was known for the general case. As a matter of fact, previously to [22] some characterizations of the Scott-Suppes representability of semiorders for the general case were already obtained,

[4] Under a different name, this concept was already implicit in [31].

[5] The concept of predecessor and successor elements is a clue to find a characterization of the Scott-Suppes representability of semiorders, as proved in [24].

but all of them needed to consider suitable extensions of the semiordered structure. The schema of these "old characterizations is the following: given a semiorder $\prec$ on a nonempty set X, the semiorder $\prec$ admits a Scott-Suppes representation if and only if there exists some suitable extension $(X', \prec')$ of the semiordered structure (i.e.: X' is a superset of X, and $\prec'$ is a semiorder on X' whose restriction to X coincides with $\prec$), accomplishing a list of conditions. (See [24] for details). In this direction, one of these characterizations can be stated as follows (see [24, 2]):

Theorem 5 *The Generalized Scott-Suppes Representability Theorem: Let X be a nonempty set endowed with a semiorder $\prec$. The semiorder $\prec$ is representable through a function $u : X \to \mathbb{R}$ and a strictly positive threshold K such that $x \prec y \iff u(x) + K < u(y)$ for all $x, y \in X$ if and only if there exists a Dedekind complete[6], without jumps[7] and neither minimal nor maximal elements, extension $(\bar{X}, \bar{\precsim}^0)$ of the totally preordered structure $(X, \precsim^0)$ that satisfies the following properties:*

1. *Let $Z = \bar{X}/\bar{\sim}^0$. Then for a given $z \in Z$ there exist two elements[8] $S(z)$ and $P(z)$ in Z such that $P(S(z)) = z = S(P(z))$, and for every $a, b \in Z$ it holds that $S(a)\bar{\precsim}^0 b \iff a\bar{\precsim}^0 P(b)$.*
2. *For every $z \in Z$ the family $\mathcal{F}_z = \{z\} \cup \{S_k(z) : k \in \mathbb{N}, k > 0\} \cup \{P_k(z) : k \in \mathbb{N}, k > 0\}$ is coinitial and cofinal[9] with respect to the totally ordered structure $(Z, \bar{\precsim}^0)$, (where $P_0(z) = z = S_0(z), S_{k+1}(z) = S(S_k(z)), P_{k+1}(z) = P(P_k(z))$ for all $k \in \mathbb{N}$).*
3. *For every $p, q \in X$, let $\bar{p}$ and $\bar{q}$ be their respective equivalent classes in $Z = \bar{X}/\bar{\sim}^0$. Then it holds that $p \prec q \iff S(\bar{p})\bar{\precsim}^0\bar{q}$.*
4. *The totally preordered structure $(\bar{X}, \bar{\precsim}^0)$ is representable. (In particular, we can extend $\prec$ to a representable semiorder $\bar{\prec}$ on $Z = \bar{X}/\bar{\sim}^0$ by declaring that $z_1 \bar{\prec} z_2 \iff S(z_1)\bar{\precsim}^0 z_2$ for all $z_1, z_2 \in Z$).*

Before having achieved in [22] the solution to the general case, the problem of searching for internal characterizations of the Scott-Suppes representability of semiorders for a large class of semiordered structures was addressed in [2]. To quote here, for the sake of completeness, one of the main results in the last reference ([2]) we shall introduce now some previous concepts and definitions. A semiorder $\prec$ defined on a nonempty set X is said to be *typical* if it does not coincide with the asymmetric part of a total preorder, or equivalently, if $\precsim$ fails to be transitive. Let $\prec$ be a typical semiorder defined on a nonempty set X. Then the semiorder $\prec$ is said to be *irreducible* (see e.g. [8]) if $\precsim^0$ is a total order on X. Henceforth we shall consider a typical semiorder $\prec$ that is irreducible, defined on a nonempty set X.

Let $(Y, \precsim_Y)$ be a Dedekind completion of $(X, \precsim^0)$, characterized by the following properties: $(Y, \precsim_Y)$ is totally ordered and Dedekind-complete, $X \subseteq Y$, the restriction of $\precsim_Y$ is $\precsim^0$, and no proper subset $Z \subsetneq Y$ such that $X \subseteq Z$ is Dedekind complete with respect to the restriction of $\precsim_Y$ to Z. Every element in $Y \setminus X$ is determined by a Dedekind cut of X. (See p. 3 in [46] for further details). It may still happen that in $(Y, \precsim_Y)$ there are jumps, where a jump is a pair of elements $y_1, y_2 \in Y$ such that $y_1 \prec_Y y_2$ and there is no $z \in Y$ such that $y_1 \prec_Y z \prec_Y y_2$. We enlarge again Y by inserting a copy of the open unit interval $U = \{x \in \mathbb{R} : 0 < x < 1\}$ between the elements that define each jump. It may also occur that Y has a smallest element m and/or a greatest element M with respect to $\precsim_Y$. In this case we put a copy of U just before m and/or a copy of U just after M. Taking the newly added elements into account, the total order $\precsim_Y$ is extended in the obvious way to the enlarged set. All this standard procedure leads to a particular Dedekind completion (without jumps and neither minimal nor maximal elements) of the totally preordered structure $(X, \precsim^0)$. Let $(T, \precsim_T)$ denote that extension, that we will consider from now on. We look for special features of $(T, \precsim_T)$ that give rise to conditions for the Scott-Suppes representation of $(X, \prec)$. To do so, associated with each $x \in X$ that is not maximal with respect to $\prec$, we consider the elements $S_+(x)$ and $S_-(x)$ in $(T, \precsim_T)$, defined as follows: $S_+(x) = \inf$ (*with respect to* $\precsim_T$) $\{y \in X : x \prec y\}$. $S_-(x) = \sup$ (*with respect to* $\precsim_T$) $\{y \in X : y \precsim x\}$. Observe that these elements $S_+(x)$, $S_-(x)$ are well defined because $\precsim_T$ is Dedekind complete. On the one hand, the set $A_x = \{y \in X : x \prec y\}$ is nonempty by hypothesis because x is not maximal. Moreover $x \prec_T y$ for every $y \in A_x$, so that A_x is bounded by below with respect to $\precsim_T$ and consequently $S_+(x) = \inf$ (*with respect to* $\precsim_T$) A_x exists. On the other hand, given the set $B_x = \{y \in X : y \precsim x\}$ we immediately observe that $x \in B_x$. Thus $B_x \neq \emptyset$. Since x is not maximal, there exists $a \in X$ such that $x \prec a$ from which we get $y \precsim x \prec a \Rightarrow x \prec^0 a \Rightarrow x \prec_T a$ for every $y \in B_x$. Therefore B_x is bounded by above, and $S_-(x) = \sup$ (*with respect to* $\precsim_T$) B_x exists.

