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Additional Information

Probabilistic solution of random homogeneous linear second-order difference equations

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Abstract

This paper deals with the computation of the first probability density function of the solution of random homogeneous linear second-order difference equations by the Random Variable Transformation method. This approach allows us to generalize the classical solution obtained in the deterministic scenario. Several illustrative examples are provided.

Keywords. Random Variable Transformation method, first probability density function, random homogeneous linear second-order difference equations

1 Motivation

Due not only to the measurements errors often required to deal with problems in many areas like Physics, Chemistry, Engineering, etc., but also the inherent complexity of the phenomena under study, randomness is a key part in modelling. These facts motivate the extension of powerful deterministic tools such as differential and difference equations to the random context. Most of the contributions in this sense focus on continuous models based on both Itô-type stochastic differential equations [11, 1] or random differential equations [13, 7]. Itô-type stochastic differential equations consider uncertainty through the white noise, which is a Gaussian and stationary stochastic process. Whereas, the consideration of randomness in dealing with random differential equations is introduced by assuming that the inputs parameters (coefficients, source term and initial/boundary conditions) can have specific probability distributions including the standard distributions such as exponential, beta and Gaussian. As it also happens in the deterministic scenario, the study of discrete models containing uncertainty in their formulation has been less prolific than continuous models. The contributions have mainly focused on the discretization of Itô-type stochastic differential equations. This approach leads to Autoregressive models (usually referred to as AR-models) [6]. Some interesting contributions in this sense are [12, 14]. Random difference equations constitute the discrete counterpart of random differential equations. Their study is motivated from the natural introduction of uncertainty in models that appear in applied areas [2] and after the discretization of random differential equations [4]. We point out that recently, in [3], authors have obtained the first probability density function to the solution of random linear first-order difference equations taking advantage of the approach to be considered in this paper.

The solution of a random difference equation is a discrete stochastic process (s.p.), say $\{Z_n : n \geq 0\}$. Notice that a remarkable difference with respect to the deterministic scenario is that now the primary goal is not only to compute the solution, but also its main statistical characteristics such as the mean function, $\mathbb{E}[Z_n]$, and the variance function, $\mathbb{V}[Z_n] = \mathbb{E}[(Z_n)^2] - (\mathbb{E}[Z_n])^2$. Though more complicate, the computation of the first probability density function (1-p.d.f.), $f_1(z, n)$, associated to the discrete solution s.p. is more desirable since

from it one can determine not only the two previous statistical moments but also any other moment

$$\mathbb{E}[(Z_n)^k] = \int_{-\infty}^{\infty} z^k f_1(z, n) dz, \quad n, k = 0, 1, 2, \dots,$$

of Z_n , as well as, the probability of specific intervals where the solution s.p. lies. In dealing with the computation of the 1-p.d.f., the Random Variable Transformation (R.V.T.) method is a powerful technique. This method has been used to study relevant continuous models [9, 8, 5, 10, 4]. In this paper we will determine the 1-p.d.f. $f_1(z, n)$ of the solution of the random homogeneous linear second-order difference equation

$$Z_{n+2} + A_1 Z_{n+1} + A_2 Z_n = 0, \quad n = 0, 1, 2, \dots, \quad Z_0 = \Gamma_0, \quad Z_1 = \Gamma_1, \quad (1)$$

by the R.V.T. technique. To provide more generality to our study, we will consider that both initial conditions, $\Gamma_0 = \Gamma_0(\omega)$, $\Gamma_1 = \Gamma_1(\omega)$ and, coefficients, $A_1 = A_1(\omega)$, $A_2 = A_2(\omega)$ are dependent continuous random variables (r.v.'s) defined on a common probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and whose joint p.d.f. $f_{\Gamma_0, \Gamma_1, A_1, A_2}(\gamma_0, \gamma_1, a_1, a_2)$ is known. Without loss of generality, in the following we will denote by

$$\begin{aligned} D_{\Gamma_0} &= \{ \gamma_0 = \Gamma_0(\omega), \omega \in \Omega : \gamma_{0,1} \leq \gamma_0 \leq \gamma_{0,2} \}, & D_{A_1} &= \{ a_1 = A_1(\omega), \omega \in \Omega : a_{1,1} \leq a_1 \leq a_{1,2} \}, \\ D_{\Gamma_1} &= \{ \gamma_1 = \Gamma_1(\omega), \omega \in \Omega : \gamma_{1,1} \leq \gamma_1 \leq \gamma_{1,2} \}, & D_{A_2} &= \{ a_2 = A_2(\omega), \omega \in \Omega : a_{2,1} \leq a_2 \leq a_{2,2} \}, \end{aligned}$$

the domains of the input parameters Γ_0 , Γ_1 , A_1 and A_2 , respectively. We permit that the left (right) endpoint of each interval of the above domains takes the value $-\infty$ ($+\infty$), hence unbounded r.v.'s are allowed. In the sequel, for the sake of clarity sample dependence for r.v.'s denoted by the above ω -notation will be omitted.

