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ON THE TRACES OF THE SOLUTIONS OF THE ANISOTROPIC HYPERBOLIC HEAT EQUATION WITH IRREGULAR HEAT SOURCES

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Abstract

We present some results about the traces of solutions of the anisotropic hyperbolic heat equation over the boundary of cylindrical open sets of types $\Omega \times]0, T[$ and $\Omega \times]0, \infty[$, when the heat sources are irregular distributions.

1. Introduction

It is well known that in many important industrial processes involving

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the irradiation of certain surfaces with laser beams of high energy, it is necessary to use a modified heat conduction equation, the so-called hyperbolic heat equation. In the very frequent case of anisotropic bodies with constant density ρ and specific heat c but with thermal conductivity depending on the position, it takes the form (see [1])

$$-\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left(\sum_{j=1}^{3}k_{ij}(\mathbf{x})\frac{\partial T}{\partial x_j}\right)+\rho c\left(\frac{\partial T}{\partial t}(\mathbf{x}, t)+\tau\frac{\partial^2 T}{\partial t^2}(\mathbf{x}, t)\right)$$

$$=\rho\left(S(\mathbf{x}, t)+\tau\frac{\partial S}{\partial t}(\mathbf{x}, t)\right), \tag{1}$$

where $T(\mathbf{x}, t)$ is the temperature at point $\mathbf{x}$ at instant t, $(k_{ij}(\mathbf{x}))$ is the symmetric thermal conductivity tensor of the material, τ is the relaxation parameter and $S(\mathbf{x}, t)$ denotes the internal heat sources in the body. If $k_{ij} = k$ for every $1 \leq i, j \leq 3,$ then we obtain the isotropic hyperbolic heat equation for a homogeneous body. The mathematical formulation of such problems leads naturally to the use of nonregular distributions in the boundary conditions of the problem or in the expression of the internal heat sources. Currently, physicists and engineers handle the arising equations in a quasi purely formal way, without any reserve about questions on existence or the underlying functional spaces. So, these problems deserve a rigorous mathematical analysis. This task has been done in [4] when the used data are regular distributions. In this paper, we begin the study in the "irregular" case.

After choosing a suitable Banach space containing the irregular heat sources arising in practice, we consider the space $\mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega \times]0, T[)$ containing the solutions of (1). In order to precise the permitted degree of irregularity in the boundary conditions of a boundary value problem associated to (1), we need information about the traces of $U \in \mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega \times]0, T[)$ over the transversal section $t = T$ and over the boundary of a domain $\Omega \times]0, T[$ as well as to study the same trace problem for domains $\Omega \times]0, \infty[$, *with a nonbounded temporal interval* (the natural framework in the investigation of the Green's function for (1)). So *the purpose of this paper is to study the traces over the boundary of the solutions of equation* (1) *in the cases of domains* $\Omega \times]0, T[$ *and* $\Omega \times]0, \infty[$.

We set the hypothesis and the nonstandard notation to be used. In the whole paper, Ω will denote *a bounded open set in* $\mathbb{R}^n$ with $\mathcal{C}^\infty$ boundary $\partial\Omega$ of dimension $n-1$ lying the interior points of Ω locally in only one side of $\partial\Omega$. The outer unit normal vector to $\partial\Omega$ will be represented by $\mathbf{n}$. For every $T > 0$, the open set $\Omega \times]0, T[$ in $\mathbf{R}^{n+1}$ will be denoted by Ω_T. Analogously, we put $\Omega_\infty := \Omega \times]0, \infty[$, $(\partial\Omega)_T := \partial\Omega \times]0, T[$ and $(\partial\Omega)_\infty := \partial\Omega \times]0, \infty[$. $R_T(f)$ will be the restriction map to Ω_T of a function f defined in Ω_∞.

All the necessary information about Sobolev spaces $H^r(\Omega)$, $H_0^r(\Omega)$, $H^{-r}(\Omega) := (H_0^r(\Omega))'$, $r \in]0, \infty[$ and the analogous spaces $H^r((\partial\Omega)_T)$, $T \in]0, \infty]$ can be found in [3]. To distinguish the role of the spatial variable $\mathbf{x} \in \Omega$ and the temporal variable $t \in]0, T[$, $T > 0$, we use the Sobolev space $H^{r,s}(\Omega_T) = H^s(]0, T[, L^2(\Omega)) \cap L^2(]0, T[, H^r(\Omega))$, $r, s \in [0, \infty[$, endowed with the standard norm of the intersection of two Banach spaces. We shall also need the subspace $H_{0,0}^{r,s}(\Omega_T)$ of $H^{r,s}(\Omega_T)$ formed by the closure of $\mathcal{D}(\Omega_T)$ in $H^{r,s}(\Omega_T)$ and the spaces

$$H_{0,}^{r,s}(\Omega_T) := H^s(]0, T[, L^2(\Omega)) \cap L^2(]0, T[, H_0^r(\Omega)),$$

$$H_{,0}^{r,s}(\Omega_T) := H_0^s(]0, T[, L^2(\Omega)) \cap L^2(]0, T[, H^r(\Omega)).$$

The topological dual of $H_{0,0}^{r,s}(\Omega_T)$ is denoted by $H^{-r,-s}(\Omega_T)$. Note that the above definitions are meaningful also in the limit case $T = \infty$. Similar rules are used to define $H_{0,}^{r,s}((\partial\Omega)_T)$ and $H_{,0}^{r,s}((\partial\Omega)_T)$.

The form of equation (1) suggests to consider general operators

$$\mathcal{A} := \mathcal{X} + \lambda \frac{\partial}{\partial t} + \tau \frac{\partial^2}{\partial t^2}, \quad \mathcal{X} := \sum_{0 \le |\alpha|, |\beta| \le 1} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial \mathbf{x}^{\alpha}} \left(a_{\alpha\beta}(\mathbf{x}) \frac{\partial^{|\beta|}}{\partial \mathbf{x}^{\beta}} \right),$$

where $\lambda \in \mathbb{R}$, $\tau > 0$ and $\mathcal{X}$ is a selfadjoint strongly elliptic operator in $\overline{\Omega}$ with real coefficients $a_{\alpha\beta}(\mathbf{x}) \in \mathcal{C}^{\infty}(\overline{\Omega})$. It is known (theorem of Aronszajn-Milman, [3, Chapter 2, Theorem 2.1]) that if $\mathcal{A}^* = \mathcal{X} - \lambda \frac{\partial}{\partial t} + \tau \frac{\partial^2}{\partial t^2}$ is the adjoint operator of $\mathcal{A}$, then there are a boundary differential operator $\Re_{\mathcal{A}} := f(\mathbf{x}) + \sum_{i=1}^{3} g_i(\mathbf{x}) \cos(x_i, \mathbf{e}_i) \frac{\partial}{\partial x_i} : H^2(\Omega) \to L^2(\partial\Omega)$ with real coefficients f, g_1, g_2, g_3 in $\mathcal{C}^2(\partial\Omega)$ and a real function $\mathfrak{f}_{\mathcal{A}}(\mathbf{x}) \in \mathcal{C}^2(\partial\Omega)$ such that $\mathfrak{f}_{\mathcal{A}}(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in \partial\Omega$ and the classical Green's formula

$$\int_0^T \left(\int_{\Omega} (v\mathcal{A}(u) - u\mathcal{A}^*(v)) d\mathbf{x} \right) dt$$

$$= \int_0^T \left(\int_{\partial\Omega} \left(u \Re_{\mathcal{A}}(v) - \mathfrak{f}_{\mathcal{A}} v \frac{\partial u}{\partial \mathbf{n}} \right) \cdot d\sigma \right) dt + \lambda \left(\int_{\Omega} [v(\mathbf{x}, t) u(\mathbf{x}, t)]_{t=0}^{t=T} d\mathbf{x} \right)$$

$$+ \tau \left(\int_{\Omega} \left[v(\mathbf{x}, t) \frac{\partial u}{\partial t}(\mathbf{x}, t) - u(\mathbf{x}, t) \frac{\partial v}{\partial t}(\mathbf{x}, t) \right]_{t=0}^{t=T} d\mathbf{x} \right)$$

holds if $u \in \mathcal{C}^{\infty}(\overline{\Omega_T})$ and $v \in H^{r,s}(\Omega_T)$, $r \geq 2$, $s \geq 2$ ([3, Chapter 4, Remark 2.2]).