In addition $S_+(x)$ and $S_-(x)$ may eventually coincide. Dually, if $x \in X$ is not minimal with respect to $\prec$, let $P_+(x)$, $P_-(x)$ be the following elements in $(T, \precsim_T)$: $P_+(x) = \sup$ (*with respect to* $\precsim_T$) $\{z \in X : z \prec x\}$. $P_-(x) = \inf$ (*with respect to* $\precsim_T$) $\{z \in X : x \precsim z\}$. As above, the elements $P_-(x)$, $P_+(x)$ are well defined and may eventually coincide. Let $\prec$ be a typical semiorder defined on a nonempty set X. We say that $\prec$ is *regular* if for any $x, y \in X$ such that $x \prec y$, there is no infinite sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x \prec x_n \prec x_{n+1} \prec y$ for all $n \in \mathbb{N}$ and, dually, there is no infinite sequence $(z_n)_{n\in\mathbb{N}} \subset X$ such that $x \prec z_{n+1} \prec z_n \prec y$ for all $n \in \mathbb{N}$. A condition that is equivalent to regularity was introduced in Axiom A1 on p. 435 of [8]. Other alternative condition was introduced in

[6] A totally preordered structure $(A, \precsim)$ is said to be *Dedekind-complete* if each nonempty subset $F \subseteq A$ that has an upper bound has a least upper bound.

[7] A *jump* of a totally preordered structure $(X, \precsim)$ is a pair of elements $a, b \in X$ such that $a \prec b$ and there is no $c \in X$ such that $a \prec c \prec b$.

[8] These elements are respectively called *sucessor* $S(z)$ of z and *predecessor* $P(z)$ of z. As aforementioned, the concept of predecessor and sucessor elements was already implicit in [67].

[9] Let $(X, \precsim)$ be a totally preordered set. A subset $Z \subseteq X$ is said to be *coinitial* (respectively, *cofinal*) in X if for every $x \in X$ there exists some $z \in Z$ such that $z \precsim x$ (respectively, such that $x \precsim z$).

[54]. A semiorder $\prec$ defined on a nonempty set X is said to be connected if the totally preordered structure $(X, \precsim^0)$ is Dedekind complete and without jumps.

The statement of the key result in [2] is in order now:

Theorem 6 *Let $\prec$ be a connected irreducible typical semiorder defined on a nonempty set X. Assume also that the following conditions are satisfied:*

1. *The semiorder $\prec$ is regular.*
2. *If $x \in X$ is not maximal with respect to $\prec$, then $x \neq S_+(x)$ and $S_+(x) \precsim x$. Dually, if $x \in X$ is not minimal with respect to $\prec$, then $P_+(x) \neq x$ and $x \precsim P_+(x)$.*
3. *If $x \in X$ is not maximal with respect to $\prec$, then $P_-(S_+(x))$ is well defined and $P_-(S_+(x)) = x$. Dually, if $x \in X$ is not minimal with respect to $\prec$, then $S_-(P_+(x))$ is well defined and $S_-(P_+(x)) = x$.*
4. *The totally ordered structure $(X, \precsim^0)$ is representable.*

Then the semiorder $\prec$ admits a Scott-Suppes representation through a real-valued function, with threshold 1.

In [22] the problem of finding an internal characterization, valid for the general case, of the Scott-Suppes representability of semiorders was finally solved[10]. The main result reads as follows:

Theorem 7 *Let $\prec$ be a typical semiorder defined on a nonempty set X. Then $\prec$ is representable in the sense of Scott and Suppes if and only if it is regular and, in addition, there exists a countable subset $D \subseteq X$ such that for every $x, y \in X$ with $x \prec y$ there exist $a, b \in D$ such that $x \precsim^0 a \prec y$ and also $x \prec b \precsim^0 y$.*

In addition, in the still unpublished paper [20] many other characterizations of the Scott-Suppes representability of a semiorders have been obtained, so completing the panorama.

Theorem 8 *Let $\prec$ be a typical semiorder defined on a nonempty set X. Then $\prec$ is representable in the sense of Scott and Suppes if and only if it is regular and, considered as an interval order, it is interval-order separable.*

Remark 1 *At least ten other alternative conditions are equivalent to the interval-order-separability of an interval order. (See Theorem 3.7 in [20]).*