This paper is organized as follows. In Section 2 we will determine the 1-p.d.f. $f_1(z, n)$ of the solution s.p. of the initial value problem (i.v.p.) (1) in two pieces distinguishing the real or complex nature of the roots of the associated characteristic equation. In Section, 3 we will give some examples where the theoretical results developed in Section 2 are illustrated. Conclusions are drawn in the closing section.

2 Computing the 1-p.d.f.

This section is devoted to compute the 1-p.d.f. of the solution of the i.v.p. (1) using the R.V.T. technique. There are several versions of this technique, below we state the version that will be applied throughout this paper [13, p.25]:

Theorem 1 (*R.V.T. technique: multi-dimensional version*). *Let $\mathbf{V} = (V_1, \dots, V_m)$ be a random vector of dimension m with joint p.d.f. $f_{\mathbf{V}}(\mathbf{v})$. Let $\mathbf{r} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a one-to-one deterministic map which is assumed to be continuous with respect to each one of its arguments, and with continuous partial derivatives. Then, the joint p.d.f. $f_{\mathbf{W}}(\mathbf{w})$ of the random vector $\mathbf{W} = \mathbf{r}(\mathbf{V})$ is given by*

$$f_{\mathbf{W}}(\mathbf{w}) = f_{\mathbf{V}}(\mathbf{s}(\mathbf{w})) |J_m|, \quad (2)$$

where $\mathbf{s}(\mathbf{w})$ is the inverse transformation of $\mathbf{r}(\mathbf{v})$: $\mathbf{v} = \mathbf{r}^{-1}(\mathbf{w}) = \mathbf{s}(\mathbf{w})$ and J_m is the Jacobian of the transformation, i.e.,

$$J_m = \det \left(\frac{\partial \mathbf{v}}{\partial \mathbf{w}} \right) = \det \begin{pmatrix} \frac{\partial v_1}{\partial w_1} & \dots & \frac{\partial v_m}{\partial w_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_1}{\partial w_m} & \dots & \frac{\partial v_m}{\partial w_m} \end{pmatrix},$$

which is assumed to be different from zero.

As in the deterministic scenario, the general solution Z_n of i.v.p. (1) depends on the real or complex character of the roots, $\alpha_1 = \alpha_1(\omega)$ and $\alpha_2 = \alpha_2(\omega)$, $\omega \in \Omega$, of the random characteristic equation: $\alpha^2 + A_1 \alpha + A_2 = 0$:

$$\alpha_1 = \frac{-A_1 + \sqrt{(A_1)^2 - 4A_2}}{2}, \quad \alpha_2 = \frac{-A_1 - \sqrt{(A_1)^2 - 4A_2}}{2}. \quad (3)$$

Let us introduce the following events \mathcal{E}_i involving the r.v.'s $A_1 = A_1(\omega)$ and $A_2 = A_2(\omega)$ and their associated probabilities p_i , $i = 1, 2, 3$:

$$\begin{aligned} p_1 &= \mathbb{P}[\mathcal{E}_1 = \{\omega \in \Omega : (A_1)^2 - 4A_2 > 0\}], \\ p_2 &= \mathbb{P}[\mathcal{E}_2 = \{\omega \in \Omega : (A_1)^2 - 4A_2 < 0\}], \\ p_3 &= \mathbb{P}[\mathcal{E}_3 = \{\omega \in \Omega : (A_1)^2 - 4A_2 = 0\}]. \end{aligned} \quad (4)$$