We need to consider a space containing the solutions of the equation $\mathcal{A} = F$ when F lies in some large enough space to contain the nonregular distributions arising in the physical applications. To this goal, we begin considering for $r \in \mathbb{N} \cup \{0\}$ the space

$$\Phi^r(\Omega) := \left\{ f \in L^2(\Omega) \mid \| f \|_{\Phi^r(\Omega)} := \left(\sum_{|\boldsymbol{\alpha}| \le r} \left\| d(\mathbf{x}, \partial\Omega)^{|\boldsymbol{\alpha}|} \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}) \right\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} < \infty \right\}, \tag{2}$$

where $d(\mathbf{x}, \partial\Omega) := \inf_{\mathbf{y}\in\partial\Omega} \| \mathbf{x} - \mathbf{y} \|$ is the distance from $\mathbf{x}$ up to the boundary $\partial\Omega$. We extend the above definition to the case $r \in]0, \infty[$ by complex interpolation setting for $r = E[r] + \theta$ in $]0, \infty[$, $0 < \theta < 1$ the space $\Phi^r(\Omega) := [\Phi^{E[r]+1}(\Omega), \Phi^{E[r]}(\Omega)]_{1-\theta}$, endowed with any canonical norm of the interpolated space (E is the integer part of r). Clearly, the continuous inclusion $H^r(\Omega) \subset \Phi^r(\Omega)$, $r > 0$ holds. Finally, we define $\Phi^r(\Omega)$ if $r < 0$ by duality setting that $\Phi^{-r}(\Omega) := (\Phi^r(\Omega))'$ for every $r \in [0, \infty[$. It can be shown that $\mathcal{D}(\Omega)$ is dense in $\Phi^r(\Omega)$ for $r \ge 0$ and that $\Phi^r(\Omega) \subset L^2(\Omega)$ $\subset \Phi^{-r}(\Omega)$ (see [3, Chapter 2], for instance). In consequence, $\mathcal{D}(\Omega)$ is dense in $\Phi^{-r}(\Omega)$ too.

In order to distinguish the behavior of temporal and spatial variables, we introduce another space. Given $0 < T$, we fix a number $T_0 < \frac{T}{2}$ and consider the function $\varphi_{T_0, T} \in \mathcal{C}^\infty(\mathbb{R})$ with compact support $[0, T]$ defined as

$$\varphi_{T_0,T}(t) = e^{-\frac{T_0^2}{T_0^2 - (t - T_0)^2}} \text{ if } 0 < t \le T_0; \quad \varphi_{T_0,T}(t) = \frac{1}{e} \text{ if } T_0 \le t \le T - T_0,$$

$$\varphi_{T_0,T}(t) = e^{-\frac{T_0^2}{T_0^2 - (t - T + T_0)^2}} \text{ if } T - T_0 \le t < T; \quad \varphi_{T_0,T}(t) := 0 \text{ if }$$

$t \in]-\infty, 0] \cup [T, \infty[$.

Clearly, $\| \varphi_{T_0, T} \|_{L^\infty(\mathbb{R})} = \frac{1}{e}$ independent of *T*. For every $r \in \mathbb{N} \bigcup \{0\}$, we define $\Phi^{r,r}(\Omega_T, \varphi_{T_0,T})$ (or simply $\Phi^{r,r}(\Omega_T)$ if there is no risk of confusion) as

$$\Phi^{r,r}(\Omega_T) := \left\{ f \in L^2(]0, T[, \Phi^r(\Omega)) \mid \| f \|_{\Phi^{r,r}(\Omega_T)} : \right.$$

$$\left. = \left(\sum_{j=0}^{r} \left\| |\varphi_0(t)|^j \frac{\partial^j f}{\partial t^j} \right\|^2_{L^2(]0,T[, \Phi^{r-j}(\Omega))} \right)^{\frac{1}{2}} < \infty \right\}.$$

As above, the definition is extended to nonnegative real numbers by interpolation defining

$$\Phi^{r,r}(\Omega_T) := [\Phi^{E[r]+1, E[r]+1}(\Omega_T), \Phi^{E[r], E[r]}(\Omega_T)]_{1-\theta}$$

for every $r \geq 0$ and providing this space with any standard norm for the interpolated space. We obtain easily the continuous inclusion $I_T : H^{r,r}(\Omega_T) \subset \Phi^{r,r}(\Omega_T)$, $r > 0$ and the estimation for its norm

$$\| I_T \| \leq \left(1 + \frac{1}{e}\right)^r (1 + \operatorname{diam}(\Omega))^r \tag{3}$$

which is independent of *T*. For $r < 0$, we define

$$\Phi^{r,r}(\Omega_T) := (\Phi^{-r,-r}(\Omega_T))'.$$

It can be deduced from [3, Chapter 4, Proposition 9.1] that $\mathcal{D}(\Omega_T)$ is dense in $\Phi^{r,r}(\Omega_T)$ if $r \geq 0$. Moreover, we have $\Phi^{r,r}(\Omega_T) \subset L^2(\Omega_T) \subset \Phi^{-r,-r}(\Omega_T) \subset \mathcal{D}'(\Omega_\infty)$.

Finally, we arrive at the desired fundamental space. We define for $r \in \mathbb{R}$,

$$D_{\mathcal{A}}^{-(r-1)}(\Omega_T) := \{u \in H^{-(r-1), -r}(\Omega_T) \mid \mathcal{A}(u) \in \Phi^{-(r+1), -(r+1)}(\Omega_T)\}$$

which becomes a Banach space endowed with the norm

$$\| u \|_{D_{\mathcal{A}}^{-(r-1)}(\Omega_T)} = \| u \|_{H^{-(r-1), -r}(\Omega_T)} + \| \mathcal{A}(u) \|_{\Phi^{-(r+1), -(r+1)}(\Omega_T)}. \quad (4)$$

The closure of $\mathcal{C}^{\infty}(\overline{\Omega_T})$ in $D_{\mathcal{A}}^{-(r-1)}(\Omega_T)$ will be denoted by $\mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega_T)$.

