On completable fuzzy metric spaces

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Abstract

In this paper we construct a non-completable fuzzy metric space in the sense of George and Veeramani which allows to answer an open question related to continuity on the real parameter t. In addition, the constructed space is not strong (non-Archimedean).

Key words: Fuzzy metric space; fuzzy metric completion; strong (non-Archimedean) fuzzy metric space.
MSC: 54A40, 54D35, 54E50

1 Introduction

Kramosil and Michalek [13] gave a notion of fuzzy metric space which could be considered as a reformulation, in the fuzzy context, of the notion of probabilistic metric space due to Menger [14]. Later, George and Veeramani [2,4] introduced and studied a notion of fuzzy metric space which constitutes a modification of the one due to Kramosil and Michalek. From now on, by fuzzy metric we mean a fuzzy metric in the sense of George and Veeramani. Several authors have contributed to the development of this theory, for instance [5,8,11,12,15–18,21]. In particular, it has been proved that the class of

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topological spaces which are fuzzy metrizable agrees with the class of metrizable topological spaces (see [3,8]) and then, some classical theorems on metric completeness and metric (pre)compactness have been adapted to the realm of fuzzy metric spaces [8]. Nevertheless, the theory of fuzzy metric completion is, in this context, very different from the classical theory of metric completion. Indeed, as it is well-known metric and Menger spaces are completable. Further, imitating the Sherwod's proof [20] one can prove that fuzzy metric spaces defined by Kramosil and Michalek are completable (other different proof can be found in [1]). In this sense non-completability is a specific feature of fuzzy metric spaces, since there are fuzzy metric spaces which are not completable [9,10]. The following characterization of completable fuzzy metric spaces was given (in a slightly different way) in [10]:

Theorem 1 *A fuzzy metric space $(X, M, *)$ is completable if and only if for each pair of Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in X the following three conditions are fulfilled:*

(c1) $\lim_n M(a_n, b_n, s) = 1$ *for some* $s > 0$ *implies* $\lim_n M(a_n, b_n, t) = 1$ *for all* $t > 0$.
(c2) $\lim_n M(a_n, b_n, t) > 0$ *for all* $t > 0$.
(c3) The assignment $t \to \lim_n M(a_n, b_n, t)$ *for each* $t > 0$ *is a continuous function on* $]0, \infty[$, *provided with the usual topology of* $\mathbb{R}$.

In [9] and [10] two non-completable fuzzy metric spaces were given in which conditions (c2) and (c1), respectively, are not satisfied. Since then the following is an open question (it was posed formally in [6] Problem 25): Does it exist a fuzzy metric space in which condition (c3) is not satisfied? In this paper we answer in a positive way this question, constructing a fuzzy metric space (Proposition 9) in which (c3) is not satisfied (Example 12). In addition, we also show that this space is an example of a non-strong fuzzy metric space.

2 Preliminaries

Definition 2 *(George and Veeramani [2]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:*

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ *if and only if* $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(GV5) $M(x, y, \_) :]0, \infty[\to]0, 1]$ *is continuous.*

Definition 3 *A fuzzy metric space $(X, M, *)$ is said to be strong (non-Archimedean) if for all $x, y, z \in X$ and all $t > 0$ satisfies*

$$M(x, z, t) \geq M(x, y, t) * M(y, z, t).$$

George and Veeramani proved in [2] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, 0 < \epsilon < 1, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X$, $\epsilon \in]0, 1[$ and $t > 0$.

Let (X, d) be a metric space and let M_d a function on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M_d, \cdot)$ is a fuzzy metric space, [2], and M_d is called the *standard fuzzy metric* induced by d. The topology τ_{M_d} coincides with the topology on X deduced from d.

