# A fast and practical method for model reduction of large scale water distribution networks

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## ABSTRACT[4]:

The paper presents a method for the reduction of network models described by a system of non-linear algebraic equations. Such models are, for example, present when modeling water networks, electrical networks and gas networks. The approach calculates a network model, equivalent to the original one, but which contains fewer components. This procedure has an advantage compared to straightforward linearization because the reduced non-linear model preserves the non-linearity of the original model and approximates the original model in a wide range of operating conditions. The method is applicable to hydraulic analysis andhas been validated by simplifying many practical water network models for optimization studies.

**Keywords**: water distribution network, full nonlinear model, full linearized model, reduced linear model, reduced nonlinear model, Gaussian elimination, large scale WDS simplification.

## INTRODUCTION

The paper presents a method for the reduction of network models described by a system of non-linear algebraic equations. The method will be formulated using an example of a water pipe network but the same arguments can be directly applied to a network of non-linear

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[4] The paper is an extended version of the conference article, Ulanicki, B., Zehnpfund, A. and Martinez, F. (1996). "Simplification of Water Network Models", In: HYDRINFORMATICS '96, *Proceedings of the 2nd International Conference on Hydroinformatics*, ETH Zurich, Switzerland, 9-13 September, vol. 2, Rotterdam, A.A.Balkema, pp.493-500.

resistors or other non-linear networks. The function of a water distribution network is to transport water from sources (rivers, boreholes etc.) to sinks (user demands). Major components of a network are reservoirs, pipes, valves and pumps. A typical water network may contain thousands of pipes and only tens of other components. In modeling, this network of pipes can be replaced by an equivalent reduced network. Control and design problems are normally solved with optimization techniques. The numerical complexity of optimization problems is much higher than the equivalent simulation problems, and consequently simplified models are required to make calculation time acceptable. It was realised (Zessler and Shamir, 1989), (Brdys and Ulanicki, 1994) that slow progress in developing optimization methods for water networks among other reasons was due to the lack of efficient model reduction methods.

There are different techniques of model reduction; the outcome of most of these methods is a hydraulic model with a smaller number of components than the prototype. The main aim of a reduced model is to preserve the nonlinearity of the original network and approximate its operation accurately under different conditions. The accuracy of the simplification depends on the model complexity and the selected method such as skeletonization (Walski et al., 2003; Saldarriaga et al., 2008), decomposition (Deuerlein, 2008), usage of artificial neural networks (ANN) metamodels (Rao and Alvarruiz, 2007; Broad et al., 2010) and variables elimination (Ulanicki et al., 1996). The skeletonization is the process of selecting for inclusion in the model only the parts of the hydraulic network that have a significant impact on the behaviour of the water distribution system (WDS) (Walski et al., 2003) e.g. use of equivalent pipes in place of numbers of pipes connected in parallel and/or in series. However the skeletonization is not a single process but several different low-level element removal processes that must be applied in series. This makes difficult the utilisation of this technique for the online optimisation purposes. In (Saldarriaga et al., 2008) authors presented an automated skeletonization methodology that can be used to achieve reduced models of WDS that accurately reproduce both, the hydraulics and non-permanent water quality parameters (chlorine residual) of the original

model. The proposed methodology was based on the resilience concept (Todini, 2000); by using the resilience index as selection criterion to remove pipes from the prototype, reduced models that simulate the hydraulics of the real network were achieved. However, the method is focused on the pipes removal only and thereby it can be mainly applied for looped pipe networks. Moreover the achievable degree of model reduction is not significant if the pressure in the simplified model is to be simulated accurately. In (Rao and Alvarruiz, 2007; Broad et al., 2010) ANNs have been successfully employed to approximate the water network model. The usage of ANN, due to time demanding training process, is not suitable for online water network optimisation where adaptation to abnormal structural changes is required. In (Deuerlein, 2008) a graph-theoretical decomposition concept of the network graph of WDS was proposed. The approach involves a several-steps decomposition to obtain a block graph of core of network graph. During that process demands of the root nodes are increased by the total demand of the connected trees to ensure that the simplified network replicates the hydraulic behaviour the total network. Also this approach due to its complexity and number of calculations involved is not applicable for online optimisation requirements.

The approach presented here is an extended version of the conference publication (Ulanicki et al 1996) and is based on mathematical formalism which finds a network model automatically in a comparatively short period of time.

The most direct way of reducing a system of algebraic equations would be by analytical elimination of some variables with the process of back substitution. Unfortunately, such general techniques do not exist for non-linear systems. The approach proposed here proceeds by the following steps: formulate the full non-linear model, linearise this model, reduce the linear model using the Gauss elimination procedure, and retrieve a reduced non-linear model from the reduced linear model. The method is applicable to hydraulic analysis especially for preparing reduced models for optimization studies. The paper has the following structure. In the Water network model formulation section a nodal model of a water network is presented. In Fundamentals it is explained how the method works on a very simple example and how well the reduced model approximate the nonlinearity of the original hydraulic model.

The following sections explain the technical details of the model reduction process which exploits properties of the non-linear and linearized models of water networks. The non-linear model is formulated using ideas from (Zehnpfund and Ulanicki 1993) and (Ulanicki et al., 1996) and this model is analogous to models of electrical networks discussed in (Balabanian and Bickart, 1969). It is shown that the Jacobian matrix of the linearized model has a special structure which enables the reduction procedure. In the Implementation section two computer implementations are described, the matrix based and the node by node. Finally the results of numerical experiments for two case studies are shown using the node by node and the matrix implementations.

## Water network models formulation

Mathematical models of water networks can be derived by analogy with electrical network models. The specific properties of water networks are determined by the non-linear head-flow relationships of its components. It is assumed that a pipe model is given by the Hazen-William formula (Williams and Hazen 1906)

$$q = q(\Delta h) = g|\Delta h|^{0.54}\ sign(\Delta h) \qquad q = q(\Delta h) = g\ S(\Delta h) \tag{1}$$

where, $\Delta h$ is the head drop, i.e. difference between the origin head and destination head, $q$ is a pipe flow, $g$ is the pipe conductance and $S(\Delta h) = |\Delta h|^{0.54} sign(\Delta h)$ is a function relating the pipe flow to the head drop between the origin and destination nodes. The pipe conductance depends on the pipe length, the pipe diameter, and the Hazen-Williams friction coefficient. The theory presented in this paper is valid for a general pipe characteristic where the flow is expressed as product of a conductance and some nonlinear function *S* of the head drop which is monotonic and crosses the origin. Hence, different explicit approximation of the Darcy-Weisbach equations can also be considered.

The topology of the network can be represented as a directed graph, where the branches are the network components and the nodes are connections between these components. Orientation of the branches is used to distinguish between different directions of a branch flow. For algebraic manipulation it is convenient to represent a network with a node branch incidence matrix $\Lambda$ (Brdys and Ulanicki 1994) which can be portioned in two blocks $\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_c \\ \mathbf{\Lambda}_f \end{bmatrix}$, for connection nodes and fixed head nodes respectively. The set of nodes connected to a given node *n* is denoted by $N_n$. A branch can be identified either by the branch index *j* or by the pair of node indices $(n,m)$. Transformation from one description to another is done with the help of the mapping $j(n,m)$, where *j* is the branch connected between nodes *n* and *m*.