2. Traces on the Boundary of a Finite Cylinder Ω_T

We begin studying the traces over transversal sections in a finite time. We formulate our results on Ω_T but with a natural and elementary change of temporal variable, they become true in sets of type $\Omega \times]T, T'[$ with $T < T'$ in $\mathbb{R}$. We need a previous lifting theorem:

Lemma 1. *Let* $r \geq 1$ *be such that* $r + \frac{1}{2} - \left(\frac{i+1}{2}\right)\left(\frac{r+1}{r+2}\right) \notin \mathbb{Z}$, $i = 0, 1$ *and let* j_0, k_0 *be the greatest elements in* $\mathbb{N} \cup \{0\}$ *such that* $0 \leq j_0 < r + \frac{1}{2}$ *and* $0 \leq k_0 < r + \frac{3}{2}$, *respectively. Given*

$$\mathbf{g} = (g_0(\mathbf{x}), g_1(\mathbf{x})) \in H_0^{r+1-\frac{1}{2}\frac{r+1}{r+2}}(\Omega) \times H_0^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega),$$

define

$$\mathbf{G}(\mathbf{g}) := (g_0(\mathbf{x}), g_1(\mathbf{x}), \ldots, g_{k_0}(\mathbf{x})) \in \prod_{k=0}^{k_0} H_0^{r+1-\left(k+\frac{1}{2}\right)\frac{r+1}{r+2}}(\Omega)$$

as

$$g_k(\mathbf{x}) := -\frac{1}{\tau}(\mathcal{X}(g_{k-2}(\mathbf{x})) - \lambda g_{k-1}(\mathbf{x})), \quad k = 2, \ldots, k_0. \quad (5)$$

Then there is a lifting

$$R : \mathbf{g} \in H_0^{r+1-\frac{1}{2}\frac{r+1}{r+2}}(\Omega) \times H_0^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega) \to H_{0,}^{r+1, r+2}(\Omega_T)$$

such that

$$\mathcal{A}^*(R(\mathbf{g})) \in H_{0,0}^{r-1,r}(\Omega_T), \tag{6}$$

$$\forall 0 \leq k \leq k_0, \quad \frac{\partial^k R(\mathbf{g})}{\partial t^k}(\mathbf{x}, 0) = 0, \quad \frac{\partial^k R(\mathbf{g})}{\partial t^k}(\mathbf{x}, T) = g_k(\mathbf{x}) \text{ in } \Omega \tag{7}$$

and

$$\forall 0 \leq |\boldsymbol{\alpha}| \leq j_0, \quad \frac{\partial^{|\boldsymbol{\alpha}|} R(\mathbf{g})}{\partial \mathbf{x}^{\boldsymbol{\alpha}}}(\mathbf{x}, t) = 0 \text{ in } (\partial\Omega)_T. \tag{8}$$

Proof. We begin checking that

$$\forall 0 \leq k \leq k_0, \quad g_k(\mathbf{x}) \in H_0^{r+1-\left(k+\frac{1}{2}\right)\frac{r+1}{r+2}}(\Omega) \tag{9}$$

proceeding by induction. (9) is true for $k = 0, 1$ by definition of $\mathbf{g}$. Assume (9) holds for every $2 \leq j \leq k \leq k_0 - 1$. Then

$$\mathcal{X}(g_{k-1}) \in H^{r-1-\left(k+1+\frac{1}{2}\right)\frac{r+1}{r+2}}(\Omega)$$

and $g_k \in H^{r+1-\left(k+\frac{1}{2}\right)\frac{r+1}{r+2}}(\Omega)$. It follows that (9) holds for $k+1$.

As a consequence of (9), we can apply [3, Chapter 4, Theorem 2.3] and the proof of [3, Chapter 1, Trace Theorem 3.2 and Remark 3.3] and the change $t = 1 - t'$ to obtain a continuous lifting

$$L : \prod_{k=0}^{k_0} H_0^{r+1-\left(k+\frac{1}{2}\right)\frac{r+1}{r+2}}(\Omega) \to H^{r+1,r+2}(\Omega_\infty)$$

such that the lifting

$$R : \mathbf{g} \in H_0^{r+1-\frac{1}{2}\frac{r+1}{r+2}}(\Omega) \times H_0^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega)$$

$$\to (R_T L(\mathbf{G}(\mathbf{g})))(\mathbf{x}, T-t) \in H^{r+1,r+2}(\Omega_T)$$

verifies

$$\forall T > 0, \quad \| R \| \leq \| L \| \tag{10}$$

(independent of *T*),

$$\forall 0 \leq k \leq k_0, \quad \frac{\partial^k R(\mathbf{g})}{\partial t^k}(\mathbf{x}, T) = g_k(\mathbf{x}) \text{ in } \Omega \tag{11}$$

and

$$\forall 1 \leq |\boldsymbol{\alpha}| \leq j_0, \quad R(\mathbf{g})(\mathbf{x}, t) = \frac{\partial^{|\alpha|} R(\mathbf{g})}{\partial \mathbf{x}^{\alpha}}(\mathbf{x}, t) = 0 \text{ in } (\partial\Omega)_T. \tag{12}$$

By [3, Chapter 1, Theorem 11.5], we obtain $R(\mathbf{g}) \in H_{0,}^{r-1, r}(\Omega_T)$ and obviously,

$$\frac{\partial R(\mathbf{g})}{\partial t}(\mathbf{x}, t) = \frac{\partial^2 R(\mathbf{g})}{\partial t^2}(\mathbf{x}, t) = 0 \text{ in } (\partial\Omega)_T. \tag{13}$$

Choosing a function $\varphi(t) \in \mathcal{C}^{\infty}([0, T])$ such that $\varphi(t) = 1$ in some neighbourhood of $t = T$ and $\varphi(t) = 0$ in some neighbourhood of $t = 0$ and switching to the new lifting $\varphi(t) R(\mathbf{g})(\mathbf{x}, t)$, it can be assumed that the initial *R* verifies, moreover,

$$\forall 0 \leq k \leq k_0, \quad \frac{\partial^k R(\mathbf{g})}{\partial t^k}(\mathbf{x}, 0) = 0 \text{ in } \Omega. \tag{14}$$

Furthermore, using [3, Chapter 4, Theorem 2.1] and the properties of traces in the space $H^{r-1, r+2}(\Omega_T)$, we obtain inductively for every $0 \leq k \leq k_0 - 2$,

$$\left(\frac{\partial^k \mathcal{X}(R(\mathbf{g}))}{\partial t^k}\right)(\mathbf{x}, 0) = \lim_{t \to 0} \left(\frac{\partial^k \mathcal{X}(R(\mathbf{g}))}{\partial t^k}\right)(\mathbf{x}, t) = \lim_{t \to 0} \mathcal{X}\left(\frac{\partial^k R(\mathbf{g})}{\partial t^k}\right)(\mathbf{x}, t)$$

$$= \mathcal{X}\left(\frac{\partial^k R(\mathbf{g})}{\partial t^k}(\mathbf{x}, 0)\right) = \mathcal{X}(g_k)(\mathbf{x})$$

in the space $H^{r-1-\left(k+\frac{1}{2}\right)\frac{r+1}{r+2}}(\Omega)$. Hence, by (5) and (11), we get $\forall 0 \leq k \leq k_0 - 2$,

$$\frac{\partial^k \mathcal{A}^*(R(\mathbf{g}))}{\partial t^k}(\mathbf{x}, T) = \mathcal{X}\left(\frac{\partial^k R(\mathbf{g})}{\partial t^k}(\mathbf{x}, t)\right) - \lambda \frac{\partial^{k+1} R(\mathbf{g})}{\partial t^{k+1}}(\mathbf{x}, T)$$

$$+ \tau \frac{\partial^{k+2} R(\mathbf{g})}{\partial t^{k+2}}(\mathbf{x}, T) = 0.$$

But, from (12), we have too $\mathcal{X}\left(\frac{\partial^j R(\mathbf{g})}{\partial \mathbf{n}^j}\right)(\mathbf{x}, t) = 0$ in $(\partial\Omega)_T$ for every $0 \leq j \leq j_0 - 2$ and from proposition [3, Chapter 4, Proposition 2.2] and (13), we obtain consecutively

$$\frac{\partial R(\mathbf{g})}{\partial \mathbf{n}}(\mathbf{x}, t) = \left(\frac{\partial}{\partial t}\frac{\partial^k R(\mathbf{g})}{\partial \mathbf{n}^k}(\mathbf{x}, t)\right) = \left(\frac{\partial^2}{\partial t^2}\frac{\partial^k R(\mathbf{g})}{\partial \mathbf{n}^k}(\mathbf{x}, t)\right) = 0$$

in $(\partial\Omega)_T$. Hence $\frac{\partial^j \mathcal{A}^*(R(\mathbf{g}))}{\partial \mathbf{n}^j}(\mathbf{x}, t) = 0$ in $(\partial\Omega)_T$ for every $0 \leq j \leq j_0 - 2$, and now, by [3, Chapter 4, Footnote to Lemma 10.1], we have $\mathcal{A}^*(R(\mathbf{g})) \in H_{0,0}^{r-1,r}(\Omega_T)$. Then, from (14) and [3, Chapter 1, Theorem 11.5], the lemma follows. □