Definition 4 *(George and Veeramani [2]), Schweizer and Sklar [19]). A sequence $\{x_n\}$ in a fuzzy metric space (X, M) is said to be* M-Cauchy, *or simply* Cauchy, *if for each $\epsilon \in]0, 1[$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. X is called* complete *if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is also said to be complete.*

Definition 5 *(Gregori and Romaguera [9]). Let (X, M) and (Y, N) be two fuzzy metric spaces. A mapping f from X to Y is called an* isometry *if for each $x, y \in X$ and $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$ and, in this case, if f is a bijection, X and Y are called* isometric. *A* fuzzy metric completion *of (X, M) is a complete fuzzy metric space (X^*, M^*) such that (X, M) is isometric to a dense subspace of X^*. X is called* completable *if it admits a fuzzy metric completion.*

Proposition 6 *(Gregori and Romaguera [9]). If a fuzzy metric space has a fuzzy metric completion then it is unique up to isometry.*

From now on $\mathbb{R}^+$ will denote the set of positive real numbers.

3 The Results

Next, we attend to the requirement of [6] Problem 25, constructing a fuzzy metric space $(X, M, *)$ in which for two Cauchy sequences $\{a_n\}$ and $\{b_n\}$ in

X the assignment $f : \mathbb{R}^+ \to]0,1]$ given by $f(t) = \lim_n M(a_n, b_n, t)$ for all $t > 0$ is a non-continuous function on $\mathbb{R}^+$, endowed with the usual topology of $\mathbb{R}$ restricted to $\mathbb{R}^+$.

We start with the following lemma.

Lemma 7 *Let $A, B, C, a, b, c \in \mathbb{R}^+$ and $u, v, w \in]0,1[$ such that $A \geq a$, $B \geq b$, $C \geq c$, and $A \geq B \cdot C$, $a \geq b \cdot c$ and $u \geq m = \max\{v, w\}$. Then*

$$Au + a(1-u) \geq (Bv + b(1-v)) \cdot (Cw + c(1-w)). \tag{1}$$

Proof. The following expressions are satisfied:

$$Au + a(1-u) \geq Am + a(1-m). \tag{2}$$

(Indeed, $Au + a(1-u) - Am - a(1-m) = (A-a)(u-m) \geq 0$).

$$Am + a(1-m) \geq (Bm + b(1-m)) \cdot (Cm + c(1-m)). \tag{3}$$

(Indeed, $Am + a(1-m) \geq BCm + bc(1-m) - (B-b)(C-c)m(1-m) = (Bm + b(1-m)) \cdot (Cm + c(1-m))$).

$$Bv + b(1-v) \leq Bm + b(1-m). \tag{4}$$

$$Cw + c(1-w) \leq Cm + c(1-m). \tag{5}$$

(Indeed, $Bm + b(1-m) - Bv - b(1-v) = (B-b)(m-v) \geq 0$. The proof of (5) is similar).

Now, using expressions (4), (5), (3) and (2), successively, we have

$$(Bv + b(1-v)) \cdot (Cw + c(1-w)) \leq (Bm + b(1-m)) \cdot (Cm + c(1-m)) \leq$$

$$Am + a(1-m) \leq Au + a(1-u).$$

Lemma 8 *Let d be the usual metric on $\mathbb{R}$ and consider on $]0,1]$ the standard fuzzy metric M_d induced by d. Then*

$$M_d(x, z, t+s) \geq M_d(x, y, t) \cdot M_d(y, z, 2s)$$

for all $x, y, z \in]0,1]$, $d(y,z) < s \leq 1$ and $0 < t \leq d(x,y)$.

Proof. Let $x, y, z \in]0,1]$, $d(y,z) < s \le 1$ and $0 < t \le d(x,y)$. We have

$(t+s)(t+d(x,y))(2s+d(y,z)) = (t+s)(2st+td(y,z)+2sd(x,y)+d(x,y)d(y,z)) \ge$

$(t+s)(2st+2sd(x,y)+2td(y,z)) \ge 2ts(t+s+d(x,y)+d(y,z)) \ge 2ts(t+s+d(x,z))$.
So,

$M_d(x,z,t+s) = \frac{t+s}{t+s+d(x,z)} \ge \frac{t}{t+d(x,y)} \cdot \frac{2s}{2s+d(y,z)} = M_d(x,y,t) \cdot M_d(y,z,2s)$.

