The final publication is available at

http://dx.doi.org/10.1016/j.nonrwa.2012.07.015

Additional Information

Solving a model for the evolution of smoking habit in Spain with homotopy analysis method

F. Guerrero [a,*], F.J. Santonja [b], R.J. Villanueva [c]

[a] *Departamento de Matemática Aplicada, Universidad de Valencia, Av. Dr. Moliner, 50; 46100-Burjassot-Valencia, Spain*
[b] *Departamento de Estadística e Investigación Operativa, Universidad de Valencia, Spain*
[c] *Instituto de Matemática Multidisciplinar, Universidad Politecnica de Valencia, Spain*

ARTICLE INFO

ABSTRACT

We obtain an approximated analytical solution for a dynamic model for the prevalence of the smoking habit in a constant population but with equal and different from zero birth and death rates. This model has been successfully used to explain the evolution of the smoking habit in Spain. By means of the Homotopy Analysis Method, we obtain an analytic expression in powers of time t which reproduces the correct solution for a certain range of time. To enlarge the domain of convergence we have applied the so-called optimal convergence-control parameter technique and the homotopy-Padé technique. We present and discuss graphical results for our solutions.

1. Introduction

Epidemic models have been widely used to study epidemiological processes such as the spread of infectious diseases. This kind of model has also been used for the spread of social habits, such as the smoking habit [1], cocaine consumption [2], alcohol consumption [3] or obesity epidemics [4].

These models consist of a system of nonlinear ordinary differential equations that can be easily solved numerically by any standard numerical method. However, these integration algorithms give approximate solutions only at discrete points and they may also give rise to numerical instabilities such as oscillations, solutions that do not correspond to the solution of the original system of nonlinear ODEs, false equilibrium states, etc. [5].

To avoid this, we are interested in obtaining a continuous solution and this can be done by means of the Homotopy Analysis Method (HAM) initially developed by Liao [6,7].

In the past, HAM has been successfully used to solve many problems in science and engineering [8–15], and also in epidemic models such as SIR [16] and SIS models [17].

The main objective of this paper is to use HAM to obtain an analytical solution for the model for the spread of the smoking habit in Spain presented in [1]. This model has been used successfully to explain the evolution of the prevalence of smoking in Spain. According to this, we are using real data for the initial values and for the parameters of the system. We assume a constant population with birth rate and death rate equal and different from zero. This fact makes the total number of individuals constant, but those individuals are continuously renewed.

We obtain an analytic expression in powers of time t which reproduces the correct solution for a certain range of time. The HAM provides a family of solutions depending on an auxiliary parameter h. The value of this parameter controls the

---

convergence region of the solution. To determine the best value of h we have used the so-called optimal convergence-control parameter technique [10,11]. Finally, to increase further the convergence domain of the analytical approximation we have applied the homotopy-Padé technique [6] with good results.

2. The mathematical model

The following model has been used successfully to predict the evolution of the prevalence of smoking in Spain and to evaluate the impact of the Spanish smoke-free law of 2006 [1].

We assume the population consists of four types of individuals, whose proportions are denoted by n (non-smokers), s (normal smokers), c (excessive smokers) and e (ex-smokers). All of them are functions of time.

$n(t)$ is the proportion of the total population who has never smoked, $s(t)$ is the proportion of people who smoke less than 20 cigarettes per day, $c(t)$ is the proportion of individuals who smoke more than 20 cigarettes per day and $e(t)$ is the proportion of ex-smokers.

The dynamics of the smokers in Spain is given by the following system of differential equations (see [1] for details)

$$n'(t) = \mu(1 - n(t)) - \beta n(t)(s(t) + c(t)), \tag{1}$$

$$s'(t) = \beta n(t)(s(t) + c(t)) + \rho e(t) + \alpha c(t) - (\gamma + \lambda + \mu)s(t), \tag{2}$$

$$c'(t) = \gamma s(t) - (\alpha + \delta + \mu)c(t), \tag{3}$$

$$e'(t) = \lambda s(t) + \delta c(t) - (\rho + \mu)e(t). \tag{4}$$

The parameters of the model assumed constant are:

- μ, birth rate in Spain.
- β, transmission rate due to social pressure to adopt smoking habit.
- ρ, rate at which ex-smokers return to smoking.
- α, rate at which an excessive smoker becomes a normal smoker by decreasing the number of cigarettes per day.
- γ, rate at which normal smokers become excessive smokers by increasing the number of cigarettes per day.
- λ, rate at which normal smokers stop smoking.
- δ, rate at which excessive smokers stop smoking.

Since the constant population has been normalized to unity, our variables satisfy that:

$$n(t) + s(t) + c(t) + e(t) = 1. \tag{5}$$

3. The Homotopy Analysis Method (HAM)

In this section the basic ideas of the HAM [6] are shown.
Consider

$$N[y(t)] = 0, \tag{6}$$

where N is any operator, $y(t)$ is the unknown solution of the system of differential equations that we want to solve and t is the independent variable. We denote $y_0(t)$ the initial guess of the exact solution. $h \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, and L an auxiliary operator that satisfies the property $L[y(t)] = 0$ when $y(t) = 0$.