Theorem 2. Traces over transversal sections. *Let* $T > 0$ *and* $r \geq 1$ *verifying the conditions of Lemma* 1. *Then the map* $Z : f \in \mathcal{C}^\infty(\overline{\Omega_T}) \to \left(f(\mathbf{x}, T), \frac{\partial f}{\partial t}(\mathbf{x}, T)\right) \in \mathcal{C}^\infty(\overline{\Omega}) \times \mathcal{C}^\infty(\overline{\Omega})$ *can be extended to a continuous linear map* (*again denoted by Z*)

$$Z : \mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega_T) \to H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega) \times H^{-(r+1)+\frac{1}{2}\frac{r+1}{r+2}}(\Omega).$$

Proof. Given $\mathbf{h} = (h_0(\mathbf{x}), h_1(\mathbf{x})) \in F := H_0^{r+1-\frac{1}{2}\frac{r+1}{r+2}}(\Omega) \times H_0^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega)$, define

$$g_0(\mathbf{x}) := \frac{1}{\tau} h_0(\mathbf{x}), \quad g_1(\mathbf{x}) := \frac{1}{\tau}(h_1(\mathbf{x}) - \lambda h_0(\mathbf{x})) \tag{15}$$

and $\{g_k(\mathbf{x})\}_{k=2}^{k_0}$ by (5). Writing $\mathbf{g} := \mathbf{g}(\mathbf{h}) = (g_0(\mathbf{x}), g_1(\mathbf{x}))$ as above, by the argumentation of Lemma 1 and using the same numbers j_0 and k_0 there are liftings

$$R : H_0^{r+1-\frac{1}{2}\frac{r+1}{r+2}}(\Omega) \times H_0^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega) \to H_{0,}^{r+1, r+2}(\Omega_T)$$

such that the equalities (6), (7) and (8) hold. We define a map

$$Z : U \in \mathcal{C}^\infty(\overline{\Omega_T}) \to (\varphi_0(\mathbf{x}), \varphi_1(\mathbf{x})) \in H^{-(r+1)+\frac{1}{2}\frac{r+1}{r+2}}(\Omega) \times H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega) \tag{16}$$

such that $\langle Z(U), \mathbf{h}\rangle = -\langle U, \mathcal{A}^*(R(\mathbf{g}(\mathbf{h})))\rangle + \langle \mathcal{A}(U), R(\mathbf{g}(\mathbf{h}))\rangle$ if $\mathbf{h} \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$. We check that Z is independent of the chosen lifting R. If we use another lifting R_1 verifying (6), (7) and (8), for $\Psi(\mathbf{g}) := R(\mathbf{g}) - R_1(\mathbf{g})$ we would have

$$\Psi(\mathbf{g})(\mathbf{x}, t) = \frac{\partial \Psi(\mathbf{g})}{\partial \mathbf{n}}(\mathbf{x}, t) = 0 \text{ in } (\partial\Omega)_T, \tag{17}$$

$$\forall 0 \le k \le k_0, \quad \Psi(\mathbf{g})(\mathbf{x}, 0) = \frac{\partial^k \Psi(\mathbf{g})}{\partial t^k}(\mathbf{x}, 0) = \Psi(\mathbf{g})(\mathbf{x}, T) = \frac{\partial^k R(\mathbf{g})}{\partial t^k}(\mathbf{x}, T) = 0 \text{ in } \Omega \tag{18}$$

and $\mathcal{A}^*(\Psi(\mathbf{g})) \in H_{0,0}^{r-1, r}(\Omega_T)$. Then, by [3, Chapter 4, Footnote of Lemma 10.1], we would have

$$\forall 0 \le |\alpha| \le j_0, \quad \frac{\partial^{|\alpha|}}{\partial \mathbf{x}^{\alpha}}\left(\mathcal{X}(\Psi(\mathbf{g})) - \lambda \frac{\partial \Psi(\mathbf{g})}{\partial t} + \tau \frac{\partial^2 \Psi(\mathbf{g})}{\partial t^2}\right)(\mathbf{x}, t) = 0$$

$$\text{in } (\partial\Omega)_T. \tag{19}$$

But, from (17), one has $\frac{\partial \Psi(\mathbf{g})}{\partial t}(\mathbf{x}, t) = \frac{\partial^2 \Psi(\mathbf{g})}{\partial t^2}(\mathbf{x}, t) = 0$ in $(\partial\Omega)_T$ and hence, using (19) in the case $|\alpha| = 0$, we obtain $\mathcal{X}(\Psi(\mathbf{g})(\mathbf{x}, t)) = 0$ in $(\partial\Omega)_T$. From (17), since $\mathcal{X}$ is elliptic in $\overline{\Omega}$, we deduce necessarily $\frac{\partial^2 \Psi(\mathbf{g})}{\partial x_i^2}(\mathbf{x}, t) = 0$ in $(\partial\Omega)_T$ for every $1 \le i \le n$. Proceeding inductively in the same way, we obtain $\frac{\partial^j \Psi(\mathbf{g})}{\partial x_i^j}(\mathbf{x}, t) = 0$ in $(\partial\Omega)_T$ for every $1 \le i \le n$ and $0 \le j \le j_0$. Again by the quoted footnote in [3], from (18), we have $\Psi(\mathbf{g}) \in H_{0,0}^{2r, r}(\Omega_T)$. Then Green's formula gives $\langle U, \mathcal{A}^*(\Psi(\mathbf{g}))\rangle - \langle \mathcal{A}(U), \Psi(\mathbf{g})\rangle = 0$ which shows the independence of R.

On the other hand, as $U \in \mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega_T)$, we have

$$|\langle Z(U), \mathbf{h}\rangle|$$

$$\le \| U \|_{H^{-(r-1), -r}(\Omega_T)} \| \mathcal{A}^*(R(\mathbf{g})) \|_{H_{0,0}^{r-1, r}(\Omega_T)}$$

$$+ \| \mathcal{A}(U) \|_{\Phi^{-(r+1), -(r+1)}(\Omega_T)} \| I_T \| \| R(\mathbf{g}) \|_{H_{0,0}^{r+1, r+2}(\Omega_T)}$$

$$\le (\| \mathcal{A} \| + \| I_T \|) \| R \| \| U \|_{\mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega_T)} \| \mathbf{g} \|_{H_0^{r+1-\frac{1}{2}\frac{r+1}{r+2}}(\Omega) \times H_0^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega)}. \tag{20}$$

Then $Z(U)$ is well defined and it turns out that, by density, Z can be continuously extended to a continuous linear map from $\mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega_T)$ into $H^{-(r+1)+\frac{1}{2}\frac{r+1}{r+2}}(\Omega) \times H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega)$.

