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Solving the random diffusion model in an infinite medium: A mean square approach

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Abstract

This paper deals with the construction of an analytic-numerical mean square solution of the random diffusion model in an infinite medium. The well-known Fourier transform method, which is used to solve this problem in the deterministic case, is extended to the random framework. Mean square operational rules to the Fourier transform of a stochastic process are developed and stated. The main statistical moments of the stochastic process solution are also computed. Finally, some illustrative numerical examples are included.

Keywords: Random diffusion; Mean square approach; Infinite medium; Analytic-numerical solution; Random Fourier transform

1 Introduction

Diffusion models involve uncertainties not only to the error measurements but also due to material defects or impurities in the case of heat diffusion [1, 2, 6]. Uncertainty also appears when in a diffusion model one considers pollutants which present impurities.

The theory and applications of random differential equations is a very active area of mathematical research and there are several different approaches, from the so-called stochastic differential approach based on the Itô calculus, to the called dishonest methods [8]. Fruitful methods approach have been recently used in [9, 10, 11, 12, 13, 14]. Random heat transfer in a finite medium have been treated in [2] by developing random perturbation method, by finite elements method in [1], and finite difference method in [20, 21]. A different random approach focussing on Brownian motion stochastic processes have also been treated in [22, 23] for the finite medium model and it is based on the

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Itô calculus. Here we follow the mean square approach developed for both the ordinary and partial differential case in [15, 16, 17, 18, 19]. This approach has two suitable properties. The first is that our solution coincides with the one of the deterministic case, i.e., when the random data are deterministic. The second, is that if $X_n(t)$ is a mean square approximation to the exact mean square solution $X(t)$, then the expectation and the variance of $X_n(t)$ converges to the expectation and the variance of $X(t)$, respectively [7].

For the sake of clarity in the presentation of the paper and thinking of applications, we assume that randomness is expressed by modeling the diffusion coefficient as a random variable (r.v.) and the initial stochastic process (s.p.) as a function that depends on a random variable. The same results are available, but with a more complicated notation by considering functions which depend on a finite number of random variables, the so-called functions with a finite degree of randomness quoted in [7, p.37]. In this paper, we consider the one-dimensional random diffusion model

$$u_t = A\,u_{xx}\,, \qquad -\infty < x < +\infty, \quad 0 < t < +\infty\,, \tag{1}$$

$$u(x,0) = \varphi(x;B)\,, \quad -\infty < x < +\infty, \tag{2}$$

where A is a positive r.v., independent of r.v. B and both satisfying additional properties to be specified later. We will also specify later the further condition to be satisfied by the s.p. $\varphi(x;B)$. Unlike to the finite medium random diffusion model, to the best of our knowledge there is a lack of reliable numerical answers to the solution of the random diffusion model in an infinite medium. This paper deals with the construction of reliable solutions to model (1)–(2) by extending to the random case the Fourier transform approach, but focussing more on the applications than in theoretical issues. It is important to point out that in the random case we are not only interested in the construction of a solution s.p. $u(x,t)$ but we also need to compute the expectation and the variance of $u(x,t)$.

This paper is organized as follows. Section 2 is devoted to some preliminaries that will clarify both the understanding and reading of the paper. Section 3 addresses the definition of mean square Fourier transform, some relevant properties and a useful example and operational rules. In Section 4, the solution of problem (1)–(2) is performed as well as its statistical moments. In Section 5 some illustrative examples are included and conclusions are drawn in Section 6.

2 Preliminaries about random mean square calculus

This section begins by reviewing some important concepts, definitions and results related to the random L_p calculus, mainly focusing on the mean square (m.s.) and mean fourth (m.f.) calculus, which correspond to $p = 2$ and $p = 4$, respectively (see [17] for further details). After a relevant class of r.v.'s that will play an important role in the development of next sections is studied.

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probabilistic space. Let $p \geq 1$ be a real number. A real r.v. U defined on $(\Omega, \mathcal{F}, \mathcal{P})$ is called of order p, if

$$\mathrm{E}\left[\,|U|^p\right] < +\infty\,,$$

where $\mathrm{E}[\cdot]$ denotes the expectation operator. The space L_p of all the real r.v.'s of order p, endowed with the norm

$$\|U\|_p = \left(\mathrm{E}\left[\,|U|^p\right]\right)^{1/p}\,, \tag{3}$$

is a Banach space, [24, p.9].

Let $\{U_n : n \geq 0\}$ be a sequence of r.v.'s of order p. We say that it is convergent in the p-th mean to the real r.v. $U \in L_p$, if

$$\lim_{n\to+\infty} \|U_n - U\|_p = 0.$$

If $p_2 \geq p_1$, then $L_{p_2} \subseteq L_{p_1}$. In addition if $\{U_n : n \geq 0\}$ is p_2-th mean convergent to $U \in L_{p_2}$, then $\{U_n : n \geq 0\}$ is also p_1-th mean convergent to $U \in L_{p_1}$, [24, p.13]. Convergences in L_2 and L_4 are usually referred to as m.s. and m.s. convergence, respectively. If U and V are r.v.'s in L_4 then by the Schwarz's inequality one gets (see [17])

$$\|UV\|_2 \leq \|U\|_4\,\|V\|_4\,. \tag{4}$$

If $\{U_n : n \geq 0\}$ is a sequence of 2-r.v.'s in L_2 m.s. convergent to $U \in L_2$, then from Theorem 4.3.1 of [7, p.88] one gets

$$\lim_{n\to\infty} \mathrm{E}[U_n] = \mathrm{E}[U]\,, \quad \lim_{n\to\infty} \mathrm{Var}[U_n] = \mathrm{Var}[U]\,, \tag{5}$$

where $\mathrm{Var}[\cdot]$ denotes the variance operator. Let T be a subset of the real line. A family $\{U(t) : t \in T\}$ of real r.v.'s of order p is said to be a s.p. of order p or, in short, a p-s.p. if

$$\mathrm{E}\left[\,|U(t)|^p\right] < +\infty, \quad \forall t \in T\,.$$

We say $\{U(t) : t \in T\}$ is p-th mean continuous at $t \in T$, if

$$\|U(t+h) - U(t)\|_p \to 0 \quad \text{as} \quad h \to 0, \quad t, t+h \in T\,.$$