5. Scott-Suppes representability of semiorders. Main techniques

Perhaps the main difficulty to get a characterization of the numerical representability of semiorders is due to the following remarkable fact: the known characterizations of the representability of other classical ordered structures such as total preorders (see e.g. the first chapter in [15]) and interval orders (see [34, 29, 61, 11, 25]) are given in terms of the existence of suitable countable subsets. A consequence is that countable total preorders and countable interval orders are (trivially) representable. But an analogous fact is no longer true for semiorders: there exist countable semiorders that are not representable (see e.g. [68, 54, 24]). Till [22] was published, another difficulty found in the search for characterizations of the representability of semiorders was due to the necessity of considering suitable extensions of the totally preordered structure $(X, \precsim^0)$ related to the semiordered structure $(X, \prec)$, instead of working directly with the given semiordered structure. (See [24, 2]). Historically, the first key result on the Scott-Suppes representability of semiorders was proved in [68], and concerns the finite case, proving that any typical semiorder $\prec$ defined on a finite nonempty set X is representable through a real valued function $F : X \to \mathbb{R}$ satisfying that $a \prec b \iff F(a) + 1 < F(b) \quad (a, b \in X)$. To prove this, the technique used in Scott-Suppes goes as follows: Consider the associated withtal order $\precsim^0$. Choose a representative element $x \in X$ of each equivalence class that $\sim^0$ defines on X. Let $Y = \{x_0, \ldots, x_k\} \subseteq X$ be the resulting set of representatives, ordered by $\prec^0$ so that $x_i \prec^0 x_{i+1} \quad (i = 0, \ldots, k)$ Associated to Y, we consider a set of rational numbers $\{a_0, a_1, \ldots, a_k\}$ whose elements are defined as follows by induction on i as follows: $a_0 = 0$; $a_i = \frac{i}{i+1}$ if $x_i \precsim x_0$; $a_i = \frac{i}{i+1}a_j + \frac{i}{i+1}a_{j-1} + 1$ if $x_i \precsim x_j$ and $x_{j-1} \prec x_i \quad (j \leq i)$. Finally define $F : X \to \mathbb{R}$ by $F(x) = a_i$, where x_i is a representative of the equivalence class of x with respect to ~ 0. A final checking to the construction shows that this provides a Scott-Suppes representation with positive threshold 1, for the semiordered structure $(X, \prec)$. The next step here would consist, obviously, in analyzing the Scott-Suppes semiorders defined on infinite sets. A former question that we could ask ourselves at this stage is: *Can we extend in a natural way the original Scott-Suppes technique and construction (see [68]) in order to represent infinite semiorders? The answer is "yes, but only for particular cases. Indeed, a direct strengthening of the above technique shows that every semiorder $\prec$ defined on an infinite set X such that the order topology that $\precsim^0$ induces on X is the discrete one, can be represented with threshold 1 through a real valued function $F : X \to \mathbb{R}$ such that $a \prec b \iff F(a) + 1 < F(b)$ $(a, b \in X)$. There are many situations of infinite semiorders that do not accomplish the above restriction (to put an example, consider the set $\mathbb{Q}$ of rational numbers endowed with the semiorder $\prec$ given by $p \prec q \iff p + 1 < q \quad (p, q \in \mathbb{Q})$, that is obviously representable through the identity function and the threshold 1). A key related question is: Is there some "germ*[11] *of non-representability for semiorders?* Based on the concept of regularity already introduced, it is clear that the set $Y = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$ of real numbers endowed with the usual strict order "< on $\mathbb{R}$, is an example of a germ of non-representability for typical ir-

[10] A similar result was pointed out to us (but not published) by our colleague Esteban Olóriz, early in 2000.

[11] For a *germ of non-representability* we mean a semiordered structure $(Z, \prec_Z)$ such that if a semiordered structure $(X, \prec)$ contains either a copy of $(Z, \prec_Z)$ or a copy of its dual structure $(Z, \prec_d)$ where $a \prec_d b \iff b \prec_Z a \quad (a, b \in Z)$, then, a fortiori, $(X, \prec)$ does not admit a Scott-Suppes representation.

reducible semiorders. (Notice that $<$ is in particular a semiorder on Y). Another example of a germ is the set $Y = \{x \in \mathbb{R} : 0 < x < 2\}$ endowed with the semiorder $\prec_Y$ given by $x \prec_Y y \iff x + 1 \leq y$ for all $x, y \in Y$. Since by definition a typical irreducible semiordered superstructure of a germ is also a germ of non-representability for typical irreducible semiorders, the most important germs are obviously the minimal ones. A germ $(Y, \prec_Y)$ is said to be *minimal* if, for every semiordered substructure $(Z, \prec_Z)$ that is not semiorder-isomorphic to $(Y, \prec_Y)$, there exist a Scott-Suppes representation by means of a real-valued function u_Z defined on Z, and threshold 1. We immediately observe that a catalogue of all possible minimal germs of non-representability induces a characterization theorem, saying that a typical irreducible semiorder $\prec$ defined on a set X admits a Scott-Suppes representation if and only if it does not contain a substructure semiorder-isomorphic to a minimal germ. For the case of interval orders, the idea of studying possible minimal germs of non-representability appears implicitly in [57] by means of the concept of a "forbidden suborder. For several other ordered structures and different kinds of representations this problem has been completely solved. (See e.g. [18] for the case of commutative totally ordered semigroups using representations by means of real-valued order-preserving monomorphisms of semigroups. See also [7] for the case of the representation of totally ordered sets through a real-valued order-preserving function). The obtention of an exhaustive list of minimal germs of non-representability for irreducible typical semiorders constitutes an open problem.

We may introduce here some result in this direction, recently proved in [20].

Theorem 9 *Let $\prec$ be an irreducible typical semiorder defined on a nonempty set X. Suppose that $\prec$ admits a representation as an interval order. Then $\prec$ admits a Scott-Suppes representation with threshold* 1 *if and only if neither it, nor its dual structure $(X, \prec_d)$, contains a substructure semiorder-isomorphic to $(Y, <)$, where $Y = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$ and "$<$ denotes the usual strict order on real numbers.*

We immediately observe that regularity is a necessary condition for the Scott-Suppes representability of a semiorder, so that an infinite sequence of any of the two kinds that appear in the definition of regularity constitutes a germ of non-representability (in the sense of Scott and Suppes) for semiorders. Actually, for the particular case of semiorders defined on a countable infinite set, the germ of non-representability that we have just introduced in the previous Theorem 9, namely $(Y, <)$, where $Y = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$ and "$<$ is the usual strict order on the real line $\mathbb{R}$ is actually minimal. The reason is that the Scott-Suppes representability of typical semiorders defined on countable infinite sets was characterized in [54], through the following key result:

Theorem 10 *Let $\prec$ denote a typical semiorder defined on a countable infinite set X. Then $\prec$ is representable in the sense of Scott and Suppes if and only if it is regular.*

Remark 2 *Indeed, we may observe that this Theorem 10 is a direct consequence of Theorem 9. However, it is of capital importance to say that the final solution to the problem of finding an internal characterization of the Scott-Suppes representability of semiorders, obtained in [22] (see Theorem 7 above) was got through a subtle and ingenious modification of the ideas expressed in Theorem 10.*