Finally, taking $\mathbf{h} \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$, by duality we have $\langle Z(U), \mathbf{h} \rangle = \langle \varphi_0, h_0 \rangle + \langle \varphi_1, h_1 \rangle$ and by classical Green's formula and the definition of Z,

$$\forall U \in \mathcal{C}^{\infty}(\overline{\Omega_T}),$$

$$\begin{aligned}\langle Z(U), \mathbf{h} \rangle = \lambda \langle U(\mathbf{x}, T), R(\mathbf{g})(\mathbf{x}, T) \rangle_{\Omega} \\ + \tau\left(\left\langle \frac{\partial U}{\partial t}(\mathbf{x}, T), R(\mathbf{g})(\mathbf{x}, T) \right\rangle_{\Omega} - \left\langle U(\mathbf{x}, T), \frac{\partial R(\mathbf{g})}{\partial t}(\mathbf{x}, T) \right\rangle_{\Omega} \right) \\ = \langle U(\mathbf{x}, T), h_1(\mathbf{x}) \rangle + \left\langle \frac{\partial U}{\partial t}(\mathbf{x}, T), h_0(\mathbf{x}) \right\rangle\end{aligned}$$

and hence $Z(U) = \left(U(\mathbf{x}, T), \frac{\partial U}{\partial t}(\mathbf{x}, T) \right)$, $\mathbf{h}$ being arbitrary in $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$. □

To study the traces over the lateral boundary $(\partial\Omega)_T$, we need another lifting theorem:

Lemma 3. *Let* $T > 0$. *Let* $r \geq 1$ *be such that* $r + \frac{3}{2} - \left(\frac{r+2}{r+1} \right) \cdot \left(k + \frac{1}{2} \right) \notin \mathbb{Z}$, $0 \leq k \leq r$ *and let* $g(\mathbf{x}, t) := (g_0(\mathbf{x}, t), g_1(\mathbf{x}, t))$. *Then there is a lifting*

$$R : \mathbf{g}(\mathbf{x}, t) \in H_{,0}^{r+\frac{1}{2}, \left(r+\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_T) \times H_{,0}^{r-\frac{1}{2}, \left(r-\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_T) \\ \to H^{r+1, r+2}(\Omega_T)$$

such that $\mathcal{A}^*(R(\mathbf{g})) \in H_{0,0}^{r-1, r}(\Omega_T)$, $R(\mathbf{g})(\mathbf{x}, t) = g_0(\mathbf{x}, t)$ *and* $\Re_{\mathcal{A}}(R(\mathbf{g}))(\mathbf{x}, t) = g_1(\mathbf{x}, t)$ *in* $(\partial\Omega)_T$ *and*

$$\forall 0 \leq k < r + \frac{3}{2}, \quad \frac{\partial^k R(\mathbf{g})}{\partial t^k}(\mathbf{x}, 0) = \frac{\partial^k R(\mathbf{g})}{\partial t^k}(\mathbf{x}, T) = 0 \text{ in } \Omega. \tag{21}$$

Proof. Let j_0, k_0 be as in Lemma 1. Given $\mathbf{g}(\mathbf{x}, t)$ as announced, define $\{g_j(\mathbf{x}, t)\}_{j=2}^{j_0}$ in $(\partial\Omega)_T$ by the same rule used in (15). By [3, Chapter 1, Theorem 3.2 and Remark 3.3] and [3, Chapter 4, Theorem 2.3], there is a continuous linear map

$$R_1 : F := H_{,0}^{r+\frac{1}{2}, \left(r+\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_T) \times H_{,0}^{r-\frac{1}{2}, \left(r-\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_T)$$

$$\to H^{r+1, r+2}(\Omega_T)$$

such that $\frac{\partial^k R_1(\mathbf{g})}{\partial t^k}(\mathbf{x}, 0) = 0$ in Ω for each $0 \le k \le k_0$ and $R_1(\mathbf{g}) = g_0(\mathbf{x}, t)$, $\Re_{\mathcal{A}}(R_1(\mathbf{g})) = g_1(\mathbf{x}, t)$ and $\frac{\partial^j R_1(\mathbf{g})}{\partial \mathbf{n}^j}(\mathbf{x}, t) = g_j(\mathbf{x}, t)$, $2 \le j \le j_0$ in $(\partial\Omega)_T$ for every $0 \le j \le j_0$. Analogously, after the variable change $t' = T - t$, using [3, Chapter 4, Theorems 2.1 and 2.3], there is another continuous linear map $R_2 : F \to H^{r+1, r+2}(\Omega_T)$ such that

$$\frac{\partial^k R_2(\mathbf{g})}{\partial t^k}(\mathbf{x}, T) = \frac{\partial^k R_1(\mathbf{g})}{\partial t^k}(\mathbf{x}, T)$$

in Ω for every $0 \le k \le k_0$ and $\Re_{\mathcal{A}}(R_2(\mathbf{g})) = 0$, $R_2(\mathbf{g}) = 0$ and $\frac{\partial^j R_2(\mathbf{g})}{\partial \mathbf{n}^j} = 0$ in $(\partial\Omega)_T$ for every $2 \le j \le j_0$.

Choosing a function $\varphi \in \mathcal{C}^\infty(\overline{\Omega_T})$ such that $\varphi(\mathbf{x}, t) = 0$ in a neighbourhood of the section $\{(\mathbf{x}, 0), \mathbf{x} \in \Omega\}$ and $\varphi(\mathbf{x}, t) = 1$ in a neighbourhood of the section $\{(\mathbf{x}, T), \mathbf{x} \in \Omega\}$, it turns out that $R(\mathbf{x}, t) := R_1(\mathbf{x}, t) - \varphi(\mathbf{x}, t) R_2(\mathbf{x}, t)$ verifies

$$\frac{\partial^k R(\mathbf{g})}{\partial t^k}(\mathbf{x}, 0) = \frac{\partial^k R(\mathbf{g})}{\partial t^k}(\mathbf{x}, T) = 0$$

in Ω for $0 \leq k \leq k_0$ and $R(\mathbf{g}) = g_0(\mathbf{x}, t)$, $\Re_{\mathcal{A}}(R(\mathbf{g})) = g_1(\mathbf{x}, t)$ and $\dfrac{\partial^j R(\mathbf{g})}{\partial \mathbf{n}^j}(\mathbf{x}, t) = g_j(\mathbf{x}, t)$, $2 \leq j \leq j_0$ in $(\partial\Omega)_T$. From now on, the proof follows as in Lemma 1. □

Theorem 4. Traces on the lateral boundary $(\partial\Omega)_T$. *Let* $r \geq 1$ *be as in Lemma* 3. *Then the map* $Z : U \in \mathcal{C}^\infty(\overline{\Omega_T}) \to \left(U, \dfrac{\partial U}{\partial \mathbf{n}}\right) \in \mathcal{C}^\infty(\overline{(\partial\Omega)_T}) \times \mathcal{C}^\infty(\overline{(\partial\Omega)_T})$ *can be continuously extended to a map* (*again denoted by* Z)

$$Z : \mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega_T) \to H^{-\left(r-\frac{1}{2}\right), -\left(r-\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_T)$$

$$\times H^{-\left(r+\frac{1}{2}\right), -\left(r+\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_T).$$

Proof. It is the same as that in Theorem 2 but starting from $\mathbf{h} := (h_0(\mathbf{x}, t), h_1(\mathbf{x}, t)) \in H_{,0}^{r+\frac{1}{2}, \left(r+\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_T) \times H_{,0}^{r-\frac{1}{2}, \left(r-\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_T)$ using Lemma 3 and defining in $(\partial\Omega)_T$ functions $g_0(\mathbf{x}, t) = -\dfrac{h_0(\mathbf{x}, t)}{\mathfrak{f}_{\mathcal{A}}(\mathbf{x})}$, $g_1(\mathbf{x}, t) = h_1(\mathbf{x}, t)$ and $\{g_k(\mathbf{x}, t)\}_{k=2}^{j_0}$ according to rule (5). By Green's formula, if $(h_0(\mathbf{x}, t), h_1(\mathbf{x}, t)) \in \mathcal{C}^\infty(\overline{(\partial\Omega)_T}) \times \mathcal{C}^\infty(\overline{(\partial\Omega)_T})$, then we have $\langle Z(U), (h_1(\mathbf{x}, t), h_0(\mathbf{x}, t))\rangle = \langle U, h_1(\mathbf{x}, t)\rangle_{(\partial\Omega)_T} + \left\langle \dfrac{\partial U}{\partial \mathbf{n}}, g_0(\mathbf{x}, t)\right\rangle_{(\partial\Omega)_T}$ and the result follows. □

3. Traces on the Boundary of an Infinite Cylinder Ω_∞

Our goal in this section is to obtain the corresponding versions of Theorems 2 and 4 in the case of domains of type Ω_∞. We need to define suitable spaces $\mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega_\infty)$ since previous definitions do not directly apply to define spaces $\Phi^{r,r}(\Omega_\infty)$.