The pairs (\mathcal{E}_i, p_i) , $i = 1, 2, 3$ indicate the events and probabilities that the random roots of the associated characteristic equation are real and distinct, complex or real and equals, respectively. Note that $p_3 = 0$ since we are assuming that A_1 and A_2 are continuous r.v.'s. Unlike what happens in the deterministic scenario, once r.v.'s A_1 and A_2 have been chosen, the roots α_1 and α_2 can be both real (and distinct) and complex whenever $0 < p_1 < 1$ and $0 < p_2 < 1$. Note that $p_1 + p_2 = 1$. In this case, the 1-p.d.f. $f_1(z, n)$ of the solution of i.v.p. (1) will be defined in two pieces, $f_{1R}(z, n)$ and $f_{1I}(z, n)$. Each one of these functions is the contribution inferred by the event \mathcal{E}_i , $i = 1, 2$, which happens with probability $p_i \in]0, 1[$, $i = 1, 2$. Both pieces $f_{1R}(z, n)$ and $f_{1I}(z, n)$ will be determined below taking advantage of R.V.T. method. If $p_1 = 1$ and $\mathbb{P}[\{\omega \in \Omega : A_1(\omega) = \hat{a}_1\}] = 1$ or, $p_2 = 1$ and $\mathbb{P}[\{\omega \in \Omega : A_2(\omega) = \hat{a}_2\}] = 1$, i.e., when the p.d.f.'s of A_1 and A_2 are given by the Dirac delta functions $\delta(A_1 - \hat{a}_1)$ and $\delta(A_2 - \hat{a}_2)$, respectively, our random approach becomes the classical results.

2.1 Real and distinct random roots

Let us assume that $p_1 > 0$. Then, the solution of the i.v.p. (1) is

$$Z_n = \frac{\alpha_1(\alpha_2)^n - \alpha_2(\alpha_1)^n}{\alpha_1 - \alpha_2} \Gamma_0 + \frac{(\alpha_1)^n - (\alpha_2)^n}{\alpha_1 - \alpha_2} \Gamma_1,$$

where α_1 and α_2 are given by (3). Now, in order to determine the 1-p.d.f. $f_1(z, n)$ of the i.v.p. (1), we set n and compute the p.d.f. of the r.v. $Z = Z_n$ by applying Theorem 1 with the following identification: $\mathbf{V} = (X_0, X_1, A_1, A_2)$, $f_{\mathbf{V}}(\mathbf{v}) = f_{\Gamma_0, \Gamma_1, A_1, A_2}(x_0, x_1, a_1, a_2)$, $m = 4$ and, $\mathbf{W} = (W_1, W_2, W_3, W_4) = \mathbf{r}(\Gamma_0, \Gamma_1, A_1, A_2)$ and its inverse mapping $\mathbf{s}(W_1, W_2, W_3, W_4)$ are given by:

$$\begin{aligned} W_1 &= r_1(\Gamma_0, \Gamma_1, A_1, A_2) = \frac{\alpha_1(\alpha_2)^n - \alpha_2(\alpha_1)^n}{\alpha_1 - \alpha_2} \Gamma_0 + \frac{(\alpha_1)^n - (\alpha_2)^n}{\alpha_1 - \alpha_2} \Gamma_1 \\ \Rightarrow \Gamma_0 &= s_1(W_1, W_2, W_3, W_4) = \left(W_1 - \frac{(\tilde{\alpha}_1)^n - (\tilde{\alpha}_2)^n}{\tilde{\alpha}_1 - \tilde{\alpha}_2} W_2 \right) \frac{\tilde{\alpha}_1 - \tilde{\alpha}_2}{\tilde{\alpha}_1(\tilde{\alpha}_2)^n - \tilde{\alpha}_2(\tilde{\alpha}_1)^n}, \\ W_2 &= r_2(\Gamma_0, \Gamma_1, A_1, A_2) = \Gamma_1 \Rightarrow \Gamma_1 = s_2(W_1, W_2, W_3, W_4) = W_2, \\ W_3 &= r_3(\Gamma_0, \Gamma_1, A_1, A_2) = A_1 \Rightarrow A_1 = s_3(W_1, W_2, W_3, W_4) = W_3, \\ W_4 &= r_4(\Gamma_0, \Gamma_1, A_1, A_2) = A_2 \Rightarrow A_2 = s_4(W_1, W_2, W_3, W_4) = W_4, \end{aligned} \quad (5)$$

where

$$\tilde{\alpha}_1 = \frac{-W_3 + \sqrt{(W_3)^2 - 4W_4}}{2}, \quad \tilde{\alpha}_2 = \frac{-W_3 - \sqrt{(W_3)^2 - 4W_4}}{2}.$$