At this point, it is noticeable that the techniques used now to represent typical semiorders on countable infinite sets may considerably differ from the Scott-Suppes original construction to represent finite semiorders. Thus, the key result stated in Theorem 5.1 is also proved in [8] through more than twelve pages full of previous lemmas, whereas the original proof in [54] is based on a deep result coming from Mathematical Logic. For instance, if we want to apply Theorem 5.1 to get a Scott-Suppes representation of the semiorder $\prec$ defined on the set $\mathbb{Q}$ of rational numbers by declaring $a \prec b \iff a + 1 \leq b \quad (a, b \in \mathbb{Q})$, a technique outlined in [54] is based on a deep classical result of Real Analysis that states that there exist an increasing real function, too hard to visualize, $F : \mathbb{R} \to \mathbb{R}$ that is continuous on each irrational number, and discontinuous on each rational number. The result introduced in the above Theorem 5.1 is only stated for the countable case. About the general infinite case, if we are dealing with a semiorder $\prec$ defined on an infinite set X that may or may not be countable, and we try to find characterizations of the Scott-Suppes representability of $\prec$, we arrive indeed, in our opinion, to the most difficult problem of the Scott-Suppes representability theory for semiorders. We have already mentioned that, despite a general characterization had already been obtained in [24], the solution achieved was not easy to be checked directly on a given semiorder $\prec$ defined on a set X, since it leans on the existence of suitable extensions of the semiordered structure $(X, \prec)$ and the associated withtally ordered structure $(X, \precsim^0)$. The problem here is that it is not easy to imagine which could be the suitable extensions (if any) that lead to the existence of a Scott-Suppes representation for $\prec$. By the way, in particular cases mainly related to topological properties of the order topology that $\precsim^0$ induces on X, there appear several characterizations that are much easier to handle: see [42, 43, 44, 17].

Moreover, several authors had already debated upon the difficulties that carries the analysis of the uncountable case (see e.g. Section 4 in [8]), sometimes avoiding the consideration of a Scott-Suppes representation (that could be very restrictive in the general case, as textually said on p. 126 of [71]), and substituting it by milder forms of representations (e.g., through different kinds of interval representations) easier to be characterized. However, the existence of a Scott-Suppes representation immediately implies the existence of many other kinds of representations analyzed in the literature (e.g.: interval representations in the sense of [37], generalized numerical representations[12] in the sense of [8]), so that it is important to get a characterization of it (as in [24]), or at least provide a set of sufficient conditions for particular cases, as in [2].

[12] Basically, a generalized numerical representation for a binary relation $\mathcal{R}$ defined on a nonempty set X consists of a function $u : X \to \mathbb{R}$ an a suitable subset S of the real plane $\mathbb{R}^2$ such that $x\mathcal{R}y \Leftrightarrow (u(x), u(y)) \in S, \ (x, y \in X)$.

Working in the direction of Theorem 5.1, we may wonder what else should be added to regularity to get a necessary and sufficient condition, so characterizing the Scott-Suppes representability of semiorders. Moreover, we should compare the different conditions that other authors have used to get characterizations of the existence of some particular kind of representation of semiorders in order to guess which could be the “extra condition that, added to regularity, could finally characterize the existence of a Scott-Suppes representation of a typical irreducible semiorder in the general (uncountable) case.

Fortunately, the key problem has recently been solved, as aforementioned through Theorem 7, Theorem 8 and Theorem 9. (See [22, 20] for further details).

Remark 3 *To complete the panorama stated by Theorem 5.1, observe that there exist semiorders that are defined on a countable set (so that they are representable as interval orders) but fail to be regular (so that they cannot admit a Scott-Suppes representation. An example is the semiorder $\prec$ defined on the real line $\mathbb{R}$ by $x \prec y \iff 3|x| < 2|y|$, for all $x, y \in \mathbb{R}$. Let $u, v : \mathbb{R} \to \mathbb{R}$ be the functions defined by $u(x) = 2|x|, v(x) = 3|x|$ for all $x \in \mathbb{R}$. It is clear that the pair (u, v) is a representation of $\prec$ as an interval order. Nevertheless, $\prec$ is not representable as a semiorder. Observe that the sequence $(2^{-n})_{n\in\mathbb{N}}$ satisfies that $0 \prec 2^{-(n+1)} \prec 2^{-n}$ for all $n \in \mathbb{N}$, so that $\prec$ is not regular.*

Another classical technique is based in the comparison of interval orders and semiorders (see [54, 38, 43, 8, 24, 16, 17]). Obviously, a semiorder is a particular case of an interval order, and a semiorder representable in the sense of Scott and Suppes is, in particular, representable as an interval order. Regrettably, the converse is not true in general as shown in the previous example (see also [24]). The technique consists on first considering a given semiorder $\prec$ as an interval order, then representing it as an interval order (when possible), and finally try to modify that representation (as an interval order) to get a new representation (now, as a semiorder) in the sense of Scott and Suppes. This technique has been used, mainly, to deal with continuous representations of semiorders defined on a topological space. See e.g. [42, 43, 44] and [17]. Thus, given a nonempty set X endowed with a topology τ, and a typical semiorder $\prec$ defined on X, a continuous Scott-Suppes representation for $\prec$ can be defined as a function $u : X \to \mathbb{R}$ that is continuous with respect to the topology τ on X and the usual Euclidean topology on the real line $\mathbb{R}$, such that $x \prec y \iff u(x) + 1 < u(y) \quad (x, y \in X)$. For the sake of completeness we quote here a result proved in [17] concerning the continuous Scott-Suppes representability of a particular class of semiorders. This result was obtained through the technique of modifying suitable representations of a semiordered structure, but considered first as an interval order, to get a new representation (now as a semiorder) in the sense of Scott and Suppes.

Theorem 11 *Let $\prec$ be a continuous semiorder without extremal elements, defined on a connected topological space (X, τ). Then there exists a continuous function $u : X \to \mathbb{R}$ such that $x \prec y \iff u(x) + 1 < u(y) \quad (x, y \in X)$ if and only if the following conditions hold:*

1. *There exists a widely dense countable subset $D \subseteq X$ for the binary relation $\prec$ considered as an interval order on X.*
2. *The associated relations $\prec^*$ and $\prec^{**}$ coincide.*
3. *The binary relation $\prec$ has no singular[13] points.*

As commented before, the technique of transformation of representations as an interval order of a semiordered structure into Scott-Suppes representations, has been used to deal with representations that involve continuity or semicontinuity. However, the use of similar techniques to get Scott-Suppes representations (continuous or not) of semiorders, in the general case (that is, when the set X where the semiorder $\prec$ is defined is uncountable and we do not take care of continuity or semicontinuity) is not common in the literature, despite there are several papers that study semiorders among interval orders (see e.g. [58, 38, 5]), most of them in the finite case. This technique could be, in our opinion, closely related to other important technique that leans on the consideration of suitable functional equations directly associated with a given semiordered structure.