Using $0 < q < 1$ as a parameter we construct the homotopy

$$(1 - q)L[\phi(t, q) - y_0(t)] - qhH(t)N[\phi(t, q)] = \hat{H}[\phi(t, q), y_0(t), H(t), h, q]. \tag{7}$$

Setting the right hand side of (7) equal to zero, we obtain the zero-order deformation equation

$$(1 - q)L[\phi(t, q) - y_0(t)] = qhH(t)N[\phi(t, q)]. \tag{8}$$

We can see that for $q = 0$ the zero-order deformation equation becomes

$$\phi(t, 0) = y_0(t), \tag{9}$$

and when $q = 1$, taking into account that $h \neq 0$ and $H(t) \neq 0$, the zero-order deformation equation is

$$\phi(t, 1) = y(t). \tag{10}$$

Then, as the parameter q increases from 0 to 1, the function $\phi(t, q)$ changes continuously from the initial $y_0(t)$ to the exact solution $y(t)$. This type of continuous variation is called deformation in homotopy.

We expand $\phi(t, q)$ by Taylor's theorem in a power series of q

$$\phi(t, q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t)q^m \tag{11}$$

where

$$y_m(t) = \frac{1}{m!} \frac{\partial^m \phi(t, q)}{\partial q^m} \bigg|_{q=0}. \tag{12}$$

Assuming that $y_0(t)$, the operator L, the parameter h and the function $H(t)$ are chosen such that

1. $\phi(t, q)$ exists for all $0 \leq q \leq 1$,
2. $y_m(t)$ exists for $m = 1, 2, \ldots$, and
3. the power series (11) of $\phi(t, q)$ is convergent for $q = 1$,

we obtain the solution series

$$\phi(t, 1) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{13}$$

Differentiating the zero-order deformation equation (8) m times with respective to q, dividing by $m!$ and setting $q = 0$, we get the mth-order deformation equation

$$L[y_m(t) - \chi_m y_{m-1}(t)] = hH(t)\mathcal{R}_m(y_{m-1}(t)) \tag{14}$$

where

$$\mathcal{R}_m(y_{m-1}(t)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(t, q)]}{\partial q^{m-1}} \tag{15}$$

with $\chi_m = 0$ for $m \leq 1$ and $\chi_m = 1$ for $m > 1$.

Eq. (15) allows us to obtain $y_m(t)$ from $y_{m-1}(t)$. This means that solving Eq. (15) we are going to obtain $y_1(t), y_2(t), \ldots$ and so on.

4. Solution of the smoking model by HAM

To obtain the analytical approximation of the model of Eqs. (1)–(4), we use HAM has shown in [6,7].

First, we take the initial approximations of $n(t)$, $s(t)$, $c(t)$ and $e(t)$ as constant functions.

$$n_0(t) = n_0, \qquad s_0(t) = s_0, \qquad c_0(t) = c_0, \qquad e_0(t) = e_0. \tag{16}$$

The HAM is based on a kind of continuous mappings

$$n(t) \rightarrow \phi_1(t, q), \qquad s(t) \rightarrow \phi_2(t, q), \qquad c(t) \rightarrow \phi_3(t, q), \qquad e(t) \rightarrow \phi_4(t, q), \tag{17}$$

such that, as q increases from 0 to 1, $\phi_i(t, q)$ moves from the initial approximation to the exact solution.

Following [16], the auxiliary linear operators are defined by

$$L_i[\phi_i(t, q)] = \frac{\partial \phi_i(t, q)}{\partial t}, \quad i = 1, 2, 3, \ldots, \tag{18}$$

satisfying that,

$$L_i[C_i] = 0 \tag{19}$$

for the integral constants C_i.

The non-linear operators N_i are defined according to Eqs. (1)–(4),

$$N_1[\phi_i(t, q)] = \frac{\partial \phi_i(t, q)}{\partial t} - \mu(1 - \phi_i(t, q)) + \beta \phi_i(t, q)(\phi_2(t, q) + \phi_3(t, q)), \tag{20}$$

$$N_2[\phi_i(t, q)] = \frac{\partial \phi_i(t, q)}{\partial t} - \beta \phi_1(t, q)(\phi_i(t, q) + \phi_3(t, q)) - \rho \phi_4(t, q) - \alpha \phi_3(t, q) + (\gamma + \lambda + \mu)\phi_i(t, q), \tag{21}$$

$$N_3[\phi_i(t, q)] = \frac{\partial \phi_i(t, q)}{\partial t} - \gamma \phi_2(t, q) + (\alpha + \delta + \mu)\phi_i(t, q), \tag{22}$$

$$N_4[\phi_i(t, q)] = \frac{\partial \phi_i(t, q)}{\partial t} - \lambda \phi_2(t, q) - \delta \phi_3(t, q) + (\rho + \mu)\phi_i(t, q). \tag{23}$$