The mathematical model of a water network can be compactly written using the node-branch incidence matrix $\Lambda$ as follows

$$\mathbf{\Lambda}_c \mathbf{q} = \mathbf{d} \qquad \text{Kirchhoff's law I for connection nodes} \qquad (2)$$

$$\mathbf{\Delta h} = \mathbf{\Lambda}^T \mathbf{h} \text{ or } \mathbf{\Delta h} = \mathbf{\Lambda}_c{}^T \mathbf{h}_c + \mathbf{\Lambda}_f{}^T \mathbf{h}_f \qquad \text{conservation of energy law} \qquad (3)$$

$$\mathbf{q} = \mathbf{q}(\mathbf{\Delta h}) \qquad \text{component law} \qquad (4)$$

where $\mathbf{q}$ = vector of branch flows, $\mathbf{d}$ = vector of nodal flows which represents demands and source flows, $\mathbf{h}_c, \mathbf{h}_f$ = vector of node heads at connection nodes and fixed grade nodes respectively, $\mathbf{\Delta h}$ = vector of branch head-drops, and $\mathbf{q}(\mathbf{\Delta h}) = (q_1(\Delta h_1),..,q_L(\Delta h_L))^T$ is a vector function where each function $q_\ell(\Delta h_\ell)$ is given by (1). The set of all components will be denoted by *L*, the set of all nodes and the set of connection nodes will be denoted by *N* and $N_c$, respectively. It is assumed that the unknown variables are the vectors of branch flows **q** and heads at connection nodes $\mathbf{h}_c$ whilst heads $\mathbf{h}_f$ at the fixed grade nodes and nodal flows **d** at the connection nodes are known.

These three equations (2), (3), (4) can be combined in different ways resulting in different models: nodal model, loop model or mixed model.

For later discussions it is convenient to use a model with vector $\mathbf{\Delta h}$ as unknown vector which is obtained by substituting Equation (4) into Equation (2):

$$\mathbf{\Lambda}_c\ \mathbf{q}(\mathbf{\Delta h}) = \mathbf{d} \tag{5}$$

The nodal model of a network involves only nodal variables; vector of nodal heads $\mathbf{h}_c$ **and** $\mathbf{h}_f$ and vector of demands **d** and is obtained by substituting Equation (3) into Equation (5).

$$\mathbf{\Lambda}_c\ \mathbf{q}(\mathbf{\Lambda}_c{}^T\mathbf{h}_c + \mathbf{\Lambda}_f{}^T\mathbf{h}_f) = \mathbf{d} \tag{6}$$

Equation (6) corresponds to $N_c$ scalar equations, each describing the mass balance at a given connection node. For a node *n* is

$$\sum_{m\in N_n} \Lambda_{n,j(n,m)} g_{n,m} S(\Delta h_{n,m}) = d_n \qquad \text{for} \quad n = 1,2,\ldots,N_c \tag{7}$$

where the terms on the left side of the equation represent the branch flows connected to the node *n*; $N_n$ is a set of nodes connected to the node *n*; $\Lambda_{n,j}$ is an element of $\mathbf{\Lambda}$ corresponding to node *n* and branch *j=j(n,m)* connected between nodes *n* and *m*, $\Delta h_{n,m}$ is the head drop between the origin and destination nodes of the branch and finally $g_{n,m}$ is a conductance of such a branch.

## Fundamentals

The fundamental idea of the model reductions is explained in Figure 1. A general approach to reduce a model is to eliminate some variables and equations by back substitution, at least such an approach works well for linear models through, for instance, the Gaussian elimination procedure. Unfortunately, this approach is not directly applicable for a general case of a nonlinear model. So the idea proposed here is to travel from a 'nonlinear world' to a 'linear world', reduce the model in the linear world and to come back to the nonlinear world. Formally, the idea proceeds in the three following steps:

1. Linearize a full nonlinear model to produce a full linearized model
2. Eliminate some variables from the full linear model using e.g. Gaussian elimination procedure to obtain a reduced linear model
3. Recover a reduced nonlinear model from the reduced linear model.

The first two steps of linearization and variable elimination are always possible. The third step of recovering the nonlinear model is possible for network models. Network models have specific features which are invariant with respect to the Gaussian elimination and hence making a return to a network nonlinear model possible. The reduced nonlinear model usually approximates the original nonlinear model over wider range of operating conditions (demands) than a linearized one.

The fundamental ideas will be illustrated using a simple three node network shown in Figure 2 before being converted into a generalized procedure. The nodal model of this simple network has the form

$$\begin{aligned} -g_{3,1}\,S(h_3-h_1)+g_{1,2}\,S(h_1-h_2)&=-d_1 \\ -g_{3,2}\,S(h_3-h_2)-g_{1,2}\,S(h_1-h_2)&=-d_2 \end{aligned} \tag{8}$$

where $g_{n,m}$=conductance of a pipe connected between nodes $n$ and $m$ and $S(h_n-h_m)$ is a branch function defined in equation 1. The first and second equations represent flow balance at nodes 1 and 2 respectively. The unknown variables are heads $h_1$ and $h_2$ at nodes 1 and 2. Node 3 is a fixed grade node with a known head $h_3$. The fundamental idea is to eliminate one variable e.g. $h_1$ to reduce the model to one unknown variable $h_2$. Unfortunately, for a system of nonlinear simultaneous equations there is no general procedure to do so. So the following approximate procedure is proposed. Linearize the model described by Equation 8 around the current operating point where unknowns are current deviations of heads $\delta h_1$ and $\delta h_2$ from the operating point caused by the known deviations of the demands from the operating point $\delta d_1$ and $\delta d_2$. Eliminate the unknown variable $\delta h_1$ e.g. using the Gaussian elimination procedure in order to obtain a linear model with the one variable $\delta h_2$. After that return to a nonlinear reduced model containing only one variable $h_2$. Of course this is only possible if there is one to one relationship between a nonlinear and linearized model. Let for the given nominal demands $d_1^0, d_2^0$ and the given fixed head $h_3^0$ the solutions to model (8) are $h_1^0$ and $h_2^0$ and this is an operating point, then the corresponding linearized model is

$$\begin{aligned}
&[g_{3,1} S_{\Delta h}(h_3^0 - h_1^0) + g_{1,2} S_{\Delta h}(h_1^0 - h_2^0)]\,\delta h_1 - g_{1,2} S_{\Delta h}(h_1^0 - h_2^0)\,\delta h_2 = -\delta d_1 \\
&- g_{1,2} S_{\Delta h}(h_1^0 - h_2^0)\,\delta h_1 + [g_{3,2} S_{\Delta h}(h_3^0 - h_2^0) + g_{1,2} S_{\Delta h}(h_1^0 - h_2^0)]\,\delta h_2 = -\delta d_2
\end{aligned} \tag{9}$$

where $S_{\Delta h}(\Delta h_{n,m})$ is the derivative of the characteristic function $S(\Delta h_{n,m})$ with respect to the head drop $\Delta h_{n,m}$ .