6. Functional equations associated with the representability of semiorders

Somewhat related to the technique of modifying representations as an interval order of a semiordered structure to get new representations in the sense of Scott and Suppes, other appealing technique, already considered in [17] (see also [1]), that can be used to represent interval orders is based on finding special solutions of some functional equations in several variables. Thus, the representability of an interval order is also characterized by means of the existence of solutions of a particular kind for a functional equation in two variables, known as the separability equation. As analyzed in [61, 11] an interval order $\prec$ defined on a nonempty set X is representable if and only if there exists a bivariate function $F : X \times X \longrightarrow \mathbb{R}$ satisfying $F(x, y) + F(y, z) = F(x, z) + F(y, y) \quad (x, y, z \in X)$ and $x \prec y \iff F(x, y) > 0 \quad (x, y \in X)$. Here the functional equation in two variables satisfied by the function F is usually called the *separability* equation (see p. 122 in [3]) because it is equivalent to say that $F(x, y) = G(x) + H(y) \quad (x, y \in X)$, for some functions $G, H : X \to \mathbb{R}$ that depend of only one variable. In the same way, with respect to the representability of semiorders in the sense of Scott and Suppes, an analogous result is available, see [23, 24, 1]:

Theorem 12 *Let X be a nonempty set endowed with a semiorder $\prec$. The following conditions are equivalent:*

1. *The semiordered structure $(X, \prec)$ is representable in the sense of Scott and Suppes.*

[13] Given an interval order $\prec$ defined on a nonempty set X, an element $x \in X$ is said to be *a singular point with respect to* $\prec$ if for every $y, z \in X$ it holds that $[(x \sim y) \wedge (x \sim z)] \Rightarrow y \sim z$.

2. *There exists a bivariate function $G : X \times X \to \mathbb{R}$ such that $x \prec y \iff G(x,y) > 0 \quad (x,y \in X)$ and, in addition, $G(x,y) + G(y,z) = G(x,z) + G(t,t)$ for every $x,y,z,t \in X$.*

3. *There exists a function $G : X \times X \to \mathbb{R}$ such that $x \prec y \iff G(x,y) > 0 \quad (x,y \in X)$ and, in addition, $G(x,y) + G(y,z) = G(x,t) + G(t,z)$ for every $x,y,z,t \in X$.*

The main problem here is that although in the case of interval orders we know (see [61]) how to construct (if there is any) those bivariate functions that are special solutions of the separability equation, and furnish a representation, it is still an open problem to do the same for the functional equations related to the Scott-Suppes representability of semiorders. Regrettably, no general way to find such suitable solutions to represent semiorders is known up-to-date. Related to this question, in [17, 1] another classical functional equation (in only one variable, in this case) has been used to characterize, in certain particular cases, the existence of Scott-Suppes representations of semiorders. An old result, obtained by the Norwegian mathematician Niels Heinrik Abel early in 1824, shows that given an open real interval I and a continuous and strictly increasing function $G : I \to I$, there exists a continuous and strictly increasing function $F : I \to \mathbb{R}$ that satisfies the so-called Abel equation (see e.g. p. 145 and ff. in [70], or pp. 64 and ff. in [69]) given by $F(G(x)) = F(x)+1 \quad (x \in I)$. A glance to this equation immediately shows that the asymmetric binary relation $\prec$ defined on I by $a \prec b \iff F(a) + 1 < F(b)$ $(a, b \in I)$ is a semiorder representable through the function F, with threshold 1. We may guess that the Scott-Suppes representability of a typical semiorder $\prec$ defined on a nonempty set X is directly related to the existence of suitable solutions of a generalized Abel equation $F(G(z)) = F(z) + 1$, where now Z is a superset of X (i.e.: $X \subseteq Z$), $G : Z \to Z$ is a map with no cycles, and $F : Z \to \mathbb{R}$ is a real-valued function. The map G is known a priori, and F is the unknown function, i.e., the solution of the equation. This kind of questions have been studied in [1]. Certain generalized Abel equations give rise to special bivariate functions that solve a separability equation and furnish a Scott-Suppes representability of a semiorder, as above. Conversely, some of those bivariate functions also give rise to generalized Abel equations. For the sake of completeness, we introduce here some ideas and results in this direction. Let X be a nonempty set, and $h : X \to X$ a (fixed) map. We say that a real-valued function $f : X \to \mathbb{R}$ satisfies the *generalized Abel equation* if it holds that $f(h(x)) = f(x)+1$, for every $x \in X$. (Observe that for a solution f to exist, a necessary condition is that the map $h : X \to X$ is fixed point free, that is $h(x) \neq x \quad (x \in X)$. Actually for every strictly positve natural number $k \in \mathbb{N}\setminus\{0\}$, the k-th iterate h^k of the function h must also be fixed point free since, otherwise, there exists $z \in X$ such that $f(z) = f(h^k(z)) = f(z) + k$, which is a contradiction. In other words, the map h has no cycles. Let us see now how can we generate total preorders, interval orders, and semiorders from particular solutions of the functional equations mentioned before.

Proposition 1 *Let X be a nonempty set.*

1. *If $F : X \times X \to \mathbb{R}$ satisfies the Sincov equation[14], then the binary relation $\precsim$ defined on X by $x \precsim y \iff F(y,x) \leq 0 \quad (x,y \in X)$ is a total preorder.*

2. *If $F : X \times X \to \mathbb{R}$ satisfies the separability equation and, in addition $F(t,t) \leq 0 \quad (t \in X)$, then the binary relation defined on X by $x \prec y \iff F(x,y) > 0 \quad (x,y \in X)$ is an interval order.*

3. *If $F : X \times X \to \mathbb{R}$ satisfies the separability equation and, in addition, there exists a non-positive real constant $K \leq 0$ such that $F(t,t) = K \quad (t \in X)$, then the binary relation defined on X by $x \prec y \iff F(x,y) > 0 \quad (x,y \in X)$ is a semiorder.*

4. *If $h : X \to X$ is a map, and $f : X \to \mathbb{R}$ is a solution of the generalized Abel equation $f(h(x)) = f(x)+1$, then the binary relation $\prec$ defined on X by $x \prec y \iff 1 < f(y) - f(x)$ is a semiorder. Moreover, this semiorder is representable in the sense of Scott and Suppes, through the real-valued function f and the positive threshold 1.*

Let us point out some relationship between the separability equation and the generalized Abel equation.