Furthermore, if there exists a s.p. $U'(t)$ of order p, such that

$$\left\| \frac{U(t+h) - U(t)}{h} - U'(t) \right\|_p \to 0 \quad \text{as} \quad h \to 0, \quad t, t+h \in T\,,$$

then we say that $\{U(t) : t \in T\}$ is p-th mean differentiable at $t \in T$ and $U'(t)$ is the p-derivative of $U(t)$.

In the particular cases that $p = 2, 4$, definitions above leads to the corresponding concepts of mean square (m.s.) and mean fourth (m.f.) continuity and differentiability. Furthermore, it is easy to prove by (4) that m.f. continuity and differentiability entail m.s. continuity and differentiability, respectively.

In accordance with [7, p.99], [27], we say that a s.p. $\{V(x) : x \in \mathbb{R}\}$ with $V(x) \in L_p$ for all x, is locally integrable in $\mathbb{R}$ if, for all finite interval $[a, b] \subset \mathbb{R}$, the integral

$$\int_a^b V(x)\, dx\,,$$

exits in L_p. We say that $\{V(x) : x \in \mathbb{R}\}$ is absolutely integrable in L_p, if

$$\int_{-\infty}^{+\infty} \|V(x)\|_p\, dx \; < +\infty\,.$$

Now, we introduce an important type of r.v.'s, L, that have played a relevant role in the m.s. solution of random ordinary differential equations (see [25] and references therein), and which will be used later. We will assume that such r.v.'s L have absolute moments with respect to the origin that increase at the most exponentially, i.e., there exist a non-negative integer n_0 and positive constants M and H such that

$$\mathrm{E}\left[\,|L|^n\right] \leq M H^n, \quad \forall n \geq n_0, \text{ i.e. } \mathrm{E}\left[\,|L|^n\right] = \mathcal{O}(H^n)\,. \tag{6}$$

From (6) and definition (3) for $p = 4$, for each $x \in \mathbb{R}$ one gets

$$\begin{aligned}
\left(\left\|e^{-Lx^2}\right\|_4\right)^4 &= \mathrm{E}\left[e^{-4Lx^2}\right] &&= \mathrm{E}\left[\sum_{n\geq 0} \frac{(-4)^n\, L^n\, x^{2n}}{n!}\right] \\
&= \sum_{n\geq 0} \frac{(-4)^n\, x^{2n}\, \mathrm{E}[L^n]}{n!} &&\leq \sum_{n\geq 0} \frac{(-4)^n\, x^{2n}\, \mathrm{E}[|L|^n]}{n!} \\
&\leq M \sum_{n\geq 0} \frac{(-4)^n\, x^{2n}\, H^n}{n!} &&= M e^{-4Hx^2}, \quad M > 0.
\end{aligned}$$

Thus

$$\left\|e^{-Lx^2}\right\|_4 \leq \tilde{M} e^{-Hx^2}, \quad x \in \mathbb{R}\,,\; \tilde{M} = \sqrt[4]{M} > 0\,. \tag{7}$$

Remark 1 *The lack of explicit formulae for the absolute moments with respect to the origin of some standard r.v.'s as well as the aim of looking for a general approach to deal with the widest range of random inputs, we are going to take advantage of censuring method (see [26, ch.V]) to show that truncated r.v.'s satisfy condition* (6)*. Let us assume a r.v. L that satisfies:*

$$l_1 \leq l = L(\omega) \leq l_2, \qquad \forall \omega \in \Omega.$$

Then

$$\mathrm{E}\left[|L|^n\right] = \int_{l_1}^{l_2} |l|^n\, f_L(l)\, \mathrm{d}l \leq H^n, \tag{8}$$

where $f_L(l)$ denotes the probability density function (p.d.f.) of r.v. L and, $H = \max\left(|l_1|\,, |l_2|\right)$. Indeed, in the case that $H > 1$, one gets

$$\int_{l_1}^{l_2} |l|^n\, f_L(l)\, \mathrm{d}l \leq H^n \int_{l_1}^{l_2} f_L(l)\, \mathrm{d}l = H^n.$$

Notice that in the last step, we have applied that the integral of the right-hand side is just 1 *because of* $f_L(l)$ *is a p.d.f. The other cases can be analyzed analogously. Substituting the integral by a sum in (8), previous reasoning remains true when* L *is a discrete r.v. As a consequence, important r.v.'s such as binomial, hypergeometric, uniform or beta satisfy condition* (6) *related to the absolute moments of* L*. Although many other unbounded r.v.'s can also verify condition* (6)*, we do not need to check it each case, since censuring their codomain suitably, we are legitimated to approximate them. Hence, truncations of r.v.'s such as exponential or gaussian satisfy condition* (6)*.. The larger the censured interval, the better the approximations. However, in practice, intervals relatively short provide very good approximations. For instance, as an illustrative example notice that the truncated interval* $[\mu - 3\sigma, \mu + 3\sigma]$ *contains the* 99.7% *of the probability mass of a gaussian r.v. with mean* μ *and standard deviation* $\sigma > 0$*.*