Proposition 9 *Let d be the usual metric on $\mathbb{R}$ restricted to $]0,1]$ and consider the standard fuzzy metric M_d induced by d. We define on $]0,1]\times]0,1]\times]0,\infty[$ the function*

$$M(x,y,t) = \begin{cases} M_d(x,y,t), & 0 < t \le d(x,y) \\ M_d(x,y,2t) \cdot \frac{t-d(x,y)}{1-d(x,y)} + M_d(x,y,t) \cdot \frac{1-t}{1-d(x,y)}, & d(x,y) < t \le 1 \\ M_d(x,y,2t), & t > 1 \end{cases}$$

Then $(]0,1], M, \cdot)$ is a fuzzy metric space.

Proof. Before starting the proof and tacking into account that

$$\frac{t-d(x,y)}{1-d(x,y)} + \frac{1-t}{1-d(x,y)} = 1$$

for all $t > 0$, we notice that the following inequalities are satisfied:

$$M_d(x,y,2t) \ge M_d(x,y,2t) \cdot \frac{t-d(x,y)}{1-d(x,y)} + M_d(x,y,t) \cdot \frac{1-t}{1-d(x,y)} \ge M_d(x,y,t) \tag{6}$$

for all $x, y \in]0,1]$ and for all $d(x,y) < t \le 1$.

Clearly, M satisfies $(GV1)$ and $(GV3)$.

It is left to the reader to verify that M satisfies $(GV2)$ and $(GV5)$.

Now, we will see that M satisfies the triangle inequality

$$M(x,z,t+s) \ge M(x,y,t) \cdot M(y,z,s)$$

for all $x, y, z \in]0,1]$ and $s, t > 0$.

We distinguish three possibilities.

(a) Suppose $0 < t + s \leq d(x, z)$.

In this case $M(x, z, t + s) = M_d(x, z, t + s)$.

Under this possibility we can consider the following cases.

(a.1) Suppose $0 < t \leq d(x, y)$ and $0 < s \leq d(y, z)$.

In this case $M(x, y, t) = M_d(x, y, t)$ and $M(y, z, s) = M_d(y, z, s)$.

Since

$$M_d(x, z, t + s) \geq M_d(x, y, t) \cdot M_d(y, z, s)$$

we have

$$M(x, z, t + s) \geq M(x, y, t) \cdot M(y, z, s).$$

(a.2) Suppose $0 < t \leq d(x, y)$ and $d(y, z) < s \leq 1$.

In this case $M(x, y, t) = M_d(x, y, t)$ and
$M(y, z, s) = M_d(y, z, 2s) \cdot \frac{s-d(y,z)}{1-d(y,z)} + M_d(y, z, s) \cdot \frac{1-s}{1-d(y,z)}$.

By Lemma 8 we have that $M_d(x, z, t+s) \geq M_d(x, y, t) \cdot M_d(y, z, 2s)$. Thus, by (6) we have that

$$M(x, z, t + s) \geq M(x, y, t) \cdot M(y, z, s).$$

(The case $d(x, y) < t \leq 1$ and $0 < s \leq d(y, z)$ is proved in a similar way.)

(b) Suppose now that $d(x, z) < t + s \leq 1$.

In this case
$M(x, z, t + s) = M_d(x, z, 2(t + s)) \cdot \frac{t+s-d(x,z)}{1-d(x,z)} + M_d(x, z, t + s) \cdot \frac{1-(t+s)}{1-d(x,z)}$.

Under this possibility we can consider the following cases.

(b.1) Suppose $0 < t \leq d(x, y)$ and $0 < s \leq d(y, z)$. In this case $M(x, y, t) =$ $M_d(x, y, t)$ and $M(y, z, s) = M_d(y, z, s)$. By (6) we have that

$$M(x, z, t + s) \geq M_d(x, z, t + s) \geq M_d(x, y, t) \cdot M_d(y, z, s)$$

and so

$$M(x, z, t + s) \geq M(x, y, t) \cdot M(y, z, s).$$

(b.2) Suppose $0 < t \leq d(x, y)$ and $d(y, z) < s \leq 1$.

In this case $M(x,y,t) = M_d(x,y,t)$ and $M(y,z,s) = M_d(y,z,2s) \cdot \frac{s-d(y,z)}{1-d(y,z)} + M_d(y,z,s) \cdot \frac{1-s}{1-d(y,z)}$.