Consider a strictly increasing and unbounded sequence $\{T_m\}_{m=0}^{\infty}$ such that $0 < T_0 < \frac{T_1}{2}$. Fixed $0 < r$, with the help of the functions φ_{T_0, T_m} defined in Section 1, we can define spaces $\Phi^{r,r}(\Omega_{T_m}, \varphi_{T_0, T_m})$, (which will be denoted in the sequel by $\Phi^{r,r}(\Omega_{T_m})$ to simplify notation) for every $m \in \mathbb{N}$. Let R_n be the map sending every measurable function in Ω_∞ to its restriction to Ω_{T_n} and let $R_{n+1,n} : \Phi^{r,r}(\Omega_{T_{n+1}}) \to \Phi^{r,r}(\Omega_{T_n})$ be the restriction map to Ω_{T_n}. Clearly, $R_{n+1}(\Phi^{r,r}(\Omega_{T_{n+1}})) \subset \Phi^{r,r}(\Omega_{T_n})$. We define

$$\Phi^{r,r}(\Omega_\infty) := \{f : \Omega \times]0, \infty[\to \mathbb{R} \mid \| f \|_{\Phi^{r,r}(\Omega_\infty)} :$$

$$= \sup_{n \in \mathbb{N}} \| R_n(f) \|_{\Phi^{r,r}(\Omega_{T_n})} < \infty\}.$$

Remark that the continuous inclusion $R_n(H_{0,0}^{r,r}(\Omega_\infty)) \subset \Phi^{r,r}(\Omega_{T_n})$ and (3) imply that the inclusion $I_\infty : H_{0,0}^{r,s}(\Omega_\infty) \subset \Phi^{r,r}(\Omega_\infty)$ holds and hence $\Phi^{r,s}(\Omega_\infty) \neq \{0\}$ becomes a nontrivial Banach space endowed with the norm $\| f \|_{\Phi^{r,r}(\Omega_\infty)}$ (see [2]). Of course, we put $\Phi^{-r,-r}(\Omega_\infty) := (\Phi^{r,r}(\Omega_\infty))'$. Finally, we define for $r \in \mathbb{R}$,

$$D^{-(r-1)}(\Omega_\infty) := \{\Theta \in H^{-(r-1),-r}(\Omega_\infty) \mid \mathcal{A}(\Phi) \in \Phi^{-(r+1),-(r+1)}(\Omega_\infty)\}$$

endowed with the topology derived from the norm

$$\| \Theta \|_{D^{-(r-1)}(\Omega_\infty)} = \| \Theta \|_{H^{-(r-1),-r}(\Omega_\infty)} + \| \mathcal{A}(\Theta) \|_{\Phi^{-(r+1),-(r+1)}(\Omega_\infty)}.$$

In order to obtain a theorem of transversal traces "at ∞", we need an adequate subspace of $D_{\mathcal{A}}^{-(r-1)}(\Omega_\infty)$ which allows the application of classical Green's formula at the side $t \to \infty$. We define $\mathcal{W}^{-(r-1)}(\Omega_\infty)$ as the subspace of $D_{\mathcal{A}}^{-(r-1)}(\Omega_\infty)$ of those functions $U \in \mathcal{C}^\infty(\overline{\Omega_\infty}) \cap D_{\mathcal{A}}^{-(r-1)}(\Omega_\infty)$ such that there exists

$$Z(U)(\mathbf{x}, t) := \lim_{t\to\infty}\left(U(\mathbf{x}, t), \frac{\partial U}{\partial t}(\mathbf{x}, t)\right) \text{ in}$$

$$H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega) \times H^{-(r+1)+\frac{1}{2}\frac{r+1}{r+2}}(\Omega). \tag{22}$$

Clearly, $\mathcal{D}(\Omega_\infty) \subset \mathcal{W}^{-(r-1)}(\Omega_\infty)$ and $\mathcal{W}(\Omega_\infty) \neq \{0\}$. We define $\mathbf{W}_{\mathcal{A}}^{-(r-1)}(\Omega_\infty)$ as the closure of $\mathcal{W}^{-(r-1)}(\Omega_\infty)$ in $D_{\mathcal{A}}^{-(r-1)}(\Omega_\infty)$.

Theorem 5. *Let* $r \geq 1$. *Then the map*

$$W : U \in \mathcal{W}^{-(r-1)}(\Omega_\infty) \to \lim_{t\to\infty}\left(U(\mathbf{x}, t), \frac{\partial U}{\partial t}(\mathbf{x}, t)\right)$$

can be extended to a continuous linear map (*again denoted by W*)

$$W : \mathbf{W}_{\mathcal{A}}^{-(r-1)}(\Omega_\infty) \to H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega) \times H^{-(r+1)+\frac{1}{2}\frac{r+1}{r+2}}(\Omega).$$

Proof. By Lemma 1, there is a continuous lifting

$$L : \mathbf{g} := (g_0(\mathbf{x}), g_1(\mathbf{x})) \in H_0^{r+1-\frac{1}{2}\frac{r+1}{r+2}}(\Omega) \times H_0^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega)$$

$$\to H^{r+1, r+2}(\Omega_1)$$

verifying (6), (7) and (8). Then

$$\Psi : \mathbf{g} \in H_0^{r+1-\frac{1}{2}\frac{r+1}{r+2}}(\Omega) \times H_0^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega) \to \Psi(\mathbf{g})(\mathbf{x}, t) := L\left(\mathbf{g}\left(\mathbf{x}, \frac{t}{1+t}\right)\right)$$

is a $H^{r+1, r+2}(\Omega_\infty)$-valued lifting verifying (7), $\dfrac{\partial^{|\alpha|}\Psi(\mathbf{g})}{\partial \mathbf{x}^\alpha}(\mathbf{x}, t) = 0$ in

$(\partial\Omega)_\infty$ for every $0 \le |\boldsymbol{\alpha}| < r + \frac{1}{2}$ and moreover, by [3, Chapter 5, Theorem 11.5], there exists $W(\Psi(\mathbf{g}))$.