3 Random Fourier operational calculus

Let $\{u(x) : x \in \mathbb{R}\}$ be a 2-s.p. m.s. absolutely integrable, i.e.,

$$\int_{-\infty}^{+\infty} \|u(x)\|_2 \, dx \; < +\infty \,, \tag{9}$$

then $\forall\, \xi \in \mathbb{R}$, the m.s. Fourier transform process of $\{u(x) : x \in \mathbb{R}\}$ is defined by

$$\mathcal{U}(\xi) = \mathfrak{F}[u(x)](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x)\, e^{-i\,\xi\, x} \; dx \,. \tag{10}$$

Note that from (9) the r.v. $\mathfrak{F}[u(x)](\xi)$ is well defined for all $\xi \in \mathbb{R}$. For the case where $\{u(x) : x \in \mathbb{R}\}$ is an absolutely integrable deterministic function, the above definition coincides with the well-known classic Fourier transform.

An important example that will be used later is related to the m.s. Fourier transform of a gaussian type s.p.

Example 1 *Let* L *be a positive r.v. satisfying* (6)*. Then*

$$\mathfrak{F}\left[e^{-Lx^2}\right](\xi) = \frac{1}{\sqrt{2\,L}}\, e^{-\frac{\xi^2}{4L}} \,, \quad \xi \in \mathbb{R} \,. \tag{11}$$

We will prove first that for each $\xi \in \mathbb{R}$ *the r.v.*

$$\mathfrak{F}\left[e^{-Lx^2}\right](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-Lx^2} e^{-i\xi x} \, dx \,, \tag{12}$$

is m.s. convergent. This is guaranteed if we prove that (12) *is m.f. convergent and it is a consequence of* (7) *because*

$$\int_{-\infty}^{+\infty} \left\| e^{-Lx^2} e^{-i\xi x} \right\|_4 \, dx = \int_{-\infty}^{+\infty} \left\| e^{-Lx^2} \right\|_4 \, dx \le \tilde{M} \int_{-\infty}^{+\infty} e^{-Hx^2} \, dx = \tilde{M} \sqrt{\frac{\pi}{H}} \,.$$

Once we have proven that (12) *is a well-defined r.v., will find a closed form representation by computing its value for each event* $\omega \in \Omega$*, i.e., the value of*

$$\left(\mathfrak{F}\left[e^{-Lx^2}\right](\xi)\right)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-L(\omega)x^2} e^{-i\xi x}\, dx, \quad \omega \in \Omega. \tag{13}$$

As $L(\omega) > 0$ *and* $\int_{-\infty}^{+\infty} e^{-L(\omega)x^2} \sin(\xi x)\, dx = 0$*, one gets*

$$\int_{-\infty}^{+\infty} e^{-L(\omega)x^2} e^{-i\xi x}\, dx = \int_{-\infty}^{+\infty} e^{-L(\omega)x^2} \cos(\xi x)\, dx\,. \tag{14}$$

Now by [27, p.61] the real integral (14) *takes the value*

$$\int_{-\infty}^{+\infty} e^{-L(\omega)x^2} \cos(\xi x)\, dx = \sqrt{\frac{\pi}{L(\omega)}}\; e^{-\frac{\xi^2}{4L(\omega)}}\,, \quad \omega \in \Omega\,. \tag{15}$$

From (13)–(15) *one gets*

$$\left(\mathfrak{F}\left[e^{-Lx^2}\right](\xi)\right)(\omega) = \frac{1}{\sqrt{2\,L(\omega)}}\, e^{-\frac{\xi^2}{4L(\omega)}}\,, \quad \omega \in \Omega\,,\ \xi \in \mathbb{R}\,. \tag{16}$$

Thus (11) *is established.*

The main fact that makes fruitful the Fourier transform approach in the deterministic framework is the relationship between the Fourier transform of a function and the Fourier transform of its derivatives. That is based on the integration by parts formula. The next result provides a random version of this property.

Let $u(x)$ be a 2-s.p. m.s. differentiable such that $u'(x)$ is m.s. continuous and let $f(x)$ be a continuously differentiable deterministic function. Then, by [7, p.96] one gets

$$\left(f(x)\,u(x)\right)' = f'(x)\,u(x) + f(x)\,u'(x)\,, \tag{17}$$

and by integrating on the interval $[0, R]$, for $R > 0$, it follows that

$$\int_0^R \left(f(x)\,u(x)\right)'\, dx = \int_0^R f'(x)\,u(x)\,dx + \int_0^R f(x)\,u'(x)\,dx\,. \tag{18}$$

By the fundamental theorem of m.s. calculus, [7, p.104], one gets

$$\int_0^R \left(f(x)\,u(x)\right)'\, dx = f(R)\,u(R) - f(0)\,u(0)\,. \tag{19}$$

Assume that $u(x)$ and $u'(x)$ are m.s. absolutely integrable on the real line and that $f(x)$ and $f'(x)$ are bounded on the real line. Then, from (18) one gets the m.s. convergence of the integral

$$\int_0^{+\infty} \left(f(x)\,u(x)\right)'\, dx\,,$$

and from (19) one gets the m.s. convergence of the limit

$$f(R)\,u(R) \xrightarrow[R\to+\infty]{\text{m.s.}} \ell\,. \tag{20}$$

Now we prove that $\ell = 0$, showing that

$$u(R) \xrightarrow[R\to+\infty]{\text{m.s.}} 0\,, \tag{21}$$

and using that $f(x)$ is bounded on the real line. By [7, p.104] we have

$$\int_0^R u'(x)\,dx = u(R) - u(0)\,. \tag{22}$$

As $u'(x)$ is m.s. absolutely integrable, the m.s. limit of the left-hand side of (22) exists, and thus there exists the m.s. limit

$$u(R) \xrightarrow[R\to+\infty]{\text{m.s.}} s\,. \tag{23}$$

As $u(x)$ is m.s. absolutely integrable on the real line, from the Cauchy condition in the Banach space L_2, the limits must be necessarily zero. Hence $s = \ell = 0$, and from (18) one gets

$$\int_0^{+\infty} f'(x)\,u(x)\,dx + \int_0^{+\infty} f(x)\,u'(x)\,dx = -f(0)\,u(0)\,. \tag{24}$$