According to zero-order deformation equation (8), a family of equations can be constructed,

$$(1 - q)L[\phi_1(t, q) - n_0(t)] = qh_1 H_1(t)N_1[\phi_1(t, q)], \tag{24}$$

$$(1 - q)L[\phi_2(t, q) - s_0(t)] = qh_2 H_2(t)N_2[\phi_2(t, q)], \tag{25}$$

$$(1 - q)L[\phi_3(t, q) - c_0(t)] = qh_3 H_3(t)N_3[\phi_3(t, q)], \tag{26}$$

$$(1 - q)L[\phi_4(t, q) - e_0(t)] = qh_4 H_4(t)N_4[\phi_4(t, q)], \tag{27}$$

satisfying the initial conditions $\phi_1(0, q) = n_0$, $\phi_2(0, q) = s_0$, $\phi_3(0, q) = c_0$ and $\phi_4(0, q) = e_0$.

Applying Eq. (13) to this case, we can write

$$\phi_1(t, 1) = n_0(t) + \sum_{m=1}^{\infty} n_m(t), \tag{28}$$

$$\phi_2(t, 1) = s_0(t) + \sum_{m=1}^{\infty} s_m(t), \tag{29}$$

$$\phi_3(t, 1) = c_0(t) + \sum_{m=1}^{\infty} c_m(t), \tag{30}$$

$$\phi_4(t, 1) = e_0(t) + \sum_{m=1}^{\infty} e_m(t), \tag{31}$$

and

$$n_m(t) = \left. \frac{1}{m!} \frac{\partial^m \phi_1(t, q)}{\partial q^m} \right|_{q=0} \tag{32}$$

$$s_m(t) = \left. \frac{1}{m!} \frac{\partial^m \phi_2(t, q)}{\partial q^m} \right|_{q=0} \tag{33}$$

$$c_m(t) = \left. \frac{1}{m!} \frac{\partial^m \phi_3(t, q)}{\partial q^m} \right|_{q=0} \tag{34}$$

$$e_m(t) = \left. \frac{1}{m!} \frac{\partial^m \phi_4(t, q)}{\partial q^m} \right|_{q=0}. \tag{35}$$

The mth-order deformation equations (14) for this particular case are:

$$L[n_m(t) - \chi_m n_{m-1}(t)] = h_1 H_1(t) \mathcal{R}_m(n_{m-1}(t)) \tag{36}$$
$$L[s_m(t) - \chi_m s_{m-1}(t)] = h_2 H_2(t) \mathcal{R}_m(s_{m-1}(t)) \tag{37}$$
$$L[c_m(t) - \chi_m c_{m-1}(t)] = h_3 H_3(t) \mathcal{R}_m(c_{m-1}(t)) \tag{38}$$
$$L[e_m(t) - \chi_m e_{m-1}(t)] = h_4 H_4(t) \mathcal{R}_m(e_{m-1}(t)) \tag{39}$$

$n_m(0) = 0, s_m(0) = 0, c_m(0) = 0$ and $e_m(0) = 0$.

Following [6] we set $H_i(t) = 1$, for $i = 1, 2, 3, 4$. We also set $h_1 = h_2 = h_3 = h_4 = h$ so that h will be the convergence-control parameter. Then, the mth-order deformation equations for $m \geq 1$ are:

$$n_m(t) = \chi_m n_{m-1}(t) + h \int_0^t d\tau \left[n'_{m-1}(\tau) - \mu(s_{m-1}(\tau) + c_{m-1}(\tau) + e_{m-1}(\tau)) \right.$$
$$\left. + \beta \sum_{k=0}^{m-1} (s_k(\tau) + c_k(\tau)) n_{m-1-k}(\tau) \right], \tag{40}$$

$$s_m(t) = \chi_m s_{m-1}(t) + h \int_0^t d\tau \left[s'_{m-1}(\tau) - \beta \sum_{k=0}^{m-1} (s_k(\tau) + c_k(\tau)) n_{m-1-k}(\tau) \right.$$
$$\left. - \rho e_{m-1}(\tau) - \alpha c_{m-1}(\tau) + (\gamma + \lambda + \mu) s_{m-1}(\tau) \right], \tag{41}$$

$$c_m(t) = \chi_m c_{m-1}(t) + h \int_0^t d\tau \left[c'_{m-1}(\tau) - \gamma s_{m-1}(\tau) + (\alpha + \delta + \mu) c_{m-1}(\tau) \right], \tag{42}$$

$$e_m(t) = \chi_m e_{m-1}(t) + h \int_0^t d\tau \left[e'_{m-1}(\tau) - \lambda s_{m-1}(\tau) - \delta c_{m-1}(\tau) + (\rho + \mu) e_{m-1}(\tau) \right]. \tag{43}$$