Remark 4 *Let X be a nonempty set. Suppose that $F : X \times X \to \mathbb{R}$ satisfies the separability equation and there exists a strictly negative $K < 0$ such that $F(t,t) = K \quad (t \in X)$. Fix an element $a \in X$. We observe that $F(x,a) + F(a,x) = F(x,x) + F(a,a) = 2K \quad (x \in X)$. Dividing by $2K$ we obtain $\frac{F(x.a)}{2K} = -\frac{F(a,x)}{2K} + 1 \quad (x \in X)$. Assume, in addition, that for every $t \in X$ there exists an element $h(t) \in X$, unique, such that $F(a, h(t)) = -F(t,a) \quad (t \in X)$. Under this hypothesis, if we define $f : X \to \mathbb{R}$ by $f(t) = -\frac{F(a,t)}{2K} \quad (t \in \mathbb{R})$ we immediately check that f satisfies the generalized Abel equation: $f(h(x)) = f(x) + 1 \quad (x \in X)$. Let now $h : X \to X$ be a map, and $f : X \to \mathbb{R}$ a solution of the generalized Abel equation $f(h(x)) = f(x) + 1 \quad (x \in X)$. Consider now the bivariate function $F : X \times X \to \mathbb{R}$ given by $F(x,y) = f(x) - f(h(y)) \quad (x,y \in X)$. It follows that: $F(x,y)+F(y,z) = f(x)-f(h(y))+f(y)-f(h(z)) = f(x)-f(h(z))+f(y)-f(h(y)) = F(x,z)+F(y,y) \quad (x,y,z \in X)$. Thus F satisfies the separability equation. Moreover, $F(t,t) = f(t) - f(h(t)) = -1$ $(t \in X)$, by hypothesis.*

Now, let us show how the problem of the representability in the sense of Scott and Suppes of semiordered structures is very close to that of defining and solving certain generalized Abel equations. To start with, consider a semiorder $\prec$ defined on a nonempty set. Suppose in addition that $\prec$ is a typical semiorder, that is, the associated binary relation $\precsim$ is not transitive (equivalently, $\precsim$ is not a total preorder). Thus, in case that $\prec$ admits a

[14] The Sincov functional equation is: $F(x,y) + F(y,z) = F(x,z) \quad (x,y,z \in X)$. See e.g. [47] for further details.

representation in the sense of Scott and Suppes through a real-valued function u and a non-negative threshold λ, we have a fortiori that λ is indeed strictly positive ($\lambda > 0$). Thus, let us consider now a typical semiorder $\prec$ defined on a nonempty set X, and assume that $\prec$ admits a representation in the sense of Scott and Suppes by means of a real-valued function $u : X \to \mathbb{R}$ such that $x \prec y \iff 1 < u(y) - u(x) \quad (x, y \in X)$. In the particular case in which u is a bijection between X and $\mathbb{R}$, we observe given $x \in X$ the element $u^{-1}(u(x)+1) \in X$ is well-defined (i.e.: such element exists and it is unique). Let $h : X \to X$ be defined by $h(x) = u^{-1}(u(x) + 1)$ $(x \in X)$. It is obvious that h is fixed point free and without cycles since, by definition of h, we have $u(h(x)) = u(x) + 1 \quad (x \in X)$. Notice also that, as a matter of fact, the function u satisfies a suitable generalized Abel equation. If u is surjective (i.e. $u(X) = \mathbb{R}$) but it is not injective, an equivalence relation $\mathcal{R}$ can be immediately defined on X by declaring $a\mathcal{R}b \iff u(a) = u(b) \quad (a, b \in X)$.