By repeating the previous steps on the interval $[-R, 0]$, $R > 0$, under the previous hypotheses on $u(x)$ and $f(x)$, it follows that

$$\int_{-\infty}^{0} f'(x)\,u(x)\,dx + \int_{-\infty}^{0} f(x)\,u'(x)\,dx = f(0)\,u(0)\,. \tag{25}$$

Adding (24) and (25) one gets

$$\int_{-\infty}^{+\infty} f'(x)\,u(x)\,dx = -\int_{-\infty}^{+\infty} u'(x)\,f(x)\,dx\,. \tag{26}$$

Summarizing, the following result has been established:

Lemma 1 (Random integration by parts formula) *Let $f(x)$ and $u(x)$ be a deterministic function and 2-s.p., respectively, such that*

(i) $f(x)$ and $f'(x)$ bounded on the real line and continuously differentiable.

(ii) $u(x)$ m.s. continuously differentiable and m.s. absolutely integrable on the real line.

(iii) $u'(x)$ is m.s. continuous and m.s. absolutely integrable on the real line.

Then (26) holds.

Assume that $u(x)$ and $u'(x)$ are 2-s.p. satisfying both the hypotheses of Lemma 1 with $f(x) = e^{-i\xi x}/\sqrt{2\pi}$. From definition (10) and Lemma 1 it follows that

$$\begin{aligned} \mathfrak{F}[u'(x)](\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u'(x)\, e^{-i\xi x}\ dx = \frac{i\,\xi}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x)\, e^{-i\xi x}\ dx \\ &= i\,\xi\ \mathfrak{F}[u(x)](\xi)\,. \end{aligned} \tag{27}$$

By applying again Lemma 1 with $u'(x)$ and by definition (10) acting on $u'(x)$ it follows that

$$\begin{aligned} \mathfrak{F}[u''(x)](\xi) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u''(x)\, e^{-i\xi x}\ dx = \frac{i\,\xi}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u'(x)\, e^{-i\xi x}\ dx \\ &= i\,\xi\ \mathfrak{F}[u'(x)](\xi)\,. \end{aligned} \tag{28}$$

From (27) and (28) one gets

$$\mathfrak{F}[u''(x)](\xi) = -\xi^2\ \mathfrak{F}[u(x)](\xi)\,. \tag{29}$$

Summarizing, if $u(x)$, $u'(x)$ and $u''(x)$ are 2-s.p. that are m.s. absolutely integrable on the real line with $u''(x)$ m.s. continuous, then (29) holds.

For the sake of clarity in the presentation, we state without proof the m.s. differentiation of integrals whose proof is a direct consequence of the proof of the deterministic case [27, p.99] and the m.s. differentiation theorem for a sequence of 2-s.p. [28].

Lemma 2 (m.s. differentiation of infinite integrals) *Let $g(x,t)$ be a 2-s.p. m.s. continuous with m.s. continuous partial derivative $\frac{\partial g}{\partial t}(x,t)$. Assume the hypotheses*

(i) $G(t) = \int_a^{+\infty} g(x,t)\,dx$ *is m.s. pointwise convergent for each $t > 0$.*

(ii) $\int_a^{+\infty} \frac{\partial}{\partial t}\left(g(x,t)\right)\,dx$ *is m.s. uniformly convergent in $[t-\delta, t+\delta]$, $\delta > 0$ for each $t > 0$.*

Then the process $G(t)$ is m.s. differentiable, and

$$G'(t) = \int_a^{+\infty} \frac{\partial}{\partial t}\left(g(x,t)\right)\,dx\,. \tag{30}$$

The last m.s. operational rule that we need to extend to the random framework is the convolution of two s.p.'s and the relationship between the Fourier transform of the convolution product with the Fourier transform of the convolution factors.

Let $f(x)$, $g(x)$ be 4-s.p.'s such that

$$\int_{-\infty}^{+\infty} \left(\|f(x)\|_4\right)^2 \, dx < +\infty \,, \quad \int_{-\infty}^{+\infty} \left(\|g(x)\|_4\right)^2 \, dx < +\infty \,. \tag{31}$$

The convolution process of f and g, denoted by $f * g$, is defined by the m.s. integral

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-y)\, g(y)\, dy \,. \tag{32}$$

Note that under hypothesis (31) and (4) together with Cauchy-Schwarz inequality for deterministic real functions one gets

$$\begin{aligned}
\int_{-\infty}^{+\infty} \|f(x-y)\, g(y)\|_2 \, dy &\leq \int_{-\infty}^{+\infty} \|f(x-y)\|_4 \, \|g(y)\|_4 \, dy \\
&\leq \left(\int_{-\infty}^{+\infty} \left(\|f(x-y)\|_4\right)^2 \, dy \right)^{1/2} \left(\int_{-\infty}^{+\infty} \left(\|g(y)\|_4\right)^2 dy \right)^{1/2} \\
&< +\infty \,.
\end{aligned}$$

