Comparison of two linearization schemes for the nonlinear bending problem of a beam pinned at both ends

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Abstract

The nonlinear bending problem of a constant cross-section simply supported beam pinned at both ends and subject to a uniformly distributed load $q(x)$ is analyzed in detail. The numerical integration of the two-point boundary value problem (BVP) derived for the nonlinear Timoshenko beam is tackled through two different linearization schemes, the multi-step transversal linearization (MTrL) and the multi-step tangential linearization (MTnL), proposed by Viswanath and Roy (2007). The fundamentals of these linearization techniques are to replace the nonlinear part of the governing ODEs through a set of conditionally linearized ODE systems at the nodal grid points along the neutral axis, ensuring the intersection between the solution manifolds (transversally in the MTrL and tangentially in the MTnL). In this paper, the solution values are determined at grid points by means of a centered finite differences method with *multipoint linear constraints* (Keller, 1969), and a simple iterative strategy. The analytical solution for this kind of bending problem, including the extensional effects, can be worked out by integration of the governing two-point BVP equations (Monleón et al., 2008). Finally, the comparison of analytical and numerical results shows the better ability of MTnL with the proposed iterative strategy to reproduce the theoretical behavior of the beam for each load step, because the restraint of equating derivatives in MTnL leads to further closeness between solution paths of the governing ODEs and the linearized ones, in comparison with MTrL. This result is opposed to the conclusion reached in Viswanath and Roy (2007), where the relative errors produced by MTrL are said to be smaller than the MTnL ones for the simply supported beam and the tip-loaded cantilever beam problems.

Key words: Multi-step transversal linearization (MTrL), Multi-step tangential linearization (MTnL), Locally transversal linearization (LTL), Timoshenko beam, Beam pinned at both ends, Nonlinear bending problem, Boundary value problem (BVP), Centered finite differences, Multi-point linear constraints

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1. Introduction

The geometrically nonlinear analysis of flexible beams, as the obtention of its physically reasonable configurations, constitutes an issue of broad technological and practical interest in scientific and engineering fields like robotics, biomechanics and aeronautics. Bending of slender rods made of steel or polymeric and plastic materials, are cases in which fairly large displacements arise without exceeding the yield strength of the material.

The numerical solution of geometrically nonlinear beams or rods is often worked out by the finite element method using, for example, lagrangian formulations (Zienkiewicz and Taylor, 1991), *co-rotational* approaches (Crisfield, 1991, 1997; Felippa and Haugen, 2005), or geometrically exact formulations (e.g. Cardona and Geradin (1988) or Simo and Vu-Quoc (1986)).

An alternative to the numerical treatment is the direct integration of the governing boundary value problem (BVP). The classical analytical solutions are usually based in the non-extensible rod assumption. Such solutions may be expressed in terms of Jacobi elliptical integrals (Love, 1944, §263). If the rod is regarded as extensible, the magnitude of displacements and strains has to be limited in order to get a explicit solution (Monleón et al., 2008).

Amongst the special cases which have been solved, we can find a tip loaded cantilever beam with constant cross-section (Bishopp and Drucker, 1945; Lee, 2002; Mattiason, 1981), a three-point or four-point loaded simply supported beam (Ohtsuki, 1986a, 1986b) and square frames with rigid (Ohtsuki and Ellyin, 2000) or two-pinned (Mattiason, 1981) joints, diagonally loaded on two opposite corners. All of them are based on the non-extensibility assumption.

The difficulties which arise in the integration of nonlinear BVPs can be overcome by carrying out a previous treatment of the system by means of a semi-analytical technique. In this way the Multi-step Transversal Linearization Method (MTrL), introduced by Ramachandra and Roy (2001a) (namely, Locally Transversal Linearization or LTL-zeroth level), allows to simplify the ODE system by replacing the nonlinear vector field by a set of conditionally linear ODE systems in grid points along the independent variable. This replacement is provided by the transversal intersection produced between both solution manifolds in every grid point.

A scope to improve the LTL-zeroth level lies in deriving the initial ODE system and equating the solution paths and its derivatives in grid points. The equality of derivatives increases the closeness between nonlinear and LTL-based solution paths. This new scheme of linearization is called LTL-first level by Ramachandra and Roy (2001a) and Tangential Linearization (MTnL) by Viswanath and Roy (2007). In the present study, a modified MTnL is applied without performing the derivation of the ODE-s system.

Later, Viswanath and Roy (2007) apply the MTrL y MTnL methods to a tip loaded cantilever beam with constant cross-section and compare the output results with the analytical ones (Mattiason, 1981). Likewise, they compare the two linearization techniques on a constant cross-section beam pinned at both ends with a uniformly distributed load.

The results included in this paper may be regarded as an extension of that obtained in Ramachandra and Roy (2001a) and Viswanath and Roy (2007) for the simply supported beam problem, with special emphasis in its coherence with the analytical solution. In the current approach, solving procedure has been modified by using a centered finite difference method with multi-point linear constraints, introduced by Keller (1969). This method has the following advantages compared to the one used by Viswanath and Roy (2007):

1. It avoids the evaluation of the Magnus series expansion, which is a time-consuming task, and
2. It makes unnecessary to solve a nonlinear system of equations by the Newton-Raphson method, ensuring a stable convergence process.

The paper is organized as follows: in section 2 the governing equations of the beam pinned at both ends are worked out. In section 3 the MTrL procedure is derived for this problem. Section 4 deals with the derivation of the MTnL of the equations. In section 5 the algorithm for the numerical solution is explained. The analytical solution of the problem can be found in section 6. Numerical results are shown in section 7, and the conclusions can be found in section 8.

2. Simply supported beam analytical approach

In this section the governing nonlinear BVP for a constant cross-section beam pinned at both ends with uniformly distributed load is analytically derived by means of a variational technique. Horizontal and vertical displacements are constrained in both ends. The axes and sign convention represented in figure 1 will be taken.

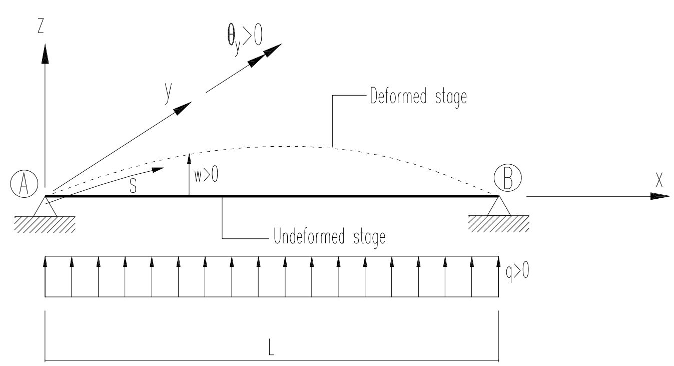


Figure 1: General diagram of simply supported beam

The main assumptions are:

1. The rod is supposed to be extensible. Coupling between extension and bending is taken into account.
2. The model includes transverse shear deformation (Timoshenko beam) of the cross-sections.
3. Moderate (but not small) displacements and rotations will be regarded – refer to the definition of the strain ε_x^* in eq. (2).

The kinematics for the current problem can be written as follows [1]:

$$u^* = u + z\theta_y \tag{1a}$$

$$w^* = w \tag{1b}$$

The non-linear terms of the Green-Lagrange strain tensor (equations (7.37) Monleón (1999)) are suitably simplified for the plane deformation of an initial straight beam [2]:

$$\begin{aligned} \varepsilon_x^* &= \frac{\partial u^*}{\partial x} + \frac{1}{2}\left(\frac{\partial w^*}{\partial x}\right)^2 = u' + z\theta_y' + \frac{1}{2}w'^2 \\ \gamma_{xz}^* &= \frac{\partial u^*}{\partial z} + \frac{\partial w^*}{\partial x} = w' + \theta_y \end{aligned} \tag{2}$$

Linear mechanical behavior is taken into account:

$$\boldsymbol{\sigma}^* = \mathbf{D}\boldsymbol{\epsilon}^* = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} \boldsymbol{\epsilon}^* \tag{3}$$

where $\boldsymbol{\sigma}^* = \{\sigma_x^* \ \tau_{xz}^*\}^t$ is the stress vector, $\boldsymbol{\epsilon}^* = \{\varepsilon_x^* \ \gamma_{xz}^*\}^t$ is the strain vector, and E and G are the Young's modulus and shear modulus, respectively.

The general expression for the Lagrangian of the problem, adopting a variational approach (for more details, see Viswanath and Roy (2007) and Monleón (1999)) will be:

$$\mathcal{L} = \mathcal{U} + \mathcal{F} + \mathcal{G} = \frac{1}{2}\int_V \boldsymbol{\epsilon}^{*T}\boldsymbol{\sigma}^* dV - \int_0^L qwdx - [\widetilde{\mathbf{f}}_A^T\mathbf{u}_A + \widetilde{\mathbf{f}}_B^T\mathbf{u}_B] \tag{4}$$

where:

$\mathcal{L}$ =	Total potential energy (Lagrangian function)
$\mathcal{U}$ =	Strain energy of the rod, calculated as $\int_0^L U dx$
	with U defined as the linear strain energy density along the neutral axis of the rod
$\mathcal{F}$ =	Potential of the uniformly distributed load
$\mathcal{G}$ =	Potential of the end reactions (zero in current case).

[1] We denote with an asterisk (*) those variables related to a generic point on the cross-section. Generalized (model) variables have no asterisk, they are related to the neutral axis and depend only of x.

[2] ()' indicates a derivative with respect to x

In the equilibrium configuration the first variation of the Lagrangian will be zero:

$$
\begin{aligned}
\delta\mathcal{L} =& \delta\mathcal{U} + \delta\mathcal{F} + \delta\mathcal{G} = \int_0^L [U_u \delta u + U_{u'} \delta u' + U_w \delta w + U_{w'} \delta w' + U_\theta \delta\theta_y + U_{\theta'} \delta\theta'_y + \\
&+ F_u \delta u + F_w \delta w + F_\theta \delta\theta_y] dx - [\widetilde{f}_{xA} \delta u_A + \widetilde{f}_{zA} \delta w_A + \widetilde{M}_A \delta\theta_{yA} + \widetilde{f}_{xB} \delta u_B + \\
&+ \widetilde{f}_{zB} \delta w_B + \widetilde{M}_B \delta\theta_{yB}] = 0
\end{aligned} \tag{5}
$$

In this expression

$U_u, U_{u'}, U_w, U_{w'}, U_\theta, U_{\theta'}$ = are the partial derivatives of the strain energy linear density, $U(u, u', w, w', \theta_y, \theta'_y)$, with respect to the generalized displacements.

F_u, F_w, F_θ = are the partial derivatives of the uniformly distributed load potential linear density F, with respect to the displacements.

Applying integration by parts and rearranging terms in eq. (5), the equilibrium equations in the Euler-Lagrange form are:

$$
\left[F_u + U_u - \frac{\partial U_{u'}}{\partial x}\right] = 0, \quad \left[F_w + U_w - \frac{\partial U_{w'}}{\partial x}\right] = 0, \quad \left[F_\theta + U_\theta - \frac{\partial U_{\theta'}}{\partial x}\right] = 0, \tag{6}
$$

and the boundary conditions are:

$$
\begin{aligned}
u_A = w_A &= 0 & u_B = w_B &= 0 \\
(U_{\theta'}|_A + \widetilde{M}_A) &= 0 & (U_{\theta'}|_B - \widetilde{M}_B) &= 0
\end{aligned} \tag{7}
$$

In the present case, eqs. (6) can be written as:

$$
\begin{cases}
H_1 u'' + H_2 \theta''_y + H_1 w' w'' = 0 \\
H_1 u'' w' + H_1 u' w'' + H_2 w'' \theta'_y + H_2 w' \theta''_y + \frac{3}{2} H_1 w'^2 w'' + \overline{H}_1 (w'' + \theta'_y) + q = 0 \\
H_2 u'' + H_3 \theta''_y + H_2 w' w'' - \overline{H}_1 (w' + \theta_y) = 0
\end{cases} \tag{8}
$$

where constants H_i and $\overline{H}_1$ have been defined as:

$$
\begin{aligned}
H_1 &= EA & H_2 &= ES_y \\
H_3 &= EI_y & \overline{H}_1 &= GA_Q
\end{aligned} \tag{9}
$$

On the other hand, assuming that cross-section centroids are on the neutral axis, conditions (7) become the following separated boundary conditions:

$$
\begin{aligned}
u(0) = w(0) &= 0 & u(L) = w(L) &= 0 \\
\theta'_y(0) &= 0 & \theta'_y(L) &= 0
\end{aligned} \tag{10}
$$

2.1. Matrix formulation of the boundary value problem

For the purpose of later linearization, the following vector of generalized displacements is defined:

$$\mathbf{u} \triangleq \begin{Bmatrix} u \\ w \\ \theta_y \\ u' \\ w' \\ \theta'_y \end{Bmatrix} = \begin{Bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{Bmatrix}. \tag{11}$$

Eq. (8) may be transformed into:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_1 & H_1 w_2 & H_2 \\ 0 & 0 & 0 & H_1 w_2 & \widetilde{H} & H_2 w_2 \\ 0 & 0 & 0 & H_2 & H_2 w_2 & H_3 \end{bmatrix} \begin{Bmatrix} u'_1 \\ w'_1 \\ \theta'_1 \\ u'_2 \\ w'_2 \\ \theta'_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\overline{H}_1 \\ 0 & 0 & \overline{H}_1 & 0 & \overline{H}_1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -q \\ 0 \end{Bmatrix}, \tag{12}$$

where

$$\widetilde{H} = \overline{H}_1 + H_1 u_2 + H_2 \theta_2 + \frac{3}{2} H_1 w_2^2. \tag{13}$$

Grouping by blocks leads to:

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}(\mathbf{u}_2) \end{bmatrix} \begin{Bmatrix} \mathbf{u}'_1 \\ \mathbf{u}'_2 \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{q} \end{Bmatrix} \tag{14}$$

$$\boxed{\begin{Bmatrix} \mathbf{u}'_1 \\ \mathbf{u}'_2 \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{B}^{-1}\mathbf{C}_{21} & \mathbf{B}^{-1}\mathbf{C}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{B}^{-1}\mathbf{q} \end{Bmatrix}}, \tag{15}$$

where:

$$\mathbf{B}(\mathbf{u}_2) = \begin{bmatrix} H_1 & H_1 w_2 & H_2 \\ H_1 w_2 & \widetilde{H} & H_2 w_2 \\ H_2 & H_2 w_2 & H_3 \end{bmatrix} \quad \mathbf{C}_{21} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{H}_1 \end{bmatrix} \quad \mathbf{C}_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\overline{H}_1 \\ 0 & \overline{H}_1 & 0 \end{bmatrix}. \tag{16}$$

Similarly, boundary conditions (10) take the form:

$$\boxed{\mathbf{M}_1 \mathbf{u}(0) + \mathbf{M}_k \mathbf{u}(L) = \mathbf{0}}, \tag{17}$$

where the matrices $\mathbf{M}_1, \mathbf{M}_k$ are defined like:

$$\mathbf{M}_1 = \left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} \mathbf{M}_{11} \\ \mathbf{0} \end{bmatrix} \qquad \mathbf{M}_k = \left[\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}_{k2} \end{bmatrix}. \tag{18}$$

The form (15) of the governing ODE system has the advantage of grouping all the nonlinearity in the matrix $\mathbf{B}(\mathbf{u}_2)$.

3. Multi-step Transversal Linearization Method (MTrL)

Multi-step Transversal Linearization Method (Ramachandra and Roy, 2001a, 2001b, 2002; Viswanath and Roy, 2007) consists of replacing the nonlinear vector field by a set of conditionally linear ODE systems in such manner that transversal intersections of the solution manifolds of the nonlinear problem and the linearized one are provided on nodal points.

The $[0, L]$ domain may be discretized on k equal intervals split in p subintervals of the same step size $h = s^j_{i+1} - s^j_i$ $\forall j = 1, 2, \cdots, k$ y $\forall i = 1, 2, \cdots, p$ (see figure 2).

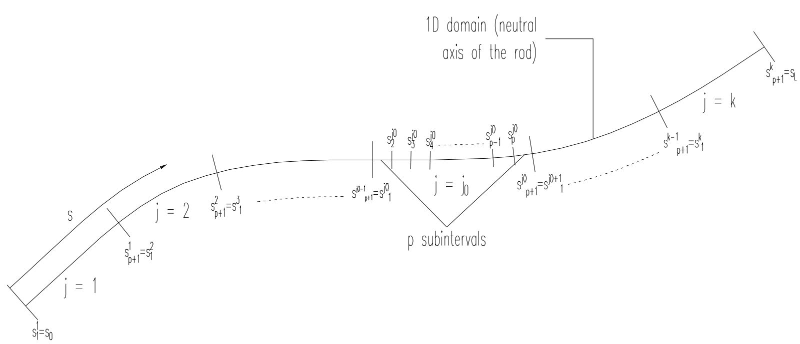


Figure 2: Discretizing of 1D domain

Equation (15) is replaced by a set of conditionally linear ODE systems in each nodal point in the so called *fully implicit form* (refer to Viswanath and Roy (2007)):

$$\begin{Bmatrix} \overline{\mathbf{u}}'_1 \\ \overline{\mathbf{u}}'_2 \end{Bmatrix} = \alpha \begin{bmatrix} \mathbf{0} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{Bmatrix} \overline{\mathbf{u}}_1 \\ \overline{\mathbf{u}}_2 \end{Bmatrix} + (1-\alpha) \begin{Bmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{Bmatrix} \tag{19}$$

where α is the so called *implicitness parameter*. The MTrL based solution at each nodal point $\overline{\mathbf{u}}_i$ approaches that of the nonlinear problem $\mathbf{u}_i$.

The equivalence of systems (15) and (19) in every nodal point leads to:

$$\mathbf{u}_{2i} = \alpha \mathbf{A}_{12i} \overline{\mathbf{u}}_{2i} + (1-\alpha)\mathbf{F}_{1i} \tag{20}$$

$$\mathbf{B}_i^{-1}\mathbf{C}_{21i}\mathbf{u}_{1i} + \mathbf{B}_i^{-1}\mathbf{C}_{22i}\mathbf{u}_{2i} + \mathbf{B}_i^{-1}\mathbf{q} = \alpha \mathbf{A}_{21i}\overline{\mathbf{u}}_{1i} + \alpha \mathbf{A}_{22i}\overline{\mathbf{u}}_{2i} + (1-\alpha)\mathbf{F}_{2i} \tag{21}$$

Forcing the transversal intersection of manifolds, $\overline{\mathbf{u}}_{1i} = \mathbf{u}_{1i}$ and $\overline{\mathbf{u}}_{2i} = \mathbf{u}_{2i}$, and equating coefficients of α we obtain from (20):

$$\mathbf{F}_{1i} = \mathbf{u}_{2i} \tag{22}$$

$$\mathbf{A}_{12i}\overline{\mathbf{u}}_{2i} - \mathbf{F}_{1i} = \mathbf{0} \rightarrow \mathbf{A}_{12i} = \mathbf{I} \tag{23}$$

similarly, from (21):

$$\mathbf{B}_i^{-1}\mathbf{C}_{21i}\mathbf{u}_{1i} + \mathbf{B}_i^{-1}\mathbf{C}_{22i}\mathbf{u}_{2i} + \mathbf{B}_i^{-1}\mathbf{q} = \mathbf{F}_{2i} \tag{24}$$

$$\mathbf{A}_{21i}\overline{\mathbf{u}}_{1i} + \mathbf{A}_{22i}\overline{\mathbf{u}}_{2i} - \mathbf{F}_{2i} = 0 \tag{25}$$

Equation (24) determines the $\mathbf{F}_{2i}$ values on every grid point. Nonetheless, condition (25) doesn't allow to establish with uniqueness every element in matrices $\mathbf{A}_{21i}$ and $\mathbf{A}_{22i}$. In the current case, it is advisable to give them the same structure as that of $\mathbf{B}^{-1}\mathbf{C}_{21}$ and $\mathbf{B}^{-1}\mathbf{C}_{22}$ in eq. (15). In order to do that, we evaluate previously:

$$\mathbf{B}^{-1}\mathbf{C}_{21} = \overline{H}_1 \begin{bmatrix} 0 & 0 & \varphi_{13} \\ 0 & 0 & \varphi_{23} \\ 0 & 0 & \varphi_{33} \end{bmatrix} \qquad \mathbf{B}^{-1}\mathbf{C}_{22} = \overline{H}_1 \begin{bmatrix} 0 