Note that the Jacobian is given by

$$J_4 = \det \left(\frac{\partial \mathbf{v}}{\partial \mathbf{w}} \right) = \frac{\tilde{\alpha}_1 - \tilde{\alpha}_2}{\tilde{\alpha}_1(\tilde{\alpha}_2)^n - \tilde{\alpha}_2(\tilde{\alpha}_1)^n} \neq 0, \quad \text{w.p. 1,}$$

where we have used that $\tilde{\alpha}_1 \neq \tilde{\alpha}_2$ with probability 1 (w.p. 1), i.e., for each $\omega \in \Omega$, $\tilde{\alpha}_1(\omega) \neq \tilde{\alpha}_2(\omega)$. Then, according to (2), the joint p.d.f. of the random vector $\mathbf{W} = (W_1, W_2, W_3, W_4)$ is given by

$$f_{\mathbf{W}}(\mathbf{w}) = f_{\Gamma_0, \Gamma_1, A_1, A_2} \left(\left(w_1 - \frac{(\tilde{\alpha}_1)^n - (\tilde{\alpha}_2)^n}{\tilde{\alpha}_1 - \tilde{\alpha}_2} w_2 \right) \frac{\tilde{\alpha}_1 - \tilde{\alpha}_2}{\tilde{\alpha}_1(\tilde{\alpha}_2)^n - \tilde{\alpha}_2(\tilde{\alpha}_1)^n}, w_2, w_3, w_4 \right) \left| \frac{\tilde{\alpha}_1 - \tilde{\alpha}_2}{\tilde{\alpha}_1(\tilde{\alpha}_2)^n - \tilde{\alpha}_2(\tilde{\alpha}_1)^n} \right|, \\ w_{1,i} \leq w_i \leq w_{2,i}, \quad 1 \leq i \leq 4.$$

Therefore, taking into account that $Z = W_1$, from the event \mathcal{E}_1 defined in (4) one obtains:

$$\begin{aligned}
f_{1R}(z, n) &= \int_{w_{2,1}}^{w_{2,2}} \int_{w_{3,1}}^{w_{3,2}} \int_{\min[w_{4,1}; \frac{(w_3)^2}{4}]}^{\min[w_{4,2}; \frac{(w_3)^2}{4}]} f_{W_1, W_2, W_3, W_4}(w_1, w_2, w_3, w_4) dw_4 dw_3 dw_2 \\
&= \int_{\gamma_{1,1}}^{\gamma_{1,2}} \int_{a_{1,1}}^{a_{1,2}} \int_{\min[a_{2,1}; \frac{(a_1)^2}{4}]}^{\min[a_{2,2}; \frac{(a_1)^2}{4}]} f_{\Gamma_0, \Gamma_1, A_1, A_2} \left(\left(z - \frac{(\alpha_1)^n - (\alpha_2)^n}{\alpha_1 - \alpha_2} \gamma_1 \right) \frac{\alpha_1 - \alpha_2}{\alpha_1(\alpha_2)^n - \alpha_2(\alpha_1)^n}, \gamma_1, a_1, a_2 \right) \\
&\quad \times \left| \frac{\alpha_1 - \alpha_2}{\alpha_1(\alpha_2)^n - \alpha_2(\alpha_1)^n} \right| da_2 da_1 d\gamma_1, \quad (6)
\end{aligned}$$

where α_1 and α_2 are defined by (3). $f_{1R}(z, t)$ is a piece of the total 1-p.d.f. $f_1(z, n)$. For the sake of clarity, we will specify the domain of z in the examples provided later, avoiding the cumbersome expressions for the general context.

2.2 Complex random roots

Let us assume that $p_2 > 0$ and let us denote by $\text{Re}(\alpha_1)$ and $\text{Im}(\alpha_1)$ the real and imaginary parts of the random root $\alpha_1 = \alpha_1(\omega)$:

$$\text{Re}(\alpha_1) = -\frac{A_1}{2}, \quad \text{Im}(\alpha_1) = \frac{\sqrt{4A_2 - (A_1)^2}}{2},$$

respectively. Then, the solution of the i.v.p. (1) is given by

$$Z_n = \frac{-r^{n+1} \sin((n-1)\theta)}{\text{Im}(\alpha_1)} \Gamma_0 + \frac{r^n \sin(n\theta)}{\text{Im}(\alpha_1)} \Gamma_1, \quad \text{where} \quad \begin{cases} r = \sqrt{(\text{Re}(\alpha_1))^2 + (\text{Im}(\alpha_1))^2}, \\ \theta = \arctan\left(\frac{\text{Im}(\alpha_1)}{\text{Re}(\alpha_1)}\right). \end{cases} \quad (7)$$