By (6) and Lemma 8 we have

$$M(x,z,t+s) \geq M_d(x,z,t+s) \geq M_d(x,y,t) \cdot M_d(y,z,2s)$$

and so

$$M(x,z,t+s) \geq M(x,y,t) \cdot M(y,z,s).$$

(The case $d(x,y) < t \leq 1$ and $0 < s \leq d(y,z)$ is proved in a similar way.)

(b.3) Suppose $d(x,y) < t \leq 1$ and $d(y,z) < s \leq 1$.

In this case $M(x,y,t) = M_d(x,y,2t) \cdot \frac{t-d(x,y)}{1-d(x,y)} + M_d(x,y,t) \cdot \frac{1-t}{1-d(x,y)}$ and $M(y,z,s) = M_d(y,z,2s) \cdot \frac{s-d(y,z)}{1-d(y,z)} + M_d(y,z,s) \cdot \frac{1-s}{1-d(y,z)}$.

Now, it is easy to verify that

$$\frac{t+s-d(x,z)}{1-d(x,z)} \geq \max\left\{\frac{t-d(x,y)}{1-d(x,y)}, \frac{s-d(y,z)}{1-d(y,z)}\right\}. \tag{7}$$

Put $u = \frac{t+s-d(x,z)}{1-d(x,z)}$, $v = \frac{t-d(x,y)}{1-d(x,y)}$, $w = \frac{s-d(y,z)}{1-d(y,z)}$, $A = M_d(x,z,2(t+s))$, $a = M_d(x,z,t+s)$, $B = M_d(x,y,2t)$, $b = M_d(x,y,t)$, $C = M_d(y,z,2s)$ and $c = M_d(y,z,s)$.

Obviously $u,v,w \in]0,1[$ and $A,B,C,a,b,c \in \mathbb{R}^+$. Now, by (7) and since $(M_d,\cdot)$ is a fuzzy metric on $\mathbb{R}$ then u,v,w,A,B,C,a,b,c fulfil the conditions of Lemma 1. Then

$$M(x,z,t+s) = Au + a(1-u) \geq (Bv + b(1-v)) \cdot (Cw + c(1-w))$$

and so

$$M(x,z,t+s) \geq M(x,y,t) \cdot M(y,z,s).$$

(c) Suppose $t+s > 1$.

In this case $M(x,z,t+s) = M_d(x,z,2(t+s))$.

Clearly, for all $x,y,z \in]0,1]$ and for all $s,t > 0$ we have that

$$M(x,y,t) \leq M_d(x,y,2t) \text{ and } M(y,z,s) \leq M_d(y,z,2s).$$

Since $M_d(x,z,2(t+s)) \geq M_d(x,y,2t) \cdot M_d(y,z,2s)$ for each $t,s > 0$ then

$$M(x,z,t+s) \geq M(x,y,t) \cdot M(y,z,s) \text{ for all } t,s > 0$$

Therefore, M satisfies the triangle inequality and hence $(]0,1], M, \cdot)$ is a fuzzy metric space.

Proposition 10 *The sequence $\{a_n\}$, where $a_n = \frac{1}{n}$ for all $n = 1, 2, \ldots$, is a Cauchy sequence in $(]0,1], M, \cdot)$.*

Proof. Fix $t > 0$. We can find $n_0 \in \mathbb{N}$ such that $\left|\frac{1}{n} - \frac{1}{m}\right| < t$ for each $m, n \geq n_0$. Then for $m, n \geq n_0$ we have

$$M(a_n, a_m, t) = \begin{cases} \frac{2t}{2t+\left|\frac{1}{n}-\frac{1}{m}\right|} \cdot \frac{t-\left|\frac{1}{n}-\frac{1}{m}\right|}{1-\left|\frac{1}{n}-\frac{1}{m}\right|} + \frac{t}{t+\left|\frac{1}{n}-\frac{1}{m}\right|} \cdot \frac{1-t}{1-\left|\frac{1}{n}-\frac{1}{m}\right|}, & 0 < t \leq 1 \\ \frac{2t}{2t+\left|\frac{1}{n}-\frac{1}{m}\right|}, & t > 1 \end{cases}$$

Hence, if $0 < t \leq 1$ we have $\lim_{n,m} M(a_n, a_m, t) = \frac{2t}{2t} \cdot t + \frac{t}{t} \cdot (1-t) = 1$, and if $t > 1$ we have $\lim_{n,m} M(a_n, a_m, t) = \frac{2t}{2t} = 1$

Then $\lim_{n,m} M(a_n, a_m, t) = 1$ for all $t > 0$. So $\{a_n\}$ is a Cauchy sequence in $(]0,1], M, \cdot)$.