Thus the convolution of two 4-s.p.'s $f(x)$ and $g(x)$ is well defined by a m.s. convergent integral. Now, taking into account the Fubini theorem in abstract normed spaces [29, p.175], section 1.85 of [27] and the proof of the Fourier transform of convolution of real functions [30, chap.7] it follows that:

If $f(x)$, $g(x)$ are m.s. continuous 4-s.p.'s satisfying (31) and

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \|f(x-y)\, g(y)\|_2 \, dx\, dy < +\infty \,, \tag{33}$$

then

$$\mathfrak{F}\left[f * g\right] = \mathfrak{F}\left[f\right] \, \mathfrak{F}\left[g\right] \,. \tag{34}$$

4 Solving the random diffusion model

This section deals with the construction of the s.p. solution of problem (1)–(2) as well as the determination of its expectation and variance. Let us assume that A and B are independent 4-r.v.'s and the moment generating function of r.v. A, $\Phi_A(t) = \mathrm{E}[e^{tA}]$, satisfies the following condition:

$$\Phi_A(t) \text{ is locally bounded about } t = 0 \,. \tag{35}$$

We also assume that the s.p. $\varphi(x; B)$ defining the initial condition (2) is m.s. absolutely integrable, m.f. continuous and satisfies that

$$\|\varphi(x; B)\|_4 = \mathcal{O}\left(e^{-\beta x^2}\right) \,, \quad -\infty < x < +\infty, \; \beta > 0 \,. \tag{36}$$

We will assume that the problem (1)–(2) admits a Fourier transformable solution s.p. $u(x,t)$ which will be denoted by

$$\mathcal{U}(t)(\xi) = \mathfrak{F}\left[u(\cdot\,,t)\right](\xi)\,. \tag{37}$$

By applying the Fourier transform to both sides of (1) and using (30) for the left-hand side of (1) and (29) for its right-hand side, it follows that

$$\left.\begin{array}{rcl} \dfrac{d}{dt}\left(\mathcal{U}(t)(\xi)\right) &=& -A\,\xi^2\,\mathcal{U}(t)(\xi)\,, \quad t>0 \\ \mathcal{U}(0)(\xi) &=& \Phi(\xi)\,, \end{array}\right\}, \tag{38}$$

where

$$\Phi(\xi) = \mathfrak{F}\left[\varphi(x;B)\right](\xi)\,, \qquad \xi\in\mathbb{R}\,. \tag{39}$$

By theorem 8 of [18], the solution of (38) is given by

$$\mathcal{U}(t)(\xi) = e^{-A\,\xi^2\,t}\;\Phi(\xi)\,, \qquad \xi \text{ fixed},\; t>0\,. \tag{40}$$

In order to guarantee further convolution properties, let us assume now that the diffusion random variable A satisfies

$$\|e^{-Ax^2}\|_4 = \mathcal{O}\left(e^{-\alpha x^2}\right)\,, \quad -\infty < x < +\infty,\; \alpha>0\,. \tag{41}$$

Notice that (41) and (36) guarantees that

$$\int_{-\infty}^{+\infty}\left(\|e^{-Ax^2}\|_4\right)^2 dx < +\infty, \qquad \int_{-\infty}^{+\infty}\left(\|\varphi(x;B)\|_4\right)^2 dx < +\infty\,, \tag{42}$$

respectively. In addition, condition (33) holds because from (4), (36) and (41) it follows that

$$\begin{aligned} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\|e^{-A(x-y)^2}\,\varphi(y;B)\|_2\;dx\,dy \;&\le\; \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\|e^{-A(x-y)^2}\|_4\;\|\varphi(y;B)\|_4\,dx\,dy \\ &\le\; W\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-\alpha(x-y)^2}\,e^{-\beta y^2}\,dx\,dy \\ &=\; W\frac{\pi}{\sqrt{\alpha\,\beta}}\,, \end{aligned} \tag{43}$$

for a positive constant W. From the convolution property (34) together with (40), (43) and Example 1 with $L = 1/(4At)$, an inverse Fourier transform of (40) is given by $\varphi(x;B) * \left[1/(\sqrt{2tA})\,e^{-\frac{x^2}{4At}}\right]$, i.e.,

$$\begin{aligned} u(x,t) &= \mathfrak{F}^{-1}\left[e^{-A\xi^2 t}\;\Phi(\xi)\right] \\ &= \varphi(x;B) * \left[\frac{1}{\sqrt{2\,t\,A}}\;e^{-\frac{x^2}{4At}}\right] \\ &= \frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\varphi(\xi;B)\;\frac{e^{-\frac{(x-\xi)^2}{4At}}}{\sqrt{A}}\;d\xi\,, \quad -\infty<x<+\infty,\; t>0\,. \end{aligned} \tag{44}$$

Summarizing, the following result has been established:

Proposition 1 *If A is a $4-$r.v. that satisfies the conditions* (35) *and* (41)*, and B is a $4-$r.v. independent of A and satisfying* (36)*, then $u(x,t)$ given by* (44) *is a m.s. solution s.p. of problem* (1)–(2)*.*