From this representation, we are ready to determine the piece $f_{1I}(z, n)$ associated to the event \mathcal{E}_2 defined in (4) that contributes to determine the 1-p.d.f. $f_1(z, n)$. This will be done by applying again Theorem 1 with the same identification for the maps r_2, r_3, r_4, s_2, s_3 and s_4 as we did in the previous case (see (5)) but now taking

$$\begin{aligned}
W_1 &= r_1(\Gamma_0, \Gamma_1, A_1, A_2) = \frac{-r^{n-1} \sin((n-1)\theta)}{\text{Im}(\alpha_1)} \Gamma_0 + \frac{r^n \sin(n\theta)}{\text{Im}(\alpha_1)} \Gamma_1, \\
\Gamma_0 &= s_1(W_1, W_2, W_3, W_4) = \left(\frac{r^n \sin(n\theta)}{\text{Im}(\alpha_1)} W_2 - W_1 \right) \frac{\text{Im}(\alpha_1)}{r^{n+1} \sin((n-1)\theta)},
\end{aligned}$$

where r and θ are defined in (7). In this case the Jacobian is given by

$$J_4 = -\frac{\text{Im}(\alpha_1)}{r^{n+1} \sin((n-1)\theta)} \neq 0, \quad \text{w.p. 1.}$$

Then, according to (2), the joint p.d.f. of the random vector $\mathbf{W} = (W_1, W_2, W_3, W_4)$ is computed as follows:

$$f_{\mathbf{W}}(\mathbf{w}) = f_{\Gamma_0, \Gamma_1, A_1, A_2} \left(\left(\frac{r^n \sin(n\theta)}{\text{Im}(\alpha_1)} w_2 - w_1 \right) \frac{\text{Im}(\alpha_1)}{r^{n+1} \sin((n-1)\theta)}, w_2, w_3, w_4 \right) \cdot \frac{|\text{Im}(\alpha_1)|}{r^{n+1} |\sin((n-1)\theta)|},$$

where $w_{1,i} \leq w_i \leq w_{2,i}$, $1 \leq i \leq 4$. Therefore, taking into account that $Z = W_2$ one gets

$$\begin{aligned}
f_{1I}(z, n) &= \int_{\gamma_{1,1}}^{\gamma_{1,2}} \int_{a_{1,1}}^{a_{1,2}} \int_{\max[a_{2,1}; \frac{(a_1)^2}{4}]}^{\max[a_{2,2}; \frac{(a_1)^2}{4}]} \frac{|\text{Im}(\alpha_1)|}{r^{n+1} |\sin((n-1)\theta)|} \\
&\quad \cdot f_{\Gamma_0, \Gamma_1, A_1, A_2} \left(\left(\frac{r^n \sin(n\theta)}{\text{Im}(\alpha_1)} \gamma_1 - z \right) \frac{\text{Im}(\alpha_1)}{r^{n+1} \sin((n-1)\theta)}, \gamma_1, a_1, a_2 \right) da_2 da_1 d\gamma_1, \quad (8)
\end{aligned}$$

where r, θ and α_1 can be expressed in terms of A_1 and A_2 taking into account (3) and (7). For the sake of clarity in the presentation, the domain of z will be determined later in the examples.

2.3 1-p.d.f.

Finally, in accordance with the results and comments made in Subsections 2.1 and 2.2, the 1-p.d.f. of the solution to i.v.p. (1) is given by

$$f_1(z, n) = f_{1R}(z, n) + f_{1I}(z, n), \quad (9)$$

where $f_{1R}(z, t)$ and $f_{1I}(z, t)$ are computed according to (6) and (8), respectively.