With these formulae we are able to compute the functions $n_m(t), s_m(t), c_m(t)$ and $e_m(t)$.
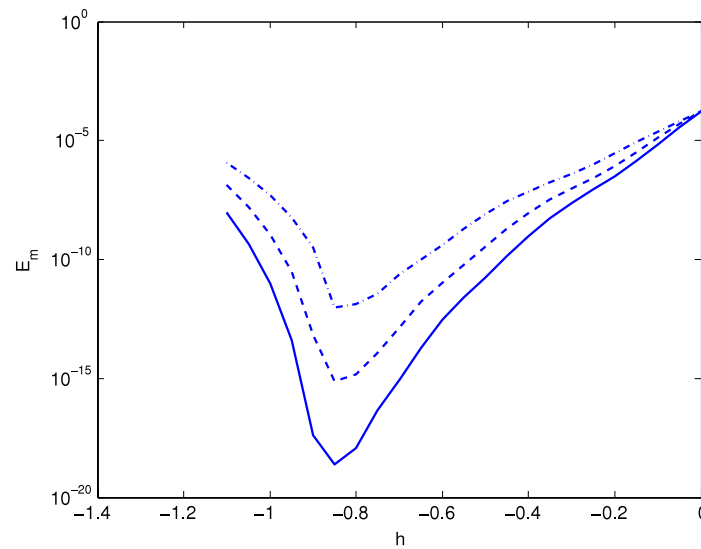
Fig. 1. Averaged residual error E_m versus h. Dash–dotted line for $m = 6$, dashed line for $m = 8$ and solid line for $m = 10$.
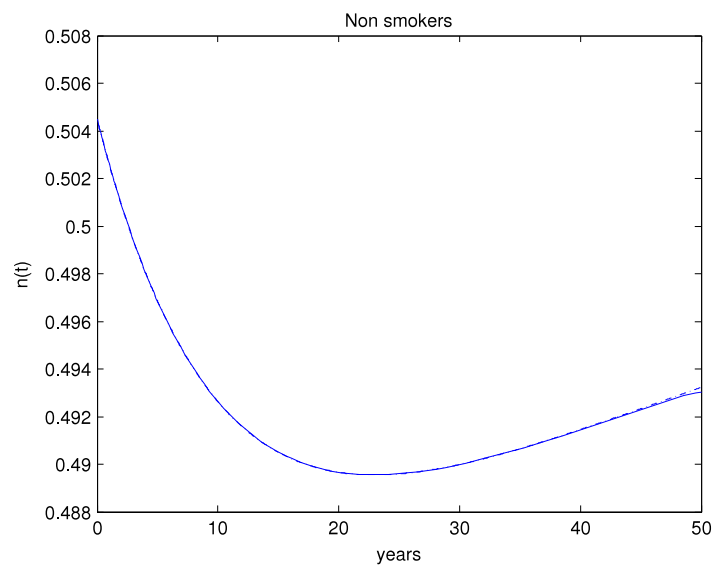

Fig. 2. Solution of HAM with 20 terms for $n(t)$ (solid line) and the *exact* solution (dash–dotted line). Note that one is on the other.

Table 1
Parameters of the model for the evolution
of the prevalence of smoking in Spain [1].

Parameter	Value (year^{-1})
μ	0.01
ρ	0.0425
β	0.0381
α	0.1244
γ	0.1175
λ	0.0498
δ	0.0498

5. Results

Using Eqs. (28)–(31) and Eqs. (40)–(43) an analytical approximation for the solution of the smoking model is obtained.

The values we use for the parameters of the model are shown in Table 1. These values have been estimated in [1] using real data for the Spanish population.
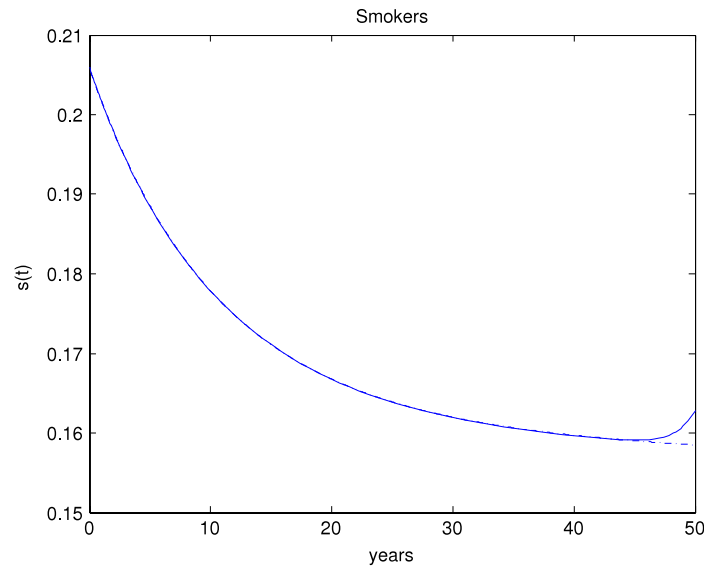
Fig. 3. Solution of HAM with 20 terms for $s(t)$ (solid line) and the *exact* solution (dash–dotted line). Differences appear for $t \geq 45$.
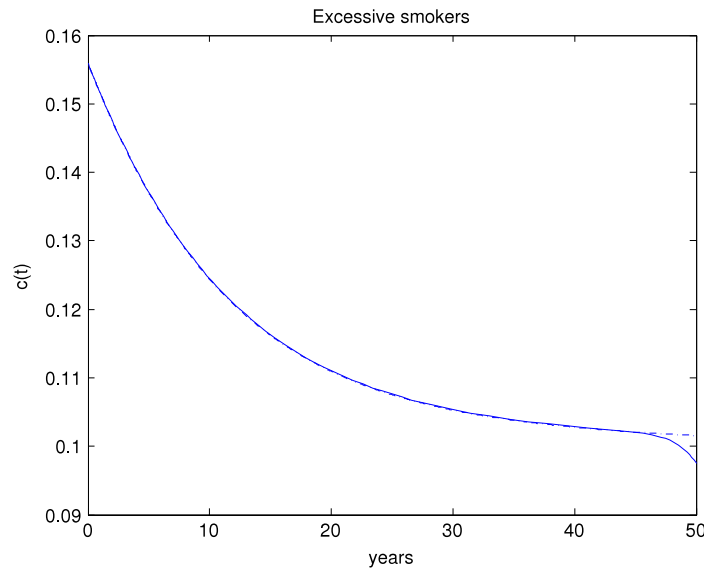