If we introduce the idea of linearized conductance

$$\begin{aligned}
&p_{1,1} = [g_{3,1} S_{\Delta h}(h_3^0 - h_1^0) + g_{1,2} S_{\Delta h}(h_1^0 - h_2^0)] \qquad p_{1,2} = g_{1,2} S_{\Delta h}(h_1^0 - h_2^0) \\
&p_{2,1} = g_{1,2} S_{\Delta h}(h_1^0 - h_2^0) \qquad p_{2,2} = [g_{3,2} S_{\Delta h}(h_3^0 - h_2^0) + g_{1,2} S_{\Delta h}(h_1^0 - h_2^0)]
\end{aligned} \tag{10}$$

then at a given operating point the linearized model can be represented as

$$\begin{aligned}
&p_{1,1}\,\delta h_1 - p_{1,2}\,\delta h_2 = -\delta d_1 \\
&- p_{2,1}\,\delta h_1 + p_{2,2}\,\delta h_2 = -\delta d_2
\end{aligned} \tag{11}$$

The linearized conductance describes how the nodal flow balance is affected by the changes of the nodal heads. For example, if head $h_2$ is changed it will changed flows in all pipes connected to node 2 which is confirmed by the fact that the linearized conductance $p_{2,2}$ depends on conductance of all pipes connected to node 2. If head $h_1$ is changed it affects the flow balance at node 2 only through flow in the branch (1, 2) which is confirmed by the fact that the linearized conductance $p_{2,1}$ depends only on the conductance pipe 1. The similar behaviour can be observed when analyzing conductance $p_{1,1}$ and $p_{1,2}$ respectively.

Let's eliminate variable $\delta h_1$ from the second equation with help of the Gauss elimination procedure

$$(p_{2,2} - \frac{p_{1,2}\, p_{2,1}}{p_{1,1}})\,\delta h_2 = -(\delta d_2 + \delta d_1 \frac{p_{2,1}}{p_{1,1}}) \tag{12}$$

and introduce notation for the conductance and the demand of the reduced linear model

$$p^r = (p_{2,2} - \frac{p_{1,2}\, p_{2,1}}{p_{1,1}}) \qquad\qquad \delta d^r = (\delta d_2 + \delta d_1 \frac{p_{2,1}}{p_{1,1}})$$

Using the new notation the reduced linear model is

$$p^r\, \delta h_2 = -\delta d^r \tag{13}$$

The two nodes are left in the model, the fixed grade node 3 and the connection node 2. It is easy to guess a nonlinear model corresponding to linear model (12), namely

$$-g^r S(h_3 - h_2) = -d^r \tag{14}$$

where $g^r = p^r \times \dfrac{1}{S_{\Delta h}(h_3^0 - h_2^0)}$ and $d^r = (d_2 + d_1 \dfrac{p_{2,1}}{p_{1,1},})$

One can check by linearization of model (14) that model (12) is obtained.

Model (14) is a reduced nonlinear model derived from original nonlinear model (8). The properties of the considered models are captured in Figure 3, where head $h_2$ is plotted as a function of a demand $d_2$ for different models. The following values are assumed for the calculations, $h_3 = 140m$, $d_1 = 40l/s$, $g_1 = g_2 = 8.716740$ and $g_3 = 0.9493$. The thick continuous line represents a full nonlinear model, the thin continuous line represent a reduced nonlinear model and the dashed line represents both a linearized full model and a reduced linear model (these two models overlap for $h_2$) . The following can be observed

- All models share the same operating point $h_{20} = 120.24\,m$
- The full nonlinear model and the reduced nonlinear model are tangent to the same linear model represented by the straight dashed line
- Reduced nonlinear model approximates very well the full nonlinear model in the whole range of operating conditions represented by a demand $d_2$.

## Linearized water network model and its properties

A linearized version of nonlinear model (6) will describe the relationship between small changes in nodal quantities, heads and demands $\delta\mathbf{h}_c, \delta\mathbf{d}$ about a given operating point defined by head $\mathbf{h}^0$ and nodal flow $\mathbf{d}^0$.

$$\mathbf{\Lambda}_c\ \mathbf{q}_{\Delta h}\ \mathbf{\Lambda}_c{}^T \times \boldsymbol{\delta}\mathbf{h}_c = \boldsymbol{\delta}\mathbf{d} \tag{15}$$

where

$$\mathbf{q}_{\Delta h} = diag\left[g_j\ S_{\Delta h}(\Delta h_j^0)\right]_{j\in L} \tag{16}$$

is a $L\times L$ diagonal matrix obtained from differentiating the vector function $\mathbf{q}(\mathbf{\Delta h})$ with respect to head losses $\mathbf{\Delta h}$ and $\Delta h_j^0$ is a head drop for components $j$ at the operating point. $\mathbf{J} = \mathbf{\Lambda}_c \mathbf{q}_{\Delta h} \mathbf{\Lambda}_c{}^T$ is called a Jacobian of model (6) and will play a fundamental role in further considerations. Linearized model (15) can now be presented as

$$\mathbf{J} \times \boldsymbol{\delta}\mathbf{h}_c = \boldsymbol{\delta}\mathbf{d} \tag{17}$$

Properties of the Jacobian matrix **J** are summarised below.

*Properties of Jacobian matrix* **J**

1. Jacobian **J** is a $N_c \times N_c$ symmetric matrix.
2. The diagonal elements of **J** are equal to

$$J_{n,n} = \sum_{m \in N_n} g_{n,m} S_{\Delta h}(h_n^0 - h_m^0) \qquad \text{for} \quad n = 1, 2, \ldots, N_c \tag{18}$$

The non-diagonal elements in a row $n$ are

$$J_{n,m} = \begin{cases} -g_{n,m} S_{\Delta h}(h_n^0 - h_m^0) & \text{for} \quad m \in N_{c,n} \\ 0 & \text{for} \quad m \notin N_{c,n} \end{cases} \tag{19}$$

where $N_{c,n}$ is a set of connection nodes connected to node $n$.

3. In a row corresponding to a node connected to a fixed grade node the diagonal element is greater than the sum of the non-diagonal elements taken with the opposite sign

$$J_{n,n} > -\sum_{m \in N_{c,n}} J_{n,m} \tag{20}$$

whilst in a row corresponding to a node not connected to a fixed grade node the diagonal element equals to the sum of the non-diagonal elements with the opposite sign.

$$J_{n,n} = -\sum_{m \in N_{c,n}} J_{n,m} \tag{21}$$

4. The matrix **J** is positive definite.

$\square$

The theorem is an implication of the special structure of the Jacobian matrix.

For a given operating point let’s introduce the notion of linearized branch conductance

$$p_{n,m} \overset{\Delta}{=} -J_{n,m} = g_{n,m} S_{\Delta h}(h_n^0 - h_m^0) \tag{22}$$

and linearized node conductance

$$p_{n,n} \overset{\Delta}{=} J_{n,n} = \sum_{m \in N_n} p_{n,m} \tag{23}$$

With these denotations the linearized model (17) can be represented in an expanded form as

$$\begin{bmatrix} p_{1,1} & -p_{1,2} & \cdots & -p_{1,N_c} \\ -p_{2,1} & p_{2,2} & \cdots & -p_{2,N_c} \\ \cdots & \cdots & \cdots & \\ -p_{N_c,1} & -p_{N_c,2} & \cdots & p_{N_c,N_c} \end{bmatrix} \times \begin{bmatrix} \delta h_1 \\ \delta h_2 \\ \cdots \\ \delta h_{N_c} \end{bmatrix} = \begin{bmatrix} \delta d_1 \\ \delta d_2 \\ \cdots \\ \delta d_{N_c} \end{bmatrix} \tag{24}$$

Conductance matrix **J** is sparse and for a row *n* elements $p_{n,m}$ are non-zero for connection nodes connected to the node *n* (i.e., $m \in N_{c,n}$), and zero for other nodes; additionally from (21) it is clear that the diagonal element is a sum of non-diagonal elements for the nodes not connected to fixed grade nodes.