3 Examples

In order to illustrate better the theoretical results obtained in the previous section, next we will compute the 1-p.d.f. $f_1(z, n)$ of the solution Z_n to i.v.p. (1) at some values of n in the three following situations:

- Case I ($p_1 \gg p_2$): The event \mathcal{E}_1 is more probable than \mathcal{E}_2 (see (4)). Roughly speaking, this can be interpreted as the *probabilistic* contribution of $f_{1R}(z, n)$ to $f_1(z, n)$ is greater than $f_{1I}(z, n)$ in the sense that real and distinct roots of the associated characteristic equation are more probable than imaginary roots.
- Case II ($p_1 \approx p_2 \approx \frac{1}{2}$): The events \mathcal{E}_1 and \mathcal{E}_2 have the same probability to occur. This can be interpreted as the *probabilistic* contribution of $f_{1R}(z, n)$ and $f_{1I}(z, n)$ to $f_1(z, n)$ are similar, in the sense that real and distinct roots and imaginary roots of the associated characteristic equation are approximately equally probable.
- Case III ($p_1 \ll p_2$): The event \mathcal{E}_1 is less probable than \mathcal{E}_2 . This can be interpreted as the *probabilistic* contribution of $f_{1R}(z, n)$ to $f_1(z, n)$ is smaller than $f_{1I}(z, n)$ in the sense that real and distinct roots of the associated characteristic equation are less probable than imaginary roots.

To illustrate these three cases, we will consider that the four random inputs Γ_0, Γ_1, A_1 and A_2 follow a joint Gaussian distribution: $\eta_i = (\Gamma_0, \Gamma_1, A_1, A_2)^T \sim N(\mu_{\eta_i}, \Sigma_{\eta_i})$, with different mean vectors μ_{η_i} , $i = 1, 2, 3$, and a common covariance matrix $\Sigma = \Sigma_{\eta_i}$ given, respectively, by

$$\mu_{\eta_i} = \begin{cases} (1, 1, -1/10, -18/25)^T & \text{if } i = 1 \text{ (Case I),} \\ (1, 1, -1, 1/4)^T & \text{if } i = 2 \text{ (Case II),} \\ (1, 1, -1/10, 18/25)^T & \text{if } i = 3 \text{ (Case III),} \end{cases} \quad \Sigma = \begin{pmatrix} 5/100 & -1/100 & 0 & -2/100 \\ -1/100 & 15/100 & 7/100 & 5/100 \\ 0 & 7/100 & 2/10 & -1/100 \\ -2/100 & 5/100 & -1/100 & 1/10 \end{pmatrix}. \quad (10)$$

Note that in the three cases, the z -domain of the 1-p.d.f. $f_1(z, n)$ is $-\infty < z < \infty$. Table 1 collects the probabilities p_1 and $p_2 = 1 - p_1$ defined by (4) that determine Cases I–III. The probability p_1 has been computed as follows:

$$p_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{(a_1)^2}{4}} f_{A_1, A_2}(a_1, a_2) da_2 da_1 \quad \text{where} \quad f_{A_1, A_2}(a_1, a_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{\Gamma_0, \Gamma_1, A_1, A_2}(\gamma_0, \gamma_1 a_1, a_2) d\gamma_0 d\gamma_1,$$

and

$$f_{\Gamma_0, \Gamma_1, A_1, A_2}(\gamma_0, \gamma_1 a_1, a_2) = f_{\eta_i}(\eta_i) = \frac{1}{4\pi^2 \sqrt{\det(\Sigma)}} e^{-\frac{1}{2}(\eta_i - \mu_{\eta_i})^T \Sigma^{-1} (\eta_i - \mu_{\eta_i})}, \quad i = 1, 2, 3,$$

corresponding to the joint p.d.f. of the Gaussian vector $\eta_i = (\Gamma_0, \Gamma_1, A_1, A_2)^T$ to each one of the Cases I–III.

In Figure 1 we have plotted the 1-p.d.f. $f_1(z, n)$ at $n = 0, 1, 2, 3, 4$, in each one of the Cases I–III according to the expressions (6), (8) and (9). In each one of these plots we observe that the 1-p.d.f. $f_1(z, n)$ seems to converge to a Dirac delta function centred at $Z = 0$: $\lim_{n \rightarrow \infty} f_1(z, n) = \delta(z)$ since $Z_n \xrightarrow{n \rightarrow \infty} 0$. This behaviour can be roughly expected by using the *averaged* deterministic difference equation associated to random difference equation (1) consisting of taking the expectation operator $\mathbb{E}[\cdot]$ of every random input, i.e.,

$$z_{n+2} + \mathbb{E}[A_1]z_{n+1} + \mathbb{E}[A_2]z_n = 0, \quad n = 0, 1, 2, \dots, \quad z_0 = \mathbb{E}[\Gamma_0], \quad z_1 = \mathbb{E}[\Gamma_1],$$

and checking that in each one of the Cases I–III the roots of the corresponding characteristic equations have modulus less than one.