For every $T > 0$, let Z_T be the function defined on $\mathcal{C}^\infty(\overline{\Omega_T})$ as in (16) with the help of the lifting $R_T(\Psi)$. Consider the continuous linear map

$$W_T := U \in \mathcal{W}^{-(r-1)}(\Omega_\infty) \to F := H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega) \times H^{-(r+1)+\frac{1}{2}\frac{r+1}{r+2}}(\Omega)$$

such that $W_T(U) := Z_T(U)(\mathbf{x}, T)$ and denote W_T again its continuous extension to $\mathbf{W}_{\mathcal{A}}^{-(r-1)}(\Omega_\infty)$. By [3, Chapter 5, Theorem 11.5], we have

$$\lim_{T\to\infty, T'\to\infty} \| W_T(U) - W_{T'}(U) \|_F = 0,$$

and so there exists $\langle Z(U), \mathbf{g}\rangle := \lim_{T\to\infty}\langle Z_T(U), \mathbf{g}\rangle$. From Banach-Steinhaus's theorem, we obtain that there exists $Z(U) = \lim_{T\to\infty} Z_T(U)$ in F. The end of the proof can be accomplished noting that if $U \in \mathcal{W}^{-(r-1)}(\Omega_\infty)$ and $Z(U) = (\varphi_0(\mathbf{x}), \varphi_1(\mathbf{x})) \in F$, given $\mathbf{h} = (h_0(\mathbf{x}), h_1(\mathbf{x})) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$, with the same argumentation of Theorem 2 and the same definition for $\mathbf{g}$ and $\{g_k\}_{k=0}^{k_0}$, by duality and by Green's formula, using (22) and (10) we have

$$\begin{aligned}
&\langle Z(U), \mathbf{h}\rangle_{\Omega\times\Omega} \\
&= \langle \varphi_0, h_0\rangle + \langle \varphi_1, h_1\rangle \\
&= \lim_{T\to\infty} \lambda\langle U(\mathbf{x}, T), \Psi(\mathbf{g})(\mathbf{x}, T)\rangle_\Omega \\
&\quad + \lim_{T\to\infty} \tau\left(\left\langle \frac{\partial U}{\partial t}(\mathbf{x}, T), \Psi(\mathbf{g})(\mathbf{x}, T)\right\rangle_\Omega - \left\langle U(\mathbf{x}, T), \frac{\partial \Psi(\mathbf{g})}{\partial t}(\mathbf{x}, T)\right\rangle_\Omega\right) \\
&= \left\langle \lim_{T\to\infty} U(\mathbf{x}, T), h_1(\mathbf{x})\right\rangle + \left\langle \lim_{T\to\infty} \frac{\partial U}{\partial t}(\mathbf{x}, T), h_0(\mathbf{x})\right\rangle.
\end{aligned}$$

□

Finally, we consider the traces over a boundary of type $(\partial\Omega)_\infty$. As above, we need a suitable subspace of $\mathcal{D}_{\mathcal{A}}^{-(r-1)}(\Omega_\infty)$. By Theorem 5, we can consider the set $\mathcal{V}^{-(r-1)}(\Omega_\infty)$ of the functions $U \in \mathcal{C}^\infty(\overline{\Omega_\infty}) \cap \mathcal{D}_{\mathcal{A}}^{-(r-1)}(\Omega_\infty)$ such that $\lim_{T\to\infty} U(\mathbf{x}, T) = \lim_{T\to\infty} \frac{\partial U}{\partial t}(\mathbf{x}, T) = 0$. We define $\mathbf{V}_{\mathcal{A}}^{-(r-1)}(\Omega_\infty)$ as the closure of $\mathcal{V}^{-(r-1)}(\Omega_\infty)$ in $D_{\mathcal{A}}^{-(r-1)}(\Omega_\infty)$. We obtain

Theorem 6. Traces over $(\partial\Omega)_\infty$. *Let* $r \geq 1$. *Then the map* $Z : U \in \mathcal{C}^\infty(\overline{\Omega_\infty}) \cap \mathcal{V}^{-(r-1)}(\Omega_\infty) \to \left(U, \frac{\partial U}{\partial \mathbf{n}}\right) \in \mathcal{C}^\infty(\overline{(\partial\Omega)_T}) \times \mathcal{C}^\infty(\overline{(\partial\Omega)_T})$ *can be continuously extended to a map* (*again denoted by Z*)

$$\mathbf{V}_{\mathcal{A}}^{-(r-1)}(\Omega_\infty) \to G_\infty := H^{-\left(r-\frac{1}{2}\right), -\left(r-\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_\infty)$$

$$\times H^{-\left(r+\frac{1}{2}\right), -\left(r+\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_\infty).$$

Proof. The proof is based on a combination of the ideas applied in Theorems 5 and 4. For this reason, we only will mention the main details. Given

$$\mathbf{g} = (g_0(\mathbf{x}, t), g_1(\mathbf{x}, t)) \in E := H_{,0}^{r+\frac{1}{2}, \left(r+\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_1)$$

$$\times H_{,0}^{r-\frac{1}{2}, \left(r-\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_1),$$

by Lemma 3, there is a continuous lifting $L : \mathbf{g} \in E \to H^{r+1, r+2}(\Omega_1)$ verifying (6), (21) and $R(\mathbf{g})(\mathbf{x}, t) = g_0(\mathbf{x}, t)$ and $\mathfrak{R}_A(R(\mathbf{g}))(\mathbf{x}, t) = g_1(\mathbf{x}, t)$ in $(\partial\Omega)_T$. Now choose a function $\varphi \in \mathcal{C}^\infty(\overline{\Omega_1})$ such that $\varphi(\mathbf{x}, t) = 1$ in a neighbourhood of the section $\{(\mathbf{x}, 0), \mathbf{x} \in \Omega\}$ and $\varphi(\mathbf{x}, t) = 0$ in a neighbourhood of the section $\{(\mathbf{x}, 1), \mathbf{x} \in \Omega\}$. If $\mathbf{h} = (h_0((x, t)), h_1((x, t)))$ $\in E$, then we choose

$$g_0(\mathbf{x}, t) := \frac{-1}{\mathfrak{f}_{\mathcal{A}}(\mathbf{x})} h_0\left(\mathbf{x}, \frac{1}{t} - 1\right), \quad g_1(\mathbf{x}, t) = h_1\left(\mathbf{x}, \frac{1}{t} - 1\right).$$

For every $T > 0$, put

$$F_T := H^{-\left(r-\frac{1}{2}\right), -\left(r-\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_T) \times H^{-\left(r+\frac{1}{2}\right), -\left(r+\frac{1}{2}\right)\frac{r+2}{r+1}}((\partial\Omega)_T)$$

and consider the map Z_T defined as in Theorem 4 with the help of the $H^{r+1, r+2}(\Omega_\infty)$-valued lifting

$$\Psi : \mathbf{g} \in E \to \Psi(\mathbf{g})(\mathbf{x}, t) = \varphi\left(\mathbf{x}, \frac{t}{1+t}\right) L\left(\mathbf{g}\left(\mathbf{x}, \frac{t}{1+t}\right)\right).$$

Denote by $Q_M(f)$ the restrictions to $M \subset (\partial\Omega)_\infty$ of a distribution f defined in $(\partial\Omega)_\infty$. Given $T < T'$, we have $Q_{]0, T[}(Z_{T'}) = Z_T$ because $\mathcal{D}(\partial\Omega)_T \subset (\partial\Omega)_{T'}$. On the other hand, there is $K > 0$ independent of T such that

$$\sup_{T>0} \| Z_T(U) \|_{F_T} = \sup_{T>0} \| Z_T(R_T(U)) \|_{F_T} \leq K \| L \| \sup_{T>0} \| R_T(U) \| \leq K \| L \| \| U \|.$$

It turns out that the distribution $Z(U) \in \mathcal{D}'((\partial\Omega)_\infty)$ coinciding with $Z_T(R_T(U))$ for every $T > 0$ actually lies in G_∞. □