Another useful interpretation of the linearized model (15) is obtained by grouping relevant terms to obtain the linearized model in terms of the head differences

$$\mathbf{\Lambda}_c \; \mathbf{q}_{\Delta h} \quad \boldsymbol{\delta\Delta}\mathbf{h} = \boldsymbol{\delta}\mathbf{d} \tag{25}$$

where $\boldsymbol{\delta\Delta}\mathbf{h} = \mathbf{\Lambda}_c{}^T \boldsymbol{\delta}\mathbf{h}_c$ is the variation of the vector of head differences $\mathbf{\Delta h}$ about the given operating point when the nodal flow at the connection nodes changes.

Equation (25) corresponds to $N_c$ scalar equations, each describing the linearized mass balance at a connection node. For nodes $n = 1,2,...,N_c$ is

$$\sum_{m \in N_n} \Lambda_{n,j(n,m)} \, p_{n,m} \times (\delta \Delta h_{n,m}) = \delta d_n \tag{26}$$

Each term on the left side of Equation (26) represents a flow in a branch connected to node *n*, and complies with the standard Ohm's law; the flow is equal to the conductance of the branch $p_{n,m}$ multiplied by the branch head difference $\delta \Delta h_{n,m}$. Model (26) is a linearized version of model (7) and clearly, there is a one to one mapping between these two models.

Both models have the same topology described by matrix $\mathbf{\Lambda}_c$ and the relationship between the non-linear branch conductance $g_{n,m}$ and the linearized branch conductance $p_{n,m}$ is given by $p_{n,m} = g_{n,m} S_{\Delta h}(\Delta h^0{}_{n,m})$.

If one wants to return from the linearized model (26) to nonlinear model (7) the following should be used.

$$\sum_{m \in N_n} \Lambda_{n,j(n,m)} \frac{p_{n,m}}{S_{\Delta h}(\Delta h^0{}_{n,m})} S(\Delta h_{n,m}) = d_n \qquad (27)$$

## Reduced linear model and its properties

The process of the Gauss-elimination (Gill et al. 1991), will be applied to the linearized model given by (15) and (24). For example, to remove node 1 from the model it is necessary to apply one step of the Gauss-elimination procedure as follows

$$\begin{bmatrix} (p_{2,2} - \frac{p_{2,1}}{p_{1,1}} p_{1,2}) & \dots & (-p_{2,N_c} - \frac{p_{2,1}}{p_{1,1}} p_{1,N_c}) \\ \dots & \dots & \dots \\ (-p_{N_c,2} - \frac{p_{N_c,1}}{p_{1,1}} p_{1,2}) & \dots & (p_{N_c,N_c} - \frac{p_{N_c,1}}{p_{1,1}} p_{1,N_c}) \end{bmatrix} \begin{bmatrix} \delta h_2 \\ \dots \\ \delta h_{N_c} \end{bmatrix} = \begin{bmatrix} \delta d_2 + \frac{p_{2,1}}{p_{1,1}} \delta d_1 \\ \dots \\ \delta d_{N_c} + \frac{p_{N_c,1}}{p_{1,1}} \delta d_1 \end{bmatrix} \qquad (28)$$

The reduced model involves variables $\delta h_2, \delta h_3, \dots, \delta h_{N_c}$, whereas variable $\delta h_1$ has been removed from the model. The demand $\delta d_1$ has been redistributed among other nodes connected to node 1 and if node 1 was not connected to a fixed grade node than the total demands in the full model and in the reduced model are the same.

$$(\delta d_2 + \frac{p_{2,1}}{p_{1,1}} \delta d_1) + \dots + (\delta d_{N_c} + \frac{p_{N_c,1}}{p_{1,1}} \delta d_1) = \delta d_1 + \delta d_2 + \dots + \delta d_{N_c}$$

The matrix **J** has a dominant diagonal so the normal Gauss-elimination is numerically stable and is equivalent to the elimination with pivoting.

If $(N_c - r)$ connection nodes are to be removed from the model then the corresponding rows have to be placed at the first $(N_c - r)$ positions in the matrix **J**, and $(N_c - r)$ steps of the Gauss-elimination procedure are to be performed. The reduced model will have $r$ nodes and the corresponding Jacobian matrix and the nodal flow vector will be denoted by $\mathbf{J}^r$ and $\mathbf{d}^r$ respectively.

With these notations the reduced $r$ model takes the form

$$\mathbf{J}^r \times \boldsymbol{\delta}\mathbf{h}_c^r = \boldsymbol{\delta}\mathbf{d}^r \tag{29}$$

where $\boldsymbol{\delta}\mathbf{h}_c^r = \begin{bmatrix} \delta h_{N_c - r+1} \\ \ldots \\ \delta h_{N_c} \end{bmatrix}$ is a vector composed of the last $r$ elements of the full vector $\boldsymbol{\delta}\mathbf{h}_c$. The properties of the linearized model described previously are invariant with respect to the Gaussian elimination procedure and consequently reduced matrix $\mathbf{J}^r$ has the same properties as matrix $\mathbf{J}$ i.e. represents a linearized model of a network.

*Properties of reduced matrix* $\mathbf{J}^r$

1. Matrix $\mathbf{J}^r$ has the same properties as matrix **J,** in particular the properties (1), (3) and (4) are true.

2. If the removed connection nodes are not connected to the fixed grade nodes then the total demands in the full model and in the reduced model are the same.

$$\sum_{n=1}^{N_c} \delta d_n = \sum_{n=1}^{N_c^r} \delta d_n^r \tag{30}$$

where $N_c^r = r - N_f$ is the number of connection nodes in the reduced model. One should remember that the fixed grade nodes are not removed and hence the following relationships are satisfied $N = N_c + N_f$ for the full model and $r = N_c^r + N_f$ for the reduced model. □

The proof can be completed with mathematical induction of which the first step has already been completed in the form of model (28).

The properties of matrix $\mathbf{J}^r$ allow to interpret reduced linear model (29) as representing a network with *r* nodes and a new topology described by matrix $\mathbf{J}^r$

$$\mathbf{J}^r = \begin{bmatrix} p_{1,1}^r & -p_{1,2}^r & \cdots & -p_{1,N_c^r}^r \\ -p_{2,1}^r & p_{2,2}^r & \cdots & -p_{2,N_c^r}^r \\ \cdots & \cdots & \cdots & \cdots \\ -p_{N_c^r,1}^r & -p_{N_c^r,2}^r & \cdots & p_{N_c^r,N_c^r}^r \end{bmatrix} \quad (31)$$

where elements $p_{i,j}^r$ play a role of a linearized conductance of the reduced model and if $p_{i,j}^r = 0$ for $i \neq j$ it means that nodes *i* and *j* are not connected. Matrix $\mathbf{J}^\mathbf{r}$ is more dense than the original matrix $\mathbf{J}$ but is much smaller and hence less time consuming to solve.

It is worth to notice that the resulting reduced model doesn't depend on the order in which the nodes for the removal are placed, however the order significantly affects the time required for the Gaussian elimination procedure.