Remark 11 *It is easy to see that $\tau_M \succ \tau_{M_d}$ and then τ_M is finer than the usual topology of $\mathbb{R}$. Then, the sequence $\{\frac{1}{n}\}$ only could converges to 0 in τ_M, but $0 \notin]0,1]$ and, in consequence, $]0,1]$ is not complete.*

Example 12 *Let $(]0,1], M, \cdot)$ the above fuzzy metric space. Consider the Cauchy sequences $\{a_n\}$ and $\{b_n\}$ where $a_n = \frac{1}{n}$ and $b_n = 1$, for $n = 1, 2, \ldots$. We will see that the assignment $t \to \lim_n M(a_n, b_n, t)$ is a well-defined non-continuous function on $]0, \infty[$, endowed with the usual topology of $\mathbb{R}$.*

Take $t \in]0,1[$. Then there exists $n_0 \in \mathbb{N}$ such that $\left|1 - \frac{1}{n}\right| > t$ for each $n \geq n_0$. Hence for each $n \geq n_0$ we have that $M(a_n, b_n, t) = \frac{t}{t+\left|1-\frac{1}{n}\right|}$ and so

$$\lim_n M(a_n, b_n, t) = \frac{t}{t+1}.$$

If $t = 1$, then $t > \left|1 - \frac{1}{n}\right|$ for all $n \in \mathbb{N}$, and so $M(a_n, b_n, t) = \frac{2}{2+\left|1-\frac{1}{n}\right|} \cdot \frac{1-\left|1-\frac{1}{n}\right|}{1-\left|1-\frac{1}{n}\right|} + \frac{1}{1+\left|1-\frac{1}{n}\right|} \cdot \frac{1-1}{1-\left|1-\frac{1}{n}\right|}$. Therefore

$$\lim_n M(a_n, b_n, 1) = \frac{2}{3}$$

And finally, take $t > 1$. Then we have that $M(a_n, b_n, t) = \frac{2t}{2t+\left|1-\frac{1}{n}\right|}$ and so

$$\lim_n M(a_n, b_n, t) = \frac{2t}{2t+1}.$$

Therefore, we can consider the function $f : \mathbb{R}^+ \to]0, 1]$ defined by

$$f(t) = \lim_n M(a_n, b_n, t)$$

for each $t > 0$. Hence this function is given by

$$f(t) = \begin{cases} \frac{t}{t+1}, & 0 < t < 1 \\ \frac{2t}{2t+1}, & t \geq 1 \end{cases}$$

As one can see f is not continuous at $t = 1$.

Remark 13 *Since M does not satisfy (c3), then by Theorem 1 the fuzzy metric space $(]0, 1], M, \cdot)$ is not completable.*

Remark 14 *The fuzzy metric space of Example 12 is not strong. Indeed, if we take $x = 1, y = \frac{1}{2}, z = \frac{9}{20} \in]0, 1]$ and $t = \frac{11}{20} > 0$, after a tedious computation one can verify that $M(x, z, t) < M(x, y, t) \cdot M(y, z, t)$.*

4 Conclusions

In this paper we have answered in affirmative way Problem 25 posed in [6] by constructing a particular non-completable fuzzy metric space which, in addition, is not strong. As a consequence, it arises the question if there exists a strong fuzzy metric space fulfilling the requirements of the mentioned problem, or in other words, if it is possible to characterize completable strong fuzzy metric spaces by conditions (c1) and (c2) of Theorem 1.

On the other hand, the way of constructing Example 12 can illustrate the reader for obtaining non-strong fuzzy metric spaces. This is interesting because in the literature many results are given for strong (non-Archimedean) fuzzy metric spaces, specially in fixed point theory. Unfortunately, only a few examples of non-strong fuzzy metric spaces are found in the literature ([7,11]) and so in many cases the necessity of being strong, for obtaining the chosen result, is not justified.

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