Fig. 4. Solution of HAM with 20 terms for $c(t)$ (solid line) and the *exact* solution (dash–dotted line). Differences appear for $t \geq 45$.

The initial values n_0, s_0, c_0 and e_0 are the real proportions for the subpopulations in Spain at the beginning of the period of time analyzed in [1].

$$n_0 = 0.5045, \qquad s_0 = 0.2059, \qquad c_0 = 0.1559, \qquad e_0 = 0.1337. \tag{44}$$

5.1. Optimal convergence

As we said above, parameter h is called the convergence-control parameter. We have to determine the value of h that provides the largest convergence domain for a given number m of terms in the analytical approximation for the solution.

Following [10], we find out the so-called optimal convergence-control parameter. This technique establishes that the optimal value for h is the one that minimizes the sum of the residual squares of the four Eqs. (1)–(4), given by:

$$E_m = \frac{1}{K} \sum_{j=0}^{K} \left\{ \left[N_1 \left(\sum_{k=0}^{m} n_k(j\Delta t) \right) \right]^2 + \left[N_2 \left(\sum_{k=0}^{m} s_k(j\Delta t) \right) \right]^2 \right.$$

$$\left. + \left[N_3 \left(\sum_{k=0}^{m} c_k(j\Delta t) \right) \right]^2 + \left[N_4 \left(\sum_{k=0}^{m} e_k(j\Delta t) \right) \right]^2 \right\}, \tag{45}$$

where we have taken $K = 20$ and $\Delta t = 1$. E_m is the so-called averaged residual error.
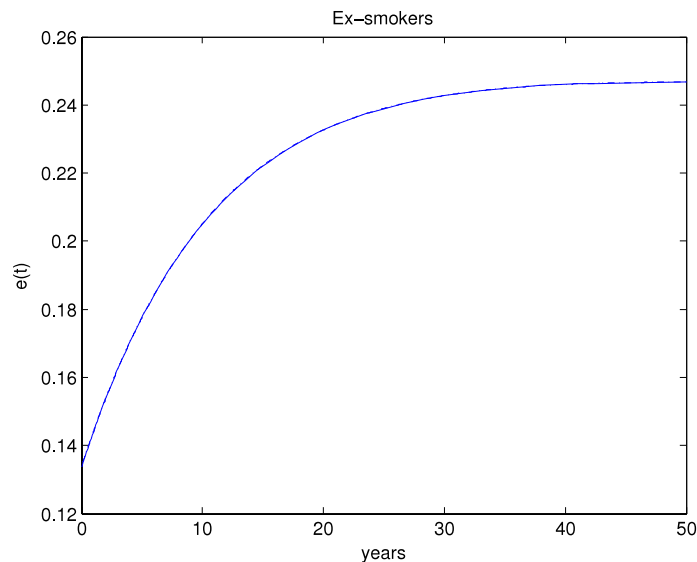
Fig. 5. Solution of HAM with 20 terms for $e(t)$ (solid line) and the *exact* solution (dash–dotted line).
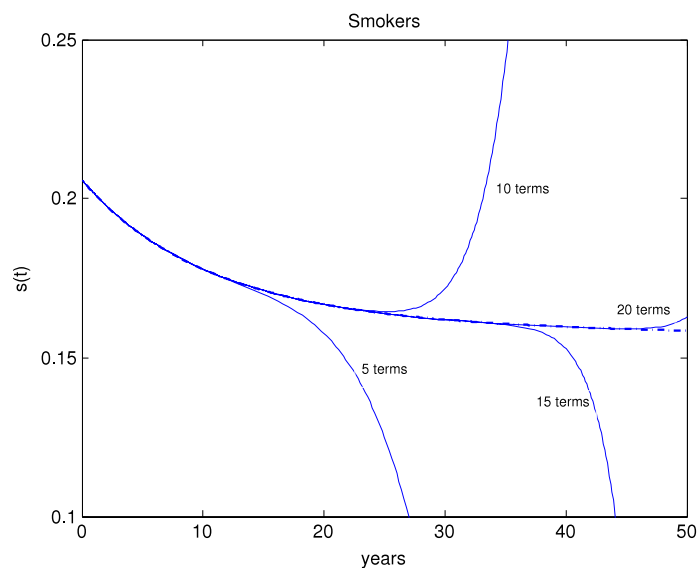

Fig. 6. Solutions of HAM with 5, 10, 15 and 20 terms for $s(t)$ (solid lines) and the *exact* solution (dash–dotted line). Note the improvement as the number of terms increases.