Using the independence of r.v.'s A and B, one gets that the expectation and the variance function of the solution s.p. are, respectively

$$\mathrm{E}[u(x,t)] = \frac{1}{2\sqrt{\pi\, t}} \int_{-\infty}^{+\infty} \mathrm{E}\left[\varphi(\xi;B)\right] \mathrm{E}\left[\frac{e^{-\frac{(x-\xi)^2}{4At}}}{\sqrt{A}}\right] d\xi\,, \tag{45}$$

$$\mathrm{E}\big[(u(x,t))^2\big] = \frac{1}{4\pi t} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{E}[\varphi(\xi_1;B)\varphi(\xi_2;B)]\, \mathrm{E}\left[\frac{1}{A}\, e^{-\frac{\left[(x-\xi_1)^2+(x-\xi_2)^2\right]}{4At}}\right] d\xi_1\, d\xi_2\,, \tag{46}$$

$$\mathrm{Var}\left[u(x,t)\right] = \mathrm{E}[(u(x,t))^2] - \left(\mathrm{E}[u(x,t)]\right)^2\ . \tag{47}$$

Remark 2 *In practice the improper integrals involved in* (45) *and* (46) *can only be computed in an approximate form by truncating the domain of integration at the interval, namely $[-N,N]$, $N>0$. Hence, in general, only approximations of the expectation and/or variance (or equivalently, standard deviation) can be achieved by*

$$\mathrm{E}[u_N(x,t)] = \frac{1}{2\sqrt{\pi\, t}} \int_{-N}^{N} \mathrm{E}\left[\varphi(\xi;B)\right] \mathrm{E}\left[\frac{e^{-\frac{(x-\xi)^2}{4At}}}{\sqrt{A}}\right] d\xi\,, \tag{48}$$

$$\mathrm{E}\big[(u_N(x,t))^2\big] = \frac{1}{4\pi t} \int_{-N}^{N}\int_{-N}^{N} \mathrm{E}[\varphi(\xi_1;B)\varphi(\xi_2;B)]\, \mathrm{E}\left[\frac{1}{A}\, e^{-\frac{\left[(x-\xi_1)^2+(x-\xi_2)^2\right]}{4At}}\right] d\xi_1\, d\xi_2\,, \tag{49}$$

$$\mathrm{Var}\left[u_N(x,t)\right] = \mathrm{E}[(u_N(x,t))^2] - \left(\mathrm{E}[u_N(x,t)]\right)^2\ , \tag{50}$$

for $N>0$ large enough. Notice that from (5)*, the m.s. convergence of $u_N(x,t)$ guarantees that these approximations $\mathrm{E}[u_N(x,t)]$ and $\mathrm{Var}\left[u_N(x,t)\right]$ converge to the exact expectation $\mathrm{E}[u(x,t)]$ and variance $\mathrm{Var}\left[u(x,t)\right]$, respectively.*

Remark 3 *Note that if A is a positive real number and $\varphi(x;B)$ is a deterministic function, then expression* (44) *coincides with the solution of the corresponding diffusion problem, see [31, p.93].*

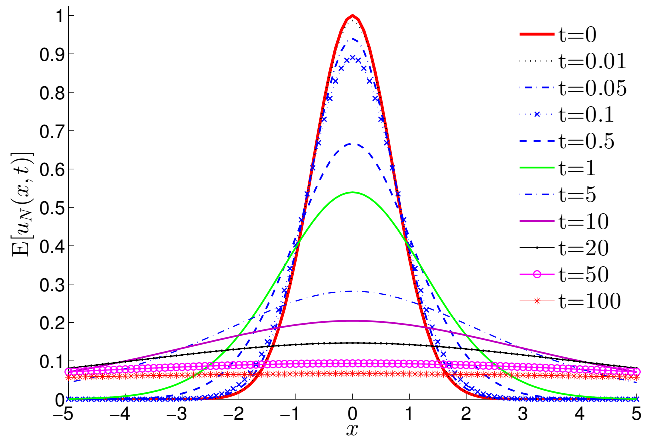
5 Numerical examples

This section is devoted to present several examples where the theoretical results previously obtained are applied. Throughout these examples randomness will be considered in both the initial condition $\varphi(x;B)$ and/or the diffusion coefficient A. Different probabilistic distributions will be assumed for the involved r.v.'s B and A.

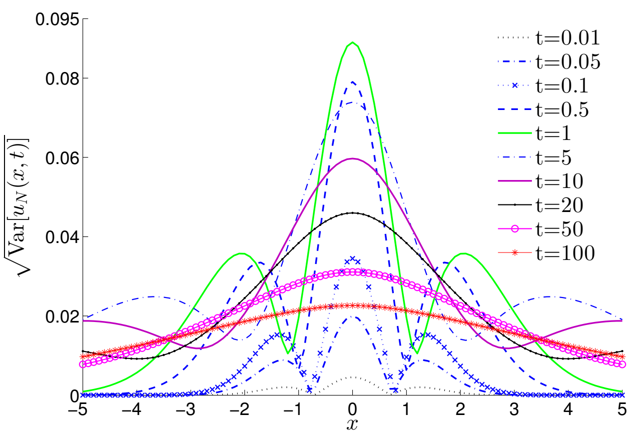
Example 2 *Consider problem* (1)–(2) *where r.v. A is assumed to follow a beta distribution of parameters $\alpha = 2$ and $\beta = 1$: $A \sim Beta(2,1)$. Notice that A is a 4-r.v. that satisfies the conditions:* (35) *and* (41) *since it is bounded (see Remark 1 and* (6)–(7)*). We consider as (deterministic) initial condition $\varphi(x;B) = \varphi(x) = e^{-x^2}$, which obviously satisfies condition* (36). *In Fig. 1, we show approximations to the expectation and the standard deviation on the spatial domain $-5 \le x \le 5$ at different time instants, both, closed to the origin: $t = 0, 0.01, 0.05, 0.1, 1$ and further $t = 5, 10, 20, 50, 100$. We have represented these approximations on this truncated spatial domain since for $t = 100$ one reaches the equilibrium temperature all over the full bar. These approximations have been computed using* (48)–(50) *with $N = 4$. This value of N corresponds until a numerical stabilization of the expectation, on the full spatial domain, is reached according to the following stopping criterium $\|\mathrm{E}[u_{N+1}(x,t)] - \mathrm{E}[u_N(x,t)]\|_\infty \le \mathcal{O}(10^{-8})$. An analogous criterium has been used for the numerical approximation to the standard deviation. In Fig. 2, we have plotted the corresponding results to the expectation and the standard deviation on the spatial domain $-100 \le x \le 100$. Here the numerical results for times close to zero are omitted because they have a different order of magnitude.*