4. An Application to Green's Formula

Previous results can be applied to obtain a generalization of classical Green's formula. To do this, we begin noting that with the same argumentation used in [5] to represent the dual space $(H^k(\Omega_T))'$, $k \in \mathbb{N}$, it can be easily shown that every $f \in L^2(\Omega_T)$ defines in a natural way two continuous linear forms $\varphi_f \in H^{-r, -s}(\Omega_T)$ and $\Phi_f \in (H^{r, s}(\Omega_T))'$ such that

$$\| \varphi_f \| = \| \Phi_f \|. \tag{23}$$

Assume that $r \geq 1$, $r + \frac{1}{2} - \left(\frac{i+1}{2}\right)\left(\frac{r+1}{r+2}\right) \notin \mathbb{Z}$, $i = 0, 1$, and $u \in \mathbf{D}^{-(r-1)}(\Omega_T)$. By Theorem 2, we have

$$\{u(\mathbf{x}, 0), u(\mathbf{x}, T)\} \subset H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega) = (H_0^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega))'.$$

Both the elements $u(\mathbf{x}, 0)$, $u(\mathbf{x}, T)$ can be extended to specific elements in $(H^s(\Omega))'$ for every $s \geq r + 1 - \frac{3}{2}\frac{r+1}{r+2}$ in the following way: choose a sequence $\{u_k\}_{k=1}^{\infty} \subset \mathcal{C}^{\infty}(\overline{\Omega_T})$ such that $u = \lim_{k\to\infty} u_k$ in $\mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega_T)$. Then $\{u_k(\mathbf{x}, 0)\}_{k=1}^{\infty} \subset \mathcal{C}^{\infty}(\overline{\Omega})$ and by Theorem 2, we have $\lim_{k\to\infty} u_k(\mathbf{x}, 0) = u(\mathbf{x}, 0)$ in $H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega)$. By (23), we obtain

$$\forall k, m \in \mathbb{N}, \ \Phi_{u_k(\mathbf{x},0)-u_m(\mathbf{x},0)} = \Phi_{u_k(\mathbf{x},0)} - \Phi_{u_m(\mathbf{x},0)} \in \left(H^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega)\right)',$$

and since $\| \Phi_{u_k(\mathbf{x},0)} - \Phi_{u_m(\mathbf{x},0)} \| = \| u_k(\mathbf{x}, 0) - u_m(\mathbf{x}, 0) \|_{H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega)}$, $\{\Phi_{u_k(\mathbf{x},0)}\}_{k=1}^{\infty}$ is a Cauchy sequence in $\left(H^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega)\right)'$ and there exists $\varphi_u^0 = \lim_{k\to\infty} \Phi_{u_k(\mathbf{x},0)}$ in $\left(H^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega)\right)'$. The desired extension is φ_u^0. In fact, if $Q : \left(H^{r+1-\frac{3}{2}\frac{r+1}{r+2}}(\Omega)\right)' \to H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega)$ is the canonical continuous quotient map, then we have

$$Q(\varphi_u^0) = \lim_{k\to\infty} Q(\Phi_{u_k(\mathbf{x},0)}) = \lim_{k\to\infty} u_k(\mathbf{x}, 0) = u(\mathbf{x}, 0) \text{ in}$$
$$H^{-(r+1)+\frac{3}{2}\frac{r+1}{r+2}}(\Omega) \tag{24}$$

and it is clear that this value is independent of the chosen sequence $\{u_k\}_{k=1}^{\infty}$. It is also clear that φ_u^0 is an extension of $u(\mathbf{x}, 0)$. If $s \geq r + 1 - \frac{3}{2}\frac{r+1}{r+2}$, then the restriction to $H^s(\Omega)$ of φ_u^0 shows our assertion. In the same way, the extension φ_u^T of $u(\mathbf{x}, T)$ can be obtained.

Under the same hypothesis and after the analogous argumentation, the traces $\frac{\partial u}{\partial t}(\mathbf{x}, 0)$ and $\frac{\partial u}{\partial t}(\mathbf{x}, T)$ of $u \in \mathbf{D}^{-(r-1)}(\Omega_T)$ can be considered as elements of $(H^s(\Omega))'$ for every $s \geq r + 1 - \frac{1}{2}\frac{r+1}{r+2}$. Analogously, it can be shown that if $r + \frac{3}{2} - \left(\frac{r+2}{r+1}\right)\left(k + \frac{1}{2}\right) \notin \mathbb{Z}$, $0 \leq k \leq r$ and $u \in \mathbf{D}^{-(r-1)}(\Omega_T)$, then the traces $u(\mathbf{x}, t)$ $\left(\text{resp. } \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}, t)\right)$ on $(\partial\Omega)_T$ can be considered as elements of $(H^{s,w}((\partial\Omega)_T))'$ for every $s \geq r - \frac{1}{2}$, $w \geq \left(r - \frac{1}{2}\right)\frac{r+2}{r+1}$ $\left(\text{resp. } s \geq r + \frac{1}{2},\ w \geq \left(r + \frac{1}{2}\right)\frac{r+2}{r+1}\right)$. In the sequel, all the considered extensions will be denoted by the same symbols representing the functional to be extended.

Now we can state

Proposition 7. A generalization of Green's formula. *Let* $r \geq 1$ *verifying* $r + \frac{1}{2} - \left(\frac{i+1}{2}\right)\left(\frac{r+1}{r+2}\right) \notin \mathbb{Z}$, $i = 0, 1$ *and* $r + \frac{3}{2} - \left(\frac{r+2}{r+1}\right)\left(k + \frac{1}{2}\right) \notin \mathbb{Z}$ *for every* $0 \leq k \leq r$. *Let* $v \in \mathcal{C}^{\infty}(\overline{\Omega_T})$ *be such that* $\mathcal{A}^*(v) \in H_{0,0}^{r-1,r}(\Omega_T)$. *Let* $u \in \mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega_T)$. *There are a boundary operator* $\mathfrak{R}_{\mathcal{A}}$ *and a function* $\mathfrak{f}_{\mathcal{A}}$ *such that*

$$\langle v(\mathbf{x}, t), (\mathcal{A}u)(\mathbf{x}, t)\rangle_{\Omega_T} - \langle u(\mathbf{x}, t), (\mathcal{A}^* v)(\mathbf{x}, t)\rangle_{\Omega_T}$$

$$= \langle u(\mathbf{x}, t), (\mathfrak{R}_{\mathcal{A}}(v))(\mathbf{x}, t)\rangle_{(\partial\Omega)_T} - \left\langle \mathfrak{f}_{\mathcal{A}}(\mathbf{x}) v(\mathbf{x}, t), \frac{\partial u}{\partial \mathbf{n}}(\mathbf{x}, t)\right\rangle_{(\partial\Omega)_T}$$

$$+ \lambda[\langle v(\mathbf{x}, t), u(\mathbf{x}, t)\rangle_{\Omega}]_{t=0}^{t=T}$$

$$+ \tau\left[\left\langle v(\mathbf{x}, t), \frac{\partial u}{\partial t}(\mathbf{x}, t)\right\rangle_{\Omega} - \left\langle u(\mathbf{x}, t), \frac{\partial v}{\partial t}(\mathbf{x}, t)\right\rangle_{\Omega}\right]_{t=0}^{t=T}, \tag{25}$$

where every bracket is understood as the duality in the corresponding dual pair.

Proof. If we choose a sequence $\{u_k\}_{k=1}^{\infty} \subset \mathcal{C}^{\infty}(\overline{\Omega_T})$ such that $u = \lim_{k\to\infty} u_k$ in $\mathbf{D}_{\mathcal{A}}^{-(r-1)}(\Omega_T)$, as $\mathcal{A}^*(v) \in H_{0,0}^{r-1,r}(\Omega_T)$, by classical Green's formula, (25) holds for v and every u_k, $k \in \mathbb{N}$. Then, taking limits when $k \to \infty$, the conclusion follows from the introduction to this section. □

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