Fig. 1 shows the values of E_m for different values of h and for $m = 6, 8$ and 10. We find out that the sum of residual squares is minimum for h about -0.85.

We present below ten terms approximations for $n(t)$, $s(t)$, $c(t)$ and $e(t)$ obtained for the optimal value $h = -0.85$

$$
\begin{aligned}
n(t)^{(10)} &= 0.5045 - 0.001999t + 0.00011026t^2 - 3.68167 \cdot 10^{-6}t^3 + 9.6534 \cdot 10^{-8}t^4 - 2.2211 \cdot 10^{-9}t^5 \\
&\quad + 4.7959 \cdot 10^{-11}t^6 - 9.8329 \cdot 10^{-13}t^7 + 1.7857 \cdot 10^{-14}t^8 - 2.4224 \cdot 10^{-16}t^9 + 1.7194 \cdot 10^{-18}t^{10} \\
s(t)^{(10)} &= 0.2059 - 0.004496t + 0.0002521t^2 - 1.2627 \cdot 10^{-5}t^3 + 6.5927 \cdot 10^{-7}t^4 - 3.4315 \cdot 10^{-8}t^5 \\
&\quad + 1.6277 \cdot 10^{-9}t^6 - 6.6248 \cdot 10^{-11}t^7 + 2.1460 \cdot 10^{-12}t^8 - 4.7663 \cdot 10^{-14}t^9 + 5.1985 \cdot 10^{-16}t^{10} \\
c(t)^{(10)} &= 0.1559 - 0.0045029t + 0.0001503t^2 + 6.5375 \cdot 10^{-7}t^3 - 4.0130 \cdot 10^{-7}t^4 + 3.0252 \cdot 10^{-8}t^5 \\
&\quad - 1.5877 \cdot 10^{-9}t^6 + 6.6325 \cdot 10^{-11}t^7 - 2.1592 \cdot 10^{-12}t^8 + 4.7935 \cdot 10^{-14}t^9 - 5.2211 \cdot 10^{-16}t^{10} \\
e(t)^{(10)} &= 0.1337 + 0.010998t - 0.0005127t^2 + 1.5655 \cdot 10^{-5}t^3 - 3.5450 \cdot 10^{-7}t^4 + 6.2838 \cdot 10^{-9}t^5 \\
&\quad - 8.7950 \cdot 10^{-11}t^6 + 9.0590 \cdot 10^{-13}t^7 - 4.7166 \cdot 10^{-15}t^8 - 2.9994 \cdot 10^{-17}t^9 + 5.4733 \cdot 10^{-19}t^{10}.
\end{aligned}
\tag{46}
$$

Figs. 2–5 show the approximate solutions with 20 terms for $n(t)$, $s(t)$, $c(t)$ and $e(t)$ compared to the so-called *exact* solution of the smoking model. We see that HAM solutions are correct for a range of about 45 years for the evolution of
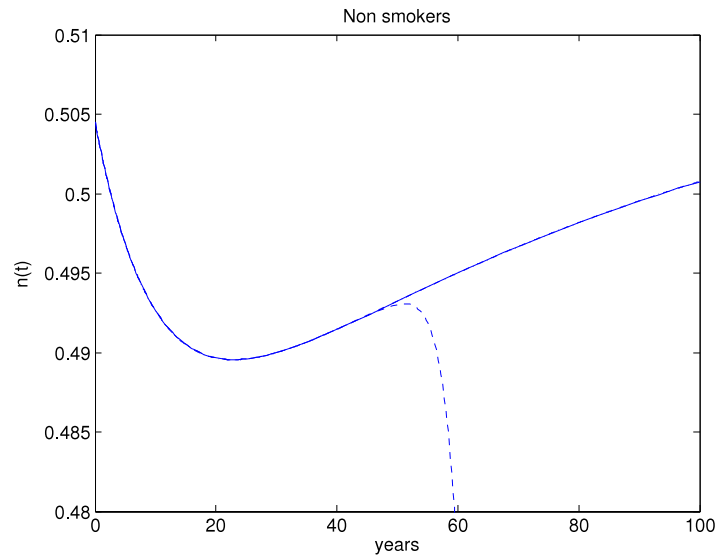
Fig. 7. Comparison between [10, 10] homotopy-Padé approximation (solid line), HAM solution with 20 terms (dashed line) and the *exact* solution (dash–dotted line) for $n(t)$.
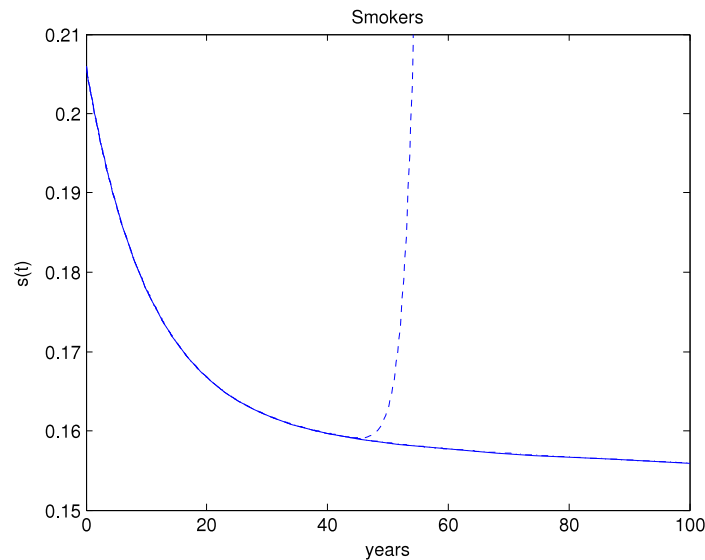