Example 3 *Now, we consider problem* (1)–(2) *being A a gamma r.v. of parameters $\alpha = 2$ and $\beta = 3$, i.e., $A \sim Gamma(2,3)$ truncated at the interval $[0,5]$. As A is a truncated r.v., conditions* (35) *and* (41) *are fulfilled. As in the previous example, we take as the (deterministic) initial condition $\varphi(x;B) = \varphi(x) = e^{-x^2}$ which satisfies condition* (36). *In Fig. 3 we show, by means of a surface, approximations on the spatial domain $-5 \le x \le 5$ to the expectations and standard deviation according to* (48)–(50) *with $N = 4$. Again, as it happened in example , one reaches the temperature steady state in the early times.*

Example 4 *In this closing example, we consider problem* (1)–(2) *with uncertainty in both the initial condition and the diffusion coefficient. We will assume that A is a gaussian r.v. of mean $\mu = 3$ and standard deviation $\sigma = 0.1$, i.e., $A \sim Normal(3, 0.1)$ truncated at the interval $[2.5\,,\,3.5]$. It satisfies conditions* (35) *and* (41). *We will assume that the initial condition is given by: $\varphi(x;B) = e^{-Bx^2}$, where B is a uniform r.v., $B \sim Un(0,1)$ which is assumed to be independent of A. As B is a truncated r.v., condition* (41) *holds. In Fig. 4 are plotted approximations to the expectation and the standard deviation on the spatial domain $-10 \le x \le 10$. In this example the approximations for the ex-*



(a)



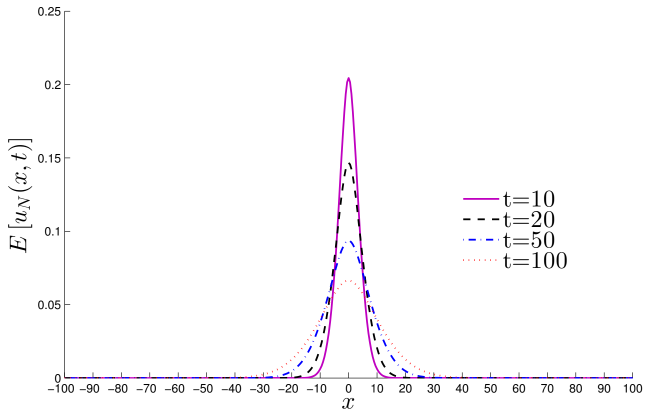
(b)

Figure 1: Approximations for the expectation $\mathrm{E}\left[u_N(x,t)\right]$ (a), and, the standard deviation $\sqrt{\mathrm{Var}\left[u_N(x,t)\right]}$ (b), on the spatial domain $-5 \le x \le 5$ at different time instants t for $N = 4$.

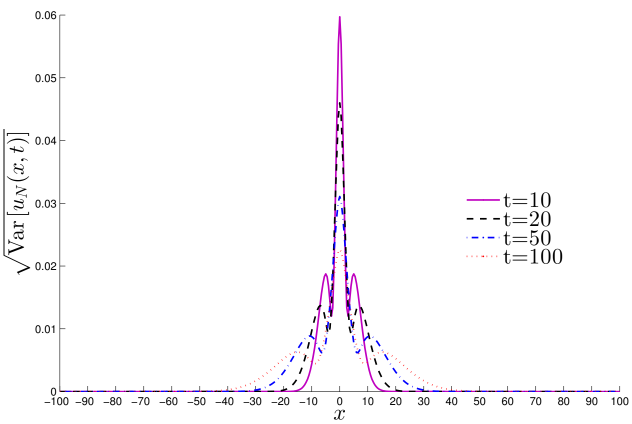
pectation are achieved for different values of N, see Table 1. Similar comments to ones made in the previous example can be noticed.

6 Conclusions

This work answers to the question if dealing with random partial differential equations models in an infinite medium, i.e., unbounded spatial variable, the



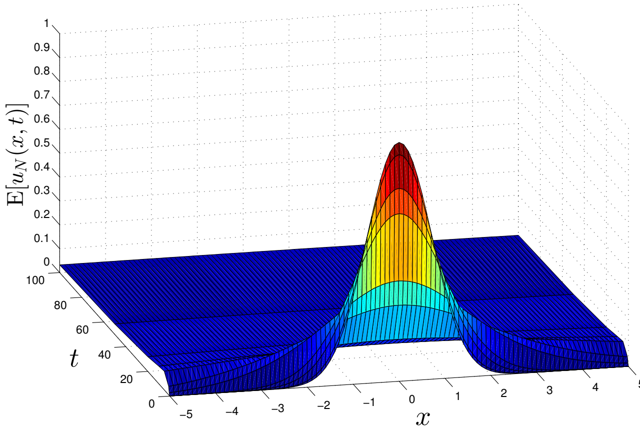
(a)