Changing order of the nodes for removal corresponds to multiplication of the incidence matrix $\mathbf{\Lambda}_c$ and respective variables by an appropriate permutation matrix $\mathbf{\Pi}$ (Gill et al. 1991). In our case the full nonlinear model (6) becomes

$$\mathbf{\Pi\Lambda}_c\mathbf{q}\left(\mathbf{\Lambda_c^T\Pi}^T\mathbf{\Pi h}_c + \mathbf{\Lambda_f^T h}_f\right) = \mathbf{\Pi d}$$

and the linearized model (15) becomes

$$(\mathbf{\Pi\Lambda}_c\mathbf{q_{\Delta h}\Lambda_c^T\Pi}^T)\mathbf{\Pi\delta h_c} = \mathbf{\Pi\delta d}$$

Considering that the permutation matrix $\Pi$ is orthogonal, $\mathbf{\Pi\Pi}^T = \mathbf{I}$ and non-singular $\mathbf{\Pi}^{-1} = \mathbf{\Pi}^T$ (Gill et al. 1991) after few manipulations applied to the two models above the original models (6) and (15) are obtained.

Moreover, the permutations are applied only to the connection nodes designated for removal and not to the connection nodes which remains in the reduced model, subsequently it can be proven that the order of connection nodes for removal doesn't affect the outcome i.e. the reduced model.

There are special rows/columns re-ordering algorithms which accelerate significantly the model reduction calculations, for instance the minimum degree ordering algorithm proposed for the first time in (Rose, 1970). The more discussion on the re-ordering is presented in the 'Node by node implementation' section.

## Recovering a reduced nonlinear model

Reduced linear model (29) is obtained by performing $N_c - r$ steps of the Gauss-elimination procedure which is equivalent to multiplying both sides of an original model (17) by a unit lower triangular matrix **M** (Gill et al. 1991).

$$\mathbf{M}\ \mathbf{J}\ \boldsymbol{\delta}\mathbf{h}_c = \mathbf{M}\ \boldsymbol{\delta}\mathbf{d} \tag{32}$$

The resulting product $\mathbf{M} \times \mathbf{J}$ has the block structure seen in the equation below

$$\begin{bmatrix} \mathbf{U} & \mathbf{Q} \\ \mathbf{0} & \mathbf{J}^r \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}\mathbf{h}_c^{N_c - r} \\ \boldsymbol{\delta}\mathbf{h}_c^r \end{bmatrix} = \mathbf{M}\ \boldsymbol{\delta}\mathbf{d} \tag{33}$$

where **U** is an $(N_c - r)\times(N_c - r)$ upper triangular matrix, and $\mathbf{J}^r$ is a $r \times r$ invertible matrix from equation (29) and **Q** is an $(N_c - r) \times (N_c - r)$ matrix block resulting the Gaussian elimination procedure .

Due to a special structure, equation (33) decomposes into two parts, one of which is a reduced linear model (29)

$$\mathbf{J}^r \boldsymbol{\delta}\mathbf{h}_c^r = \mathbf{M}^{(r)}\ \boldsymbol{\delta}\mathbf{d} \tag{34}$$

where $\mathbf{M}^{(r)}$ denotes the last $r$ rows of matrix $\mathbf{M}$. By comparing right side of equations (29) and (34) the demand of the reduced model can be expressed as

$$\boldsymbol{\delta}\mathbf{d}^r = \mathbf{M}^{(r)}\ \boldsymbol{\delta}\mathbf{d} \tag{35}$$

Since matrix $\mathbf{J}^r$ of the reduced linear model has the same properties as the matrix of the full linear model it can be considered to represent a linear network of conductance elements. The reduced model has *r* nodes and a new topology determined by $\mathbf{J}^r$. A non-zero entry at a position $(n,m)$ indicates a branch between nodes *n* and *m*, a zero entry indicates no connection between these nodes. If two nodes were connected in the original model the branch orientation between these two nodes stays the same in the reduced model. If two nodes were not connected the branch orientation is given by the sign of the head difference at them and after that applied consistently in all equations.

The new node-branch incidence matrix can be denoted by $\mathbf{\Lambda}^r$, with $N_c{}^r$ and $L^r$ signifying the number of nodes and branches respectively in the reduced model. There has been established, in the section on the properties of a linearized model, one to one mapping between linearized model (26) and non-linear model (7) in the form of equation (27). Applying the same format the following reduced nonlinear model is obtained.

$$\sum_{m \in N_n^r} \Lambda_{n,j}^r \; g_{n,m}^r \; S(\Delta h_{n,m}) = d_n^r \qquad \text{for } n = N_c - r + 1, \ldots, N_c \tag{36}$$

with $$g_{n,m}^r = \frac{p_{n,m}^r}{S_{\Delta h}(\Delta h_{n,m}^0)} \tag{37}$$

where $\Lambda^r{}_{n,j}$ =elements of a topology matrix $\mathbf{\Lambda}^r$ of the reduced model, $N_n^r$ =a set of nodes connected to a node *n* in the reduced model, $j = j(n,m)$ is an index of a component connected between nodes *n* and *m*, $\Delta h_{n,m}^0 = h_n^0 - h_m^0$ corresponds to the original operating point and $d_n^r$ = an element of the demand vector $\mathbf{d}^r = \mathbf{M}^{(r)}\mathbf{d}$ with $\mathbf{M}^{(r)}$ defined in equations (34) and (35).

Model (36) can be presented in a vector form as

$$\mathbf{\Lambda}_c^r \, \mathbf{q}^r(\Delta \mathbf{h}^r) = \mathbf{M}^{(r)} \, \mathbf{d} \tag{38}$$

where $\boldsymbol{\Delta}\mathbf{h}^r = (\boldsymbol{\Lambda}_c^r)^T \mathbf{h}_c^r + (\boldsymbol{\Lambda}_f)^T \mathbf{h}_f$ or in terms of vector $\mathbf{h}_c^r$ as

$$\boldsymbol{\Lambda}_c^r \mathbf{q}^r((\boldsymbol{\Lambda}_c^r)^T \mathbf{h}_c^r + (\boldsymbol{\Lambda}_f)^T \mathbf{h}_f) = \mathbf{M}^{(r)}\mathbf{d} \qquad (39)$$

where $\mathbf{q}^r(\boldsymbol{\Delta}\mathbf{h}^r) = (q_1(\Delta h_1),..,q_{L^r}(\Delta h_{L^r}))^T$ is a $L^r$ vector function describing the non-linear branch law for all new components $j = 1,2,..., L^r$ given by equation (1), and the elements of $\mathbf{M}^{(r)}$ are simply multipliers applied to the original set of nodal demands that produce an equivalent set of demands at the nodes remaining in the reduced modelThe results of the above discussion are collected together below.

*Properties of the reduced nonlinear model*

The reduced nonlinear model represented by equation (38) or (39) and the full nonlinear model represented by equation (5) or (6) are ‘tangent’ to one another at the operating point which means that:

1. Linearization of a model (38) leads to a reduced linear model (29) obtained by variable elimination of a full linearized model (17).
2. The full nonlinear model (5) and the reduced nonlinear model (38) have the same operating point with respect to the last *r* components of vector $\mathbf{h}^0$.
3. The difference between the solution (heads) of a full nonlinear model (5) and a reduced nonlinear model (38) is of a second order

The proof of property 1 can be done by checking the steps of the linearization procedure starting with a model (38). Property 2 follows from the manner the reduced nonlinear model has been constructed (equation (37)) around the given operating point. Property 3 is a consequence that both models have identical linearized models (with respect to the last *r* components) and the first order terms in the Taylor expansion of both models cancel one another. Although the formal proof is important from a practical perspective one should also notice that the good accuracy is not only local around the operating point but also stretches over wide range of demands, in the simple case of Fig 2 from $d_2 = 20\, l/s$ to $d_2 = 60\, l/s$.

## Implementation

The presented model reduction algorithm can be implemented as a computer program using a formal Gaussian elimination procedure applied to a Jacobian matrix **J** or using a 'node by node' elimination rules which will be explained later in this section.

### *Matrix implementation*

Normally water network models are implemented as data files used by simulation packages such as Epanet (Rossman 2000). The model reduction software can be linked to a simulator and work by reading in a simulation file with an original model and generating a file with a reduced simulation model. The matrix implementation involves five steps:

- Preparing a full nonlinear model
- Preparing an operating point
- Preparing a Jacobian matrix $\mathbf{J} = \mathbf{\Lambda}_c \mathbf{q}_{\Delta h} \mathbf{\Lambda}_c{}^T$
- Applying the Gaussian elimination procedure to Jacobian
- Generating a reduced nonlinear model

The purpose of the first step is to define a set of nodes to be removed from the model and reordering all nodes so the nodes to be removed are at the beginning and the fixed grade nodes at the end of the respective arrays. The prepared model is simulated to generate an operating point at which the model is linearized. At this operating point a Jacobian matrix is evaluated and subsequently a reduced Jacobian matrix is calculated. From the reduced Jacobian matrix the topology, the values of the pipe conductance and new allocation of demands of the reduced model can be obtained. Having this information a file containing a reduced nonlinear model can be generated.

The matrix Gaussian elimination approach has been employed to reduce models for many applications such as optimal pressure control (Ulanicka et al. 2001) and optimal scheduling (Bounds et al. 2006). In the scheduling study the model was reduced from 4388 to 414 components and the simplification process took approximately 2 minutes on a Pentium 4 2.2GHz PC. In the pressure control study the model has been reduced from 5332 to 1118 components, the significant number of nodes was preserved to maintain the structure of the system which included 24 subsystems (zones).

### *Node by node implementation*

There is a strict correspondence between symmetric positive definite matrices and graph theory and the two views complement one another in solving important network problems. The reduction procedure can be translated into a set of rules and implemented as a computer program which operates directly on the water network graph. Consider a network shown in Figure 4a and assume that node 1 is selected for removal from the network model. One should take the following steps:

a) Calculate the pipe linear conductance, $p_{n,m}$, for all pipes connected to node 1, according to equation (22);

b) Calculate node 1 nodal conductance, $p_{1,1}$, according to equation (23);

c) Calculate the new conductance between each pair of nodes connected to node 1. The new conductance between nodes $n_1$ and $n_2$ is

$$p^r_{n_1,n_2} = p_{n_1,n_2} + \frac{p_{1,n_1} p_{1,n_2}}{p_{1,1}} \qquad (40)$$

Moreover, if there was no branch between two nodes, a new branch appears between these nodes with a respective conductance. An additional conductance between nodes $n_3$ and $n_1$ is

$$p^r_{n_3,n_1} = \frac{p_{1,n_3} p_{1,n_1}}{p_{1,1}}$$

and between nodes $n_3$ and $n_2$ is

$$p^r_{n_3,n_2} = \frac{p_{1,n_3} p_{1,n_2}}{p_{1,1}}$$

The formula (40) can be interpreted as a parallel connection of $p_{n_1,n_2}$ and a composite branch which in turn comprises the series connection of $p_{n_1,1}$ and $p_{1,n_2}$. However, when calculating an equivalent conductance for the series connection the product $p_{n_1,1}$ $p_{1,n_2}$ is divided by the nodal conductance $p_{1,1}$ rather than by the sum of these two branches conductance.

d) Demand $d_1$ is redistributed between nodes connected to node 1 proportionally to the conductance of each branch, so the new demands at the $N_1$ nodes are

$$d^r_{n_1} = d_{n_1} + \frac{p_{1,n_1}}{p_{1,1}} d_1, \quad d^r_{n_2} = d_{n_2} + \frac{p_{1,n_2}}{p_{1,1}} d_1, \quad d^r_{n_3} = d_{n_3} + \frac{p_{1,n_3}}{p_{1,1}} d_1, \qquad (41)$$

The resulting network model is depicted in Figure 4b.

After the first step a new partially reduced model is obtained and a next node for elimination can be selected. The procedure is repeated many times until the required level of reduction is achieved. Once all required nodes are removed, the nonlinear model may be obtained. The new pipes conductance should be translated into length, diameter and roughness coefficient. The length can be assumed to be equal to the distance between the two nodes concerned, roughness can assume a standard value $C = 100$ and the diameter can be evaluated from the calculated value of the conductance and remaining assumed values of the length and the roughness. Also the flow rate through the new pipes can be computed, if needed, following equation (27),

$$q_{n,m} = \frac{p_{n,m}}{S_{\Delta h}(\Delta h^{0}{}_{n,m})} S(\Delta h_{n,m}) \tag{42}$$

being oriented from the node of higher head to the node of lower head .

The experience achieved during solving many case studies indicates that the following recommendation should be followed for both implementation methods. All fixed head nodes and control components, including all pumps and regulating valves, should be kept in the reduced model and as a consequence its end nodes. Also the nodes connected to fixed grade nodes must be kept to avoid redistribution of their nodal demands, which in turn will improve the accuracy of the storage trajectories. Also nodes with multiple demands, emitters or injection flow must be preserved. The demand pattern of a removed node must be the same of the adjacent nodes; in other case it has to be kept. This applies to nodes with unusual demands and its adjacent nodes. However simple throttle valves which are not controlled can be reduced by assimilating its properties to an equivalent pipe. If the network has a complex structure with many subsystems (zones) it maybe worthwhile to preserve the boundary nodes in order to maintain the major structure of the model. Also nodes of particular interest (e.g minimum pressure) can be kept additionally.

The operating point should be representative for normal operations of the network and should be chosen for average demand conditions while keeping at least one pumping unit working at each pumping station in order to avoid zero flow pipes. Before parallel pipes are removed, an equivalent pipe should be introduced by summing their conductance. During the reduction process the addition of new pipes of very low conductance compared with the nodal conductance of the joined nodes can be avoided, thus reducing the computing time. However tiny values for the nodal conductance must be avoided to reduce the error propagation, which is solved by fixing a minimum value (e.g. $10^{-10}$ $ft^2/s$). For large networks the reduction time tends to increase exponentially at the last stages. There is a significant scope to accelerate the model reduction process by re-ordering the nodes. The general problem of finding the best ordering is an NP-complete problem (George and Liu, 1989) but there are very efficient heuristic algorithms. The nodes can be pre-ordered in advance before the reduction starts (static re-ordering proposed by Cuthill and McKee, 1969) and dynamically (on-line) during the reduction process, for instance using minimum degree ordering algorithm proposed by Rose (1970) in his PhD.

George and Liu (1989) in their review paper suggest to apply two stages, first to fix the initial ordering with a static approach before passing it to a dynamic ordering routine (e.g. minimum degree ordering). Preliminary experience with water networks indicates that the optimized ordering can reduce the computing time more than 1000 times for big networks. At the end of the reduction procedure there are still many pipes with very low conductance (relatively to other pipes), such pipes can be removed from the model.

The accuracy of the reduced model over the wide range of changes in demands or in the control elements settings has not been proven formally but has been illustrated on many examples shown in the paper and other practical applications. It is important to remember to keep all control elements, including all pumps, valves and pipes with check valves or pipes directly controlled by rules in the reduced model.

# Case studies

The model reduction procedure, described above, was validated using a large number of real world networks. Two case studies are presented here, the first model is a small benchmark model and the second is a large-scale real water distribution system in UK. The first model was reduced using the node-by-node implementation whilst the second using the matrix approach.

***Case study 1***

This is a small scale "Network 1" of the EPANet examples (Rossman 2000) which consists of 12 pipes, 9 junctions, one pump, one tank and one reservoir as depicted in Figure 5a. Demands at the junctions vary according to a 24-hour demand pattern of which the first time step was used for the reduction procedure since it is equal to the average demand. The model was reduced to two junctions and two pipes as shown in Figure 5b. Junction 10 was not removed since it is connected to a pump and junction 12 was not removed since it is connected, by a pipe, to a non-demand junction, the tank. Therefore the pipe connected to the tank was not changed. The properties of the pipe connecting junctions 10 and 12 were changed as shown in Figure 5b. The total demand of the model was redistributed between junctions 10 and 12 to be 140.34 GPM and 959.66 GPM respectively. When comparing the water levels of the tank, over a period of 24 hours, between the full and reduced models it was found that the maximum deviation of the reduced model was 0.01ft.

***Case study 2***

The network is a typical large-scale regional network supplying many towns and cities with the schematic shown in Figure 6a. The model of the network includes 3535 nodes, 3279 pipes, 10 tanks, 7 reservoirs and 418 valves as illustrated in Table 1. The model reduction was required to calculate optimal pump and valve schedules for energy optimization, since the original model was too big to accomplish the optimization task.

The full model was subjected to the reduction procedure. Initially the calculations were carried out on an Intel i7 980X six-core processor without the use of parallel computing, i.e. only one CPU core was utilized; the calculation time was 1 hour and 35 minutes. Subsequently, a version of the algorithm which employs parallel computing was run on the same machine and the calculation time was reduced to 12 minutes. Finally using the minimum degree ordering  the computing time was reduced to few seconds. The schematic of the reduced model is depicted in Figure 6b, it contains 1023 nodes and 1340 pipes, keeping the tanks, reservoirs and valves. This corresponds to a reduction of 3.46 times in number of nodes and of 2.45 times in number of pipes as summarized in Table 1. The original model contains a significant number of valves (418). Some of these valves are permanently open and some are permanently closed; if they were replaced by equivalent pipes before carrying out the reduction, the ratio would be even higher. Extended period simulations were carried out for both full and reduced models with identical input data. The results are presented in Table 2 and in Figures 7 – 10.

The net tank flow balance was used to compare simulation results from the original and the reduced model. The tank flow for each tank was integrated over time horizon of 24 hours and denoted by $N_o$ for the original model and $N_r$ for the reduced model, $N_o$ and $N_r$ correspond also to the difference between the initial and the final volume of the tank in the respective models. The difference $d = N_o - N_r$ and the relative error $\frac{d}{V_t} \times 100\%$ with respect to the tank volume $V_t$ were used as a measure of quality of the reduced model and are presented in Table 2. For eight tanks, T1, T2, T4, T5, T6, T7, T8 and T9 the relative error is smaller than 1%. The smallest error is for T5 and is equal to 0.0016%, while the biggest error is for T3 and is equal to 6.6971%. The relative error between total mass balance in tanks in the original and simplified model is equal to 1.7909 % and is smaller than 2%. In order to improve accuracy for the ‘underperforming’ tanks, T3 and T12 it would be necessary to preserve more nodes in the neighborhood of these tanks. Additionally, the results are presented in graphical form as head trajectories for selected tanks. The head trajectories for the biggest tank T1 with capacity of 36 Ml are displayed in Fig. 7. The least accurate is T3 and the most accurate is T5 with trajectories displayed in Fig. 8 and Fig. 9, respectively. For comparison, the head trajectories for an ‘averagely accurate’ tank T9 are depicted in Fig. 10. The comparison of head at an important critical connection node is depicted in Figure 11, the approximation error is less than 0.1%. The flow patterns from individual sources in both models were also almost identical this is consistent with the method being invariant with respect to demands and spatial flow distribution.

## CONCLUSIONS

The method presented in the paper performs the model reduction by transferring the problem into a linear domain and then back into the non-linear domain and is well suited to hydraulic optimization studies. The user can select the nodes to be preserved in the model and the algorithm calculates the topology and the parameters of the components of the reduced network. The method is invariant with respect to the total load and operating point defined by the nodal variables. From the algorithmic point of view the method is very simple and fast. A model containing many thousands of components can be reduced in a matter of tens of minutes. The method is also very robust and has direct physical interpretation. The algorithm can be implemented on a computer or be executed manually and is very similar to finding an equivalent resistance for water network models. The case studies indicate that the reduced models are valid in a wide range of operating conditions, and are more accurate than straightforward linear models. The method was used to prepare many models for pressure control and optimal scheduling studies. Recently it has been applied with success by (Shamir and Salomons, 2008) to optimize the operation in real-time of Haifa water distribution network using the reduced model to speed up hydraulic calculations. The method has been applied to many practical case studies with astonishingly good results. If accuracy with respect to the tank trajectory was not satisfactory, it was rectified by the selection of additional nodes for the reduced model. Existing experience indicates that for the three important variables, tank trajectory, pump station flow and minimum pressure, it was always possible to achieve an error smaller than 2%. The future work will focus on implementation more efficient re-ordering algorithms and on the on-line implementation of the software where models can be reduced in real time to reflect changes in the water distribution system due to both planned and unexpected events.

**ACKNOWLEDGEMENTS**

This research was supported by EPSRC grant GR/N26005, and by the Spanish Ministry of Science and Technology, grant BIA2004-06444.

**Notation**

The following symbols are used in this paper:

| | | |
|---|---|---|
| $\mathbf{d}$ | = | vector of demands at connection nodes |
| $\mathbf{d}^0$ | = | value of the demand vector at the operating point |
| $\mathbf{d}^r$ | = | vector of demands in the reduced nonlinear model |
| $g_{n,m}$ | = | conductance of a pipe connected between nodes *n* and *m* |
| $\mathbf{h}_c$ | = | vector of head at connection nodes |
| $\mathbf{h}_c^0$ | = | value of the head vector at the operating point |
| $\mathbf{h}_f$ | = | vector of head at fixed grade nodes |

$\mathbf{J}$ = Jacobian of the full model

$\mathbf{J}^r$ = Jacobian of the reduced model

$j(n,m)$ = identifier of a pipe connected between nodes $n$ and $m$

$L$ = number of pipes

$L^r$ = number of pipes of the reduced model

$\mathbf{M}$ = Gaussian elimination matrix

$\mathbf{M}^{(r)}$ = matrix composed of last r rows of the Gaussian elimination matrix

$N_c$ = number of connection nodes in the full model

$N_f$ = number of fixed grade nodes in the full model

$N_n$ = a set of nodes connected to a node $n$ in the full model

$N_c^r$ = number of connection nodes in the reduced model

$N_n^r$ = a set of nodes connected to a node $n$ in the reduced model

$p_{n,m}$ = linearized conductance in the full model between nodes $n$ and $m$

$p_{n,m}^r$ = linearized conductance in the reduced model between nodes $n$ and $m$

$\mathbf{q}(\boldsymbol{\Delta}\mathbf{h})$ = vector of component flows as a function of the head drop in the full model

$\mathbf{q}^r(\boldsymbol{\Delta}\mathbf{h}^r)$ = vector of component flows as a function of the head drop in the reduced model

$S(\Delta h_{n,m)})$ = characteristic function of a pipe equation

$S_{\Delta h}(\Delta h_{n,m})$ = derivative of the characteristic function of a pipe equation

$\boldsymbol{\Delta}\mathbf{h}$ = vector of head drops in the full model

$\boldsymbol{\delta}\mathbf{d}$ = deviation of the demand in the full model from the operating point $\mathbf{d}^0$

$\boldsymbol{\delta}\mathbf{d}^r$ = deviation of the demand in the reduced model from the operating point

$\boldsymbol{\delta}\mathbf{h}_c$ = deviation of the heads at connection nodes from the operating point in the full model

$\boldsymbol{\delta}\mathbf{h}_c^r$ = deviation of the heads at connection nodes from the operating point in the reduced model

$\boldsymbol{\delta\Delta}\mathbf{h}$ = deviation of the head drop vector from the operating point in the full model

| | | |
|---|---|---|
| $\mathbf{\Lambda}$ | = | topology matrix in the full model |
| $\mathbf{\Lambda}_c$ | = | topology matrix corresponding to connection nodes in the full model |
| $\mathbf{\Lambda}_f$ | = | topology matrix corresponding to fixed grade nodes in the full model |
| $\mathbf{\Lambda}_c^r$ | = | topology matrix corresponding to connection nodes in the reduced model |
| $\mathbf{\Pi}$ | = | permutation matrix |

## REFERENCES


Balabanian, N. and Bickart, T. A., (1969). *Electrical Network Theory*, John Wiley and Sons, New York.

Bounds, P.L.M., Kahler, J. and Ulanicki, B. (2006). "Efficient energy management of a large-scale water supply system", *Civil Engineering and Environmental Systems*, Taylor & Francis, Vol. 23, No. 3 pp. 209 - 220.

Brdys, M. A. and Ulanicki, B., (1994). *Operational Control of Water Systems*, Prentice Hall International, New York, London, Toronto, Sydney, Tokyo, Singapore.

Broad, D.R., Maier, H.R., and Dandy, G.C. (2010). "Optimal operation of complex water distribution systems using metamodels." *Journal of Water Resources Planning and Management*, 136(4):433–443.

Cuthill, E. and McKee, J. (1969). "Reducing the bandwidth of sparse symmetric matrices"in Proc. 24th Nat. Conf. ACM, pages 157–172.

Deuerlein, J.W. (2008). "Decomposition model of a general water supply network graph." *Journal of Hydraulic Engineering*, 134(6):822–832.

Gill, P.E., Murray W. and Wright M.H. (1991). *Numerical Linear Algebra and Optimization, Volume 1,* Addison-Wesley, Redwood City, California.

George, A. and W.H. Liu, J.W.H. (1989). "*The Evolution of the minimum degree ordering algorithm*", SIAM Review, Vol. 31, No. 1, pp. 1-19.

Rao, Z. and Alvarruiz, F. (2007). "Use of an artificial neural network to capture the domain knowledge of a conventional hydraulic simulation model." *Journal Of Hydroinformatics*, 9(1):15–24.

Rose, D. J. (1970), "Symmetric elimination on sparse positive definite systems and potential flow network problem", Ph.D. thesis, Harvard University.

Rossman L.A. (2000). *EPANET 2 Users Manual,* US Environmental Protection Agency, Cincinnati, OH 45268.

Saldarriaga, J.G., Ochoa, S., Rodrguez, D., and Arbelez, J. (2008). "Water distribution network skeletonization using the resilience concept." in *10th Annual Water Distribution Systems Analysis Conference WDSA2008*, pages 852–864, Kruger National Park, South Africa.

Shamir, U. and Salomons, E. (2008). "Optimal Real-Time Operation of Urban Water Distribution Systems Using Reduced Models", *Journal of Water Resources Planning and Management*, ASCE, 134 (2), 181-185.

Todini, E.(2000). "Looped water distribution networks design using a resilience index based heuristic approach." *Urban Water*, 2(2):115–122.

Ulanicka, K., Bounds, P.L.M., Ulanicki, B. and Rance, J.P. (2001). "Pressure control of a large scale water distribution network with interacting water sources - a case study". In: *Proceedings of the 6th International Conference on Computing and Control for the Water Industry*, Ulanicki, B. et. al. (Eds), vol.2, RSP, Baldock - England, pp.41-53.

Ulanicki, B., Zehnpfund, A. and Martinez, F. (1996). "Simplification of Water Network Models", In: HYDRINFORMATICS '96, *Proceedings of the 2nd International Conference* *on Hydroinformatics*, ETH Zurich, Switzerland, 9-13 September, vol. 2, Rotterdam, A.A.Balkema, pp.493-500.

Walski, T.M., Chase, D.V., Savic, D.A., Grayman, W.M., Beckwith, S. and Koelle, E. (2003), *Advanced water distribution modeling and management*. Number v. 1. Haestead Press, 2003. ISBN 9780971414129.

Williams, G.S. and A. Hazen, 1906, *Hydraulic Tables*, John Wiley & Sons, New York.

Zehnpfund, A. and Ulanicki, B. (1993), *Water Network Model Simplification*, Research Report No 10, Water Software Systems, De Montfort University, Leicester, UK.

Zessler, U. and Shamir, U. (1989), *Optimal Operation of Water Distribution Systems*, Journal of Water Resources Planning and Management, 115(6).

**Table 1**. Statistics of Case study 2 model

| components | nodes | pipes | tanks | reservoirs | pumps | valves |
|---|---|---|---|---|---|---|
| | **Original model** | | | | | |
| | 3535 | 3279 | 10 | 7 | 19 | 418 |
| | **Reduced model** | | | | | |
| | 1023 | 1340 | 10 | 7 | 19 | 418 |
| reduction ratio | 3.46 | 2.45 | | | | |

Table 2**. Difference in the tank mass balance between the original and the reduced model**

| Tank | Tank volume $V_t$ | Difference in tank mass balance $d=N_o-N_r$ [Ml] | Relative error $d/V_t*100$ [%] |
|---|---|---|---|
| T1 | 36 | 0.0486 | 0.1351 |
| T2 | 24.3 | 0.0061 | 0.025 |
| T3 | 21.39 | 1.4325 | 6.6971 |
| T4 | 11.4 | 0.0324 | 0.2842 |
| T5 | 11.1 | 0.0002 | 0.0016 |
| T6 | 1.2 | 0.0104 | 0.8677 |
| T7 | 6.1 | 0.0009 | 0.014 |
| T8 | 11.6 | 0.0007 | 0.0063 |
| T9 | 21.8 | 0.0291 | 0.1336 |
| T12 | 27.3 | 1.5229 | 5.5784 |
| Total | 172.19 | 3.0838 | 1.7909 |