Fig. 8. Comparison between [10, 10] homotopy-Padé approximation (solid line), HAM solution with 20 terms (dashed line) and the *exact* solution (dash–dotted line) for $s(t)$.

the prevalence of smoking. The *exact* solution is obtained by the fourth order Runge–Kutta method with a mesh width of $h = 0.001$ years.

Fig. 6 shows HAM solutions for $s(t)$ with 5, 10, 15 and 20 terms to show the convergence to the exact solution. We see how, as the number of terms increases, the HAM solution is valid for a larger period of time. The HAM solutions for $n(t)$, $c(t)$ and $e(t)$ have similar behaviour. For more about convergence of HAM solutions see [6].

5.2. Homotopy-Padé technique results

The homotopy-Padé technique was developed to accelerate the convergence of the HAM solution [6]. The idea is to apply the traditional Padé technique to Eq. (11) to get the $[m, m]$ Padé-approximant with respect to the parameter q and then set $q = 1$. The result that we obtain is a fraction with one polynomial of order t^m in the numerator and the other polynomial of order t^m in the denominator.

$$[m, m] \rightarrow R_{m,m} = \frac{p_0 + p_1 t + p_2 t^2 + \cdots + p_m t^m}{1 + s_1 t + s_2 t^2 + \cdots + s_m t^m}. \tag{47}$$

Since we have calculated the HAM solution with 20 terms, we can obtain the [10, 10] homotopy-Padé approximation. Figs. 7–10 show the results of the [10, 10] homotopy-Padé approximations using the HAM solution with 20 terms presented in Figs. 2–5.
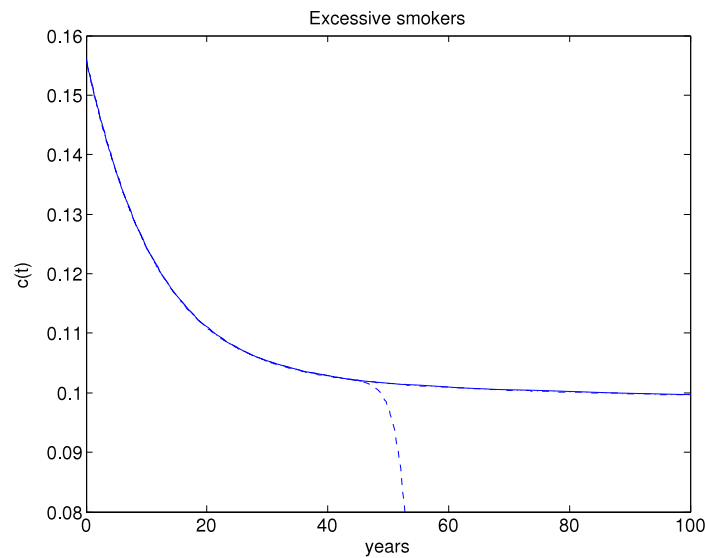
Fig. 9. Comparison between [10, 10] homotopy-Padé approximation (solid line), HAM solution with 20 terms (dashed line) and the *exact* solution (dash–dotted line) for $c(t)$.
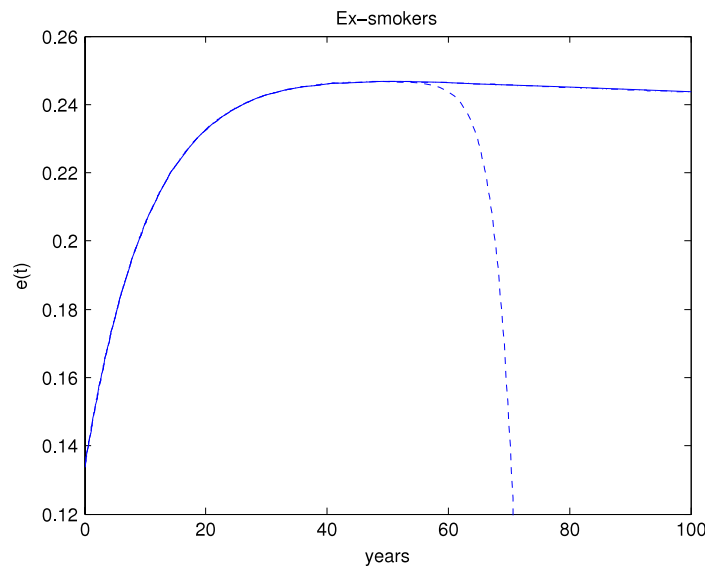