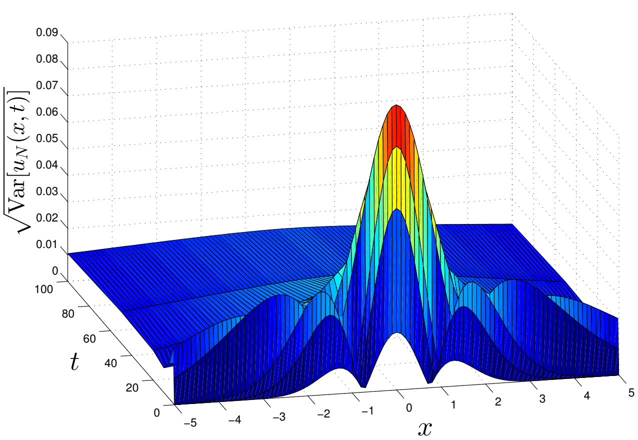
(b)

Figure 2: Approximations for the expectation $\mathrm{E}[u_N(x,t)]$ (a), and, the standard deviation $\sqrt{\mathrm{Var}[u_N(x,t)]}$ (b), on the spatial domain $-100 \le x \le 100$ at different time instants t for $N = 4$.

well-developed integral transform technique used in the deterministic case can be extended to the random framework in a reliable way. The successful result covers the apparent lack of techniques in the random case and initiates the development of an emergent research line to the construction of reliable numerical solutions of random models in unbounded domains. The random diffusion model in an infinite medium is tested in this paper. Finally, it is interesting to point out that the proposed approach could be fruitful to extend other deterministic techniques that have demonstrated to be useful in dealing with continuous models, such as



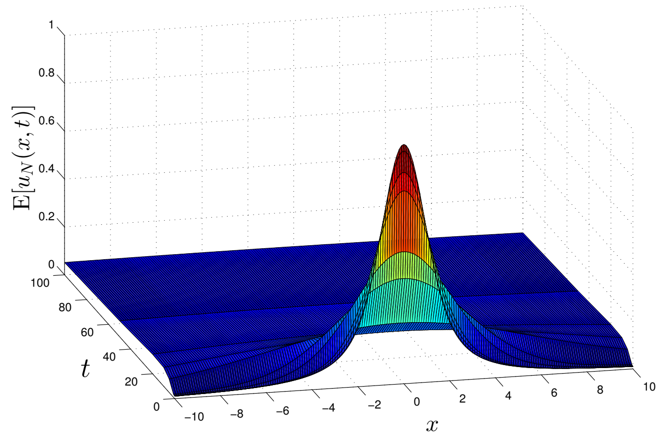
(a)



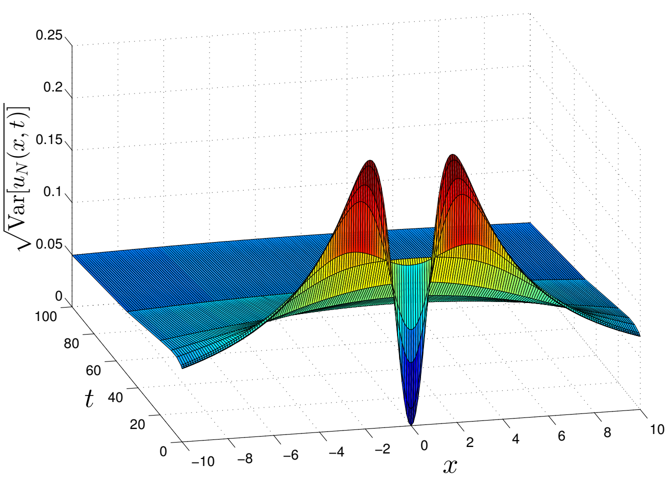
(b)

Figure 3: Approximations for the expectation $\mathrm{E}\left[u_N(x,t)\right]$ (a), and, the standard deviation $\sqrt{\mathrm{Var}\left[u_N(x,t)\right]}$ (b), on the spatial domain $-5 \leq x \leq 5$ and on the time domain $0 \leq t \leq 100$ for $N = 4$.

the Laplace transform method, see [3, 4, 5]. However, it is important to remark that the extension to the random framework requires previously the proof of the operational properties of the transform method used.



(a)



(b)

Figure 4: Approximations for the expectation $\mathrm{E}\left[u_N(x,t)\right]$ (a), and, the standard deviation $\sqrt{\mathrm{Var}\left[u_N(x,t)\right]}$ (b), on the spatial domain $-10 \leq x \leq 10$ and on the time domain $0 \leq t \leq 100$ for different values of N listed at the Table 1.

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$\tilde{t}$	N	$\|\mathrm{E}[u_{N+1}(x,\tilde{t})] - \mathrm{E}[u_N(x,\tilde{t})]\|_\infty \leq \mathcal{O}(10^{-8})$
0.01	12	4.44089×10^{-16}
0.05	13	1.41019×10^{-10}
0.1	14	6.60175×10^{-10}
0.5	17	8.46860×10^{-8}
1	20	4.65570×10^{-8}
5	30	7.38228×10^{-8}
10	37	6.86097×10^{-8}
20	45	9.33242×10^{-8}
50	60	9.17950×10^{-8}
100	74	9.95197×10^{-8}

Table 1: Values of N to compute in Example 4 the numerical approximations of $\mathrm{E}\left[u_N(x,\tilde{t})\right]$ and $\sqrt{\mathrm{Var}\left[u_N(x,\tilde{t})\right]}$ given by (48)–(50) at some time instants $\tilde{t}$ using the stopping criterium: $\|\mathrm{E}[u_{N+1}(x,\tilde{t})] - \mathrm{E}[u_N(x,\tilde{t})]\|_\infty \leq \mathcal{O}(10^{-8}), \quad -10 \leq x \leq 10\,.$

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