Case	p_1	p_2
I	0.991959	0.00804064
II	0.538897	0.461103
III	0.0202495	0.979751

Table 1: Probabilities p_1 and p_2 that determine Cases I-III when the inputs Γ_0, Γ_1, A_1 and A_2 have a joint Gaussian distribution: $\eta_i = (\Gamma_0, \Gamma_1, A_1, A_2)^T \sim N(\mu_{\eta_i}, \Sigma)$ with $\mu_{\eta_i}, i = 1, 2, 3$ and Σ given by (10).

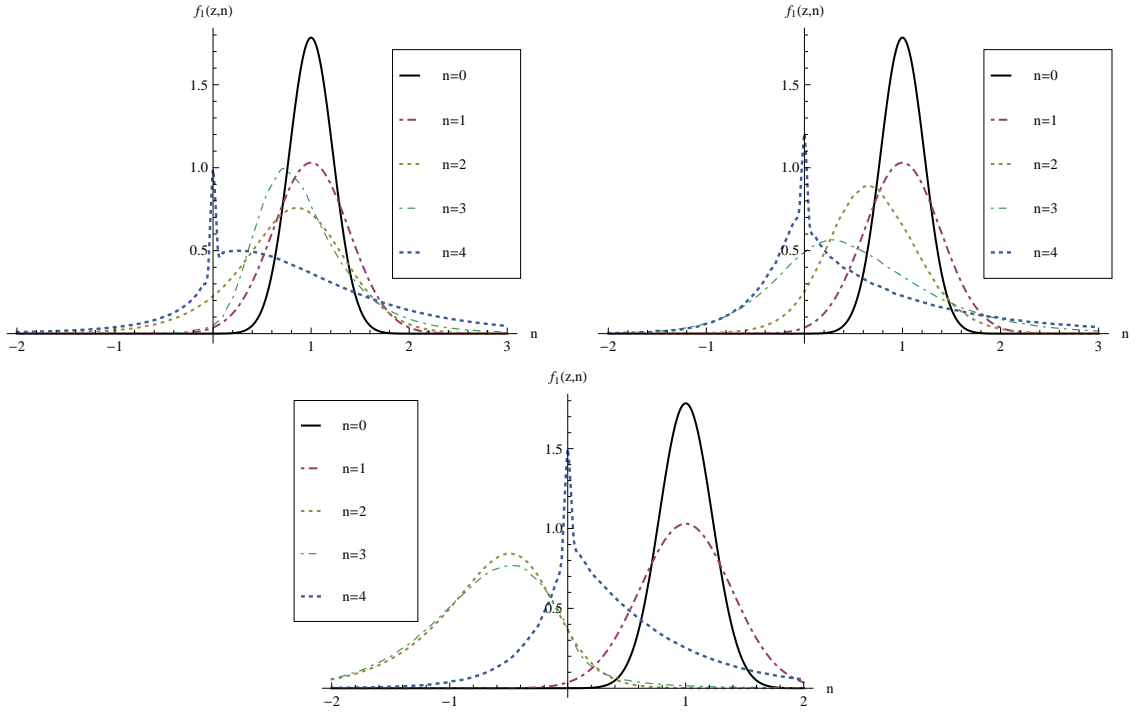


Figure 1: Plots of the 1-p.d.f. $f_1(z, n)$ of the solution Z_n to i.v.p. (1) in Case I (top left), Case II (top right) and Case III (bottom) at different values of $n = 0, 1, 2, 3, 4$.

4 Conclusions

In this paper we have provided general explicit formulae to the 1-p.d.f. of the solution stochastic process to a random homogeneous linear second-order difference equation in the general case where the involved random inputs are statistically dependent. The study has been based on the Random Variable Transformation technique. In this way, we can compute not only the mean and variance of the solution stochastic process but also provide a full probabilistic description of the solution in every discrete time instant. In addition, we have shown through the theoretical development that the study here presented generalizes its deterministic counterpart which is illustrated by examples. Finally, note that our analysis can be extended readily to the non-homogeneous case.

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