Fig. 10. Comparison between [10, 10] homotopy-Padé approximation (solid line), HAM solution with 20 terms (dashed line) and the *exact* solution (dash–dotted line) for $e(t)$. The Padé approximation is on the *exact* solution.

The convergence domain has been greatly enlarged. The HAM solution with 20 terms has a range of validity of 45 years. However, the [10, 10] homotopy-Padé solution is indistinguishable from the *exact* solution for a range of 100 years. In fact, its range of validity is about 180 years.

6. Conclusions

The analytical time power series that we have obtained for the four subpopulations proves that HAM is a very useful tool to solve this type of mathematical models for the spread of social habits.

Evaluating the optimal value of the control parameter we have obtained an analytical approximation for the solution with 20 terms which is valid for a range of time of more than 45 years. The range of validity of the approximation can be enlarged simply by adding more terms to the series. However, it is more efficient to enlarge the domain of convergence by means of the homotopy-Padé technique. We have been able to compute an analytical approximation which is valid for a period of more than 180 years.

Since the main purpose of this model is to predict the evolution of a social habit, we do not need a very large range of validity since social behaviours are continuously changing. In our smoking model, the parameters are assumed constant, and this fact cannot be valid for a very long period of time. This fact may make even an approximate solution with only five terms useful for practical purposes. Fig. 6 shows that a five term solution is a valid approximation for more than ten years.

References

[1] F. Guerrero, F.J. Santonja, R.J. Villanueva, Analysing the effect of Spanish smoke-free legislation of year 2006: a new method to quantify its impact using a dynamic model, Int. J. Drug Policy 22 (2011) 247–251.

[2] E. Sánchez, R.J. Villanueva, F.J. Santonja, M. Rubio, Predicting cocaine consumption in Spain. A mathematical modeling approach, Drugs Edu. Prevent. Policy 18 (2) (2011) 105–108.

[3] F.J. Santonja, E. Sánchez, M. Rubio, J.L. Morera, Alcohol consumption in Spain and its economic costs: a mathematical modeling approach, Math. Comput. Modelling 52 (2010) 999–1003.

[4] F.J. Santonja, R.J. Villanueva, L. Jódar, G. González, Mathematical modeling of social obesity epidemic in the region of Valencia, Spain, Math. Comput. Model. Dyn. Syst. 16 (1) (2010) 23–34.

[5] J.D. Lambert, Numerical Methods for Ordinary Differential Systems: The Initial Value Problem, John Wiley and Sons, Chichester, UK, 1991.

[6] S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall, CRC Press, Boca Raton, 2003.

[7] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 174 (2004) 499–513.

[8] S.J. Liao, An explicit totally analytic approximation of Blasius viscous flow problems, Internat. J. Non-Linear Mech. 34 (1999) 759–778.

[9] S.J. Liao, A new branch of solutions of boundary-layer flows over an impermeable stretched plate, Int. J. Heat Mass Transfer 48 (2005) 2529–3259.

[10] S.J. Liao, An optimal homotopy-analysis approach for strongly nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simul. 15 (2010) 2003–2016.

[11] S.J. Liao, Homotopy Analysis Method in Nonlinear Differential Equations, Springer & Higher Education Press, Heidelberg, 2012.

[12] M. Ayub, A. Rasheed, T. Hayat, Exact flow of a third grade fluid past a porous plate using homotopy analysis method, Internat. J. Engrg. Sci. 41 (2003) 2091–2103.

[13] T. Hayat, M. Khan, Homotopy solutions for a generalized second-grade fluid past a porous plate, Nonlinear Dynam. 42 (2005) 395–405.

[14] S. Abbasbandy, The application of homotopy analysis method to nonlinear equations arising in heat transfer, Phys. Lett. A 360 (2006) 109–113.

[15] A. Sami, M.S.M. Noorani, I. Hashim, Solving systems of ODEs by homotopy analysis method, Commun. Nonlinear Sci. Numer. Simul. 13 (2008) 2060–2070.

[16] F. Awawdeh, A. Adawi, Z. Mustafa, Solutions of the SIR models of epidemics using HAM, Chaos Solitons Fractals 42 (2009) 3047–3052.

[17] H. Khan, R.N. Mohapatra, K. Vajravelu, S.J. Liao, The explicit series solution of SIR and SIS epidemic models, Appl. Math. Comput. 215 (2009) 653–669.