Let $X_{\mathcal{R}}$ denote the quotient set $X/\mathcal{R}$. Denote by $x_{\mathcal{R}}$ the equivalence class corresponding to a given element $x \in X$. Define on $X_{\mathcal{R}}$ the binary relation $\prec_{\mathcal{R}}$ given by $x_{\mathcal{R}} \prec_{\mathcal{R}} y_{\mathcal{R}} \iff x \prec y \iff 1 < u(y) - u(x) \quad (x, y \in \mathbb{R})$ and the real-valued function $u_{\mathcal{R}} : X_{\mathcal{R}} \to \mathbb{R}$ given by $u_{\mathcal{R}}(x_{\mathcal{R}}) = u(x) \quad (x \in X)$. It is straightforward to see now that $\prec_{\mathcal{R}}$ is a typical semiorder on $X_{\mathcal{R}}$ such that $x_{\mathcal{R}} \prec_{\mathcal{R}} y_{\mathcal{R}} \iff 1 < u_{\mathcal{R}}(y_{\mathcal{R}}) - u_{\mathcal{R}}(x_{\mathcal{R}}) \quad (x, y \in \mathbb{R})$. Moreover, $u_{\mathcal{R}}$ is a bijection. Let $h_{\mathcal{R}} : X_{\mathcal{R}} \to X_{\mathcal{R}}$ be defined by $h_{\mathcal{R}}(x) = u_{\mathcal{R}}^{-1}(u_{\mathcal{R}}(x_{\mathcal{R}}) + 1)$ $(x \in X)$. We get $u_{\mathcal{R}}(h_{\mathcal{R}}(x_{\mathcal{R}})) = u_{\mathcal{R}}(x_{\mathcal{R}}) + 1 \quad (x_{\mathcal{R}} \in X_{\mathcal{R}})$, so that $u_{\mathcal{R}}$ also satisfies a generalized Abel equation. If u is injective but it is not surjective (i.e: $u(X) \subsetneq \mathbb{R}$) we enlarge the set X in the following way: for every $\alpha \in \mathbb{R} \setminus u(X)$ we add an extra element x_α to X. Let $\bar{X}$ denote the enlarged set $X \cup \{x_\alpha : \alpha \in \mathbb{R} \setminus u(X)\}$. Consider now the real-valued function $\bar{u} : \bar{X} \to \mathbb{R}$ given, for every $t \in \bar{X}$, by $\bar{u}(t) = u(t)$ if $t \in X$; $\bar{u}(t) = \alpha$ if $t = x_\alpha$ for some $\alpha \in \mathbb{R} \setminus u(X)$. Now it is plain that $\bar{u}$ is a bijection. Finally, define a binary relation $\bar{\prec}$ on $\bar{X}$ by declaring that $s \bar{\prec} t \iff 1 < \bar{u}(t) - \bar{u}(s) \quad (s, t \in \bar{X})$. We may easily check that $\bar{\prec}$ is a semiorder whose restriction to X is $\prec$. Let $\bar{h} : \bar{X} \to \bar{X}$ be defined by $\bar{h}(t) = \bar{u}^{-1}(\bar{u}(t) + 1)$ $(t \in \bar{X})$. It follows that $\bar{u}(\bar{h}(t)) = \bar{u}(t) + 1 \quad (t \in \bar{X})$, so that $\bar{u}$ satisfies another generalized Abel equation. When u is neither injective nor surjective, first we enlarge the set X to a set $\bar{X}$ obtained by adding an extra element x_α for each $\alpha \in \mathbb{R} \setminus u(X)$. Define $\bar{u}$ and $\bar{\prec}$ as in the previous case, and observe that $\bar{u}$ is now surjective but not yet injective. Consequently, we consider the quotient set $\bar{X}_{\mathcal{R}} = \bar{X}/\mathcal{R}$ through the equivalence $\mathcal{R}$ that $\bar{u}$ defines on $\bar{X}$. As before, in the quotient set $\bar{X}_{\mathcal{R}}$ we may define in the natural way a real-valued function $\bar{u}_{\mathcal{R}}$ and a map $\bar{h}_{\mathcal{R}} : \bar{X}_{\mathcal{R}} \to \bar{X}_{\mathcal{R}}$ such that $\bar{u}_{\mathcal{R}}(\bar{h}_{\mathcal{R}}(t)) = \bar{u}_{\mathcal{R}}(t) + 1 \quad (t \in \bar{X}_{\mathcal{R}})$ getting again a solution of a generalized Abel equation.

7. Other techniques: Scott-Suppes representability of semiorders with special properties

Another related question concerns the Scott-Suppes representability of typical semiorders defined on a set with special properties. Among the special properties, we may consider two important classes, namely topological properties (e.g.: continuity) and algebraic properties (e.g.: semiorders defined on some kind of algebraic structure as, for instance, a semigroup, a group, or a vector space). In these situations, we will be looking for representations that also feature special characteristics (e.g: continuity, in the topological case, or preservation of the algebraic structure through some homomorphism, in the algebraic case). About the topological case, up-to-date no general characterization of the continuous representability of a semiorder is known yet. However, some partial results have been obtained, mainly for the case of semiorders defined on connected topological spaces. See [42, 43, 44, 17] for a further account. About the algebraic case, there are some classical works that consider semiorders defined on a nonempty set endowed with some algebraic structure (see e.g. [73], where semiorders on mixture spaces have been considered). However, in our opinion, no systematic study of the Scott-Suppes representability of semiorders defined on classical algebraic structures (e.g.: semigroups, monoids, groups) has been made yet in the specialized literature. To put an example, we could consider a typical semiorder $\prec$ defined on a nonempty set X endowed with, say, an associative binary operation $\bar{+}$ (i.e.: $(X, \bar{+})$ is a semigroup) and try to find a Scott-Suppes representation through a real-valued function $u : X \to \mathbb{R}$ such that $x \prec y \iff u(x)+1 < u(y) \quad (x, y \in X)$ and, in addition, u is an algebraic homomorphism of semigroups from $(X, \bar{+})$ into the additive group of real numbers $(\mathbb{R}, +)$, that is: $u(x \bar{+} y) = u(x) + u(y)$ for every $x, y \in X$. Studies on the algebraic representability of other classical ordered structures (e.g.: total preorders on semigoups or interval orders on cones) have already appeared in the literature (see e.g. [18, 19]). But for the case of semiorders this is a line of research to be explored in next future.

8. Semiorders and fuzzy numbers

We can think on representations of binary relations through functions that take values on some set of fuzzy numbers, instead of real-valued functions (see e.g. [16]). In this sense, we may understand the Scott-Suppes representability of semiorders by means of a different kind of representations that involve functions on a suitable set of fuzzy numbers. Basically a fuzzy real number can be understood as a function $F : \mathbb{R} \longrightarrow [0, 1]$ satisfying some “axioms or properties imposed a priori. However, the precise definitions may vary. (See e.g. [51, 30, 60]). In what concerns fuzzy sets and fuzzy numbers we shall follow the definitions and notations of [60]. Thus, U being a set (usually called *universe*) and $\mathcal{L}$ being a lattice with greatest element $\bar{1}$ and smallest element $\bar{0}$, a *fuzzy set* is a function $A : U \longrightarrow \mathcal{L}$. The *support* of A is the set $Supp(A) = \{x \in U : A(x) \neq \bar{0}\}$, and the *kernel* is $\ker(A) = \{x \in U : A(x) = \bar{1}\}$. A fuzzy set A is said to be *normal* if it has nonempty kernel. Given $\alpha \in \mathcal{L}$, the *α-cut* of the fuzzy set A is the set $A_\alpha = \{x \in U : A(x) \wedge \alpha = \alpha\}$, where $\wedge$ denotes the latticial operation in $\mathcal{L}$. In the particular case in which $U = \mathbb{R}$, $\mathcal{L} = [0, 1] \subset \mathbb{R}$ and $x \wedge y = \min\{x, y\}$, the fuzzy set A is said to be *convex* if for every $\alpha \in [0, 1]$ the α-cut A_α is a convex subset of $\mathbb{R}$. In this case, a *fuzzy number* is a normal convex fuzzy set such that the function $A : \mathbb{R} \longrightarrow [0,1]$ is piecewise continuous, and there exist points $a_1 \leq a_0 \leq b_0 \leq b_1 \in \mathbb{R} \cup \{-\infty, +\infty\}$ with the following properties:

1. $a_1 \in \mathbb{R} \cup \{-\infty\}$ and $b_1 \in \mathbb{R} \cup \{+\infty\}$,
2. $a_0, b_0 \in \mathbb{R}$,
3. $Supp(A) \subseteq [a_1, b_1] \cap \mathbb{R}$,
4. the function A is increasing on $[a_1, a_0] \cap \mathbb{R}$ and decreasing on $[b_0, b_1] \cap \mathbb{R}$,
5. $a_1 \in \mathbb{R} \Rightarrow A(a_1) = 0$ and, similarly, $b_1 \in \mathbb{R} \Rightarrow A(b_1) = 0$,
6. $[a_0, b_0] \subseteq \ker(A)$.

Particular cases of fuzzy numbers are:

(i) The *ordinary* (or “non-fuzzyÕÕ) *real numbers*, where a real number $r \in \mathbb{R}$ is interpreted in the obvious way by means of its characteristic function $A_r : \mathbb{R} \to [0,1]$ where $A_r(s) = 0$ if $r \neq s$ and $A_r(r) = 1$.

(ii) The *triangular fuzzy numbers* that are those for which $a_1, b_1 \in \mathbb{R}$, $a_0 = b_0$, $A(ta_1 + (1-t)a_0) = 1 - t$ and also $A(ta_0 + (1-t)b_1) = t$ for every $t \in [0,1]$.

A triangular fuzzy number is said to be *symmetric* if $a_0 = \frac{a_1+b_1}{2}$. An element of the set $\mathcal{ST}$ of symmetric triangular fuzzy numbers can be identified by the numbers a_0 and a_1 that appearing in its definition, so that we can denote an element of $\mathcal{ST}$ as $\{a_0, a_1\}$. We can identify a symmetric triangular fuzzy number $\{a_0, a_1\} \in \mathcal{ST}$, where $b_1 = 2a_0 - a_1$, with the interval $[a_1, b_1]$ of real numbers, and, accordingly, define a binary relation $\mathcal{R}_I$ on $\mathcal{ST}$ as follows: $\{a_0, a_1\}\mathcal{R}_I\{a'_0, a'_1\} \iff a_1 \leq b'_1$. It is straightforward to see now that $\mathcal{R}_I$ is an interval order on $\mathcal{ST}$. Obviously the structure $(\mathcal{ST}, \mathcal{R}_I)$ is representable as an interval order through the pair of functions $F, G : \mathcal{ST} \longrightarrow \mathbb{R}$ given by $F(\{a_0, a_1\}) = a_1$; $G(\{a_0, a_1\}) = b_1 = 2a_0 - a_1$, for every $\{a_0, a_1\} \in \mathcal{ST}$. Thus we can easily obtain the following result on representability of interval orders (see [16] for further details).

Proposition 2 *Let X be a nonempty set endowed with an interval order $\mathcal{R}$. Then, the following statements are equivalent:*

(i) *$\mathcal{R}$ is representable.*

(ii) *The structure $(X, \mathcal{R})$ is representable in $(\mathcal{ST}, \mathcal{R}_I)$ in the sense that there exists a function $H : X \to \mathcal{ST}$ such that $x\mathcal{R}y \iff H(x)\mathcal{R}_I H(y) \quad (x, y \in X)$.*

Consider now the following particular subset of $\mathcal{ST}$, which we call the set of *symmetric triangular fuzzy numbers of unitary base*, which we denote by $\mathcal{STU}$ and consists of all the elements $\{a_0, a_0 - \frac{1}{2}\}$. We endow $\mathcal{STU}$ with the ordering $\mathcal{R}_I$ as above. The corresponding result on the Scott-Suppes representability of semiorders follows now, interpreted in this new context.

Proposition 3 *The following assertions are equivalent for a set X endowed with a semiorder $\mathcal{R}$ that is not a total preorder:*

(i) *$(X, \mathcal{R})$ is representable (as a semiorder) through a function $U : X \to \mathbb{R}$ and a strictly positive threshold $K > 0$, such that $x\mathcal{R}y \iff U(x) \leq U(y) + K$, for every $x, y \in X$,*

(ii) *$(X, \mathcal{R})$ is representable in $(\mathcal{STU}, \mathcal{R}_I)$ in the sense that there exists a function $H : X \to \mathcal{STU}$ such that $x\mathcal{R}y \iff H(x)\mathcal{R}_I H(y)$, for every $x, y \in X$.*

9. Towards fuzzy semiorders

Till this point, all the analysis made in the present manuscript is understood in the crisp setting. However, we could think on possible extensions to the fuzzy setting of the concept of a semiorder. This is also a new line for future research, that has very recently been introduced in the specialyzed literature (see e.g. [27, 49]). At this stage, it is important to say that the same question had already been considered for interval orders in [28], trying to introduce the concept of a fuzzy interval order. In that interesting work, it was proved that several equivalent definitions of the concept of an interval order (in the crisp setting) may fail to be equivalent in the fuzzy setting. In [28] the authors considered five equivalent definitions of the (crisp) concept of an interval order, as well as their generalizations to the fuzzy setting, where it is proved that they are not equivalent, in general. A similar study for semiorders has recently started (see [27]). As in the case of interval orders analyzed in [28], it also happens that many equivalent definitions of the concept of a semiorder (in the crisp case) are no longer equivalent when extended to the fuzzy setting. The hierarchy among this possible definitions (at least twelve) has also been studied in [49]. Consequently, moving towards fuzzy semiorders we could adopt several possible non-equivalent definitions, so giving origin to different theories of fuzzy semiorders that would obviously depend on the definitions adopted.

10. Open questions

To conclude the survey, we recall several of the main open problems related to the Scott-Suppes representability of semiorders.

1. Find a characterization (for the general case) of the continuous representability of semiorders defined on topological spaces.
2. Determine to what extent the Scott-Suppes representability of a semiorder is related to the existence of solutions of generalized Abel equations.
3. Analyze the numerical representabiity of semiorders on algebraic structures.
4. Study the numerical representability of suitable classes of fuzzy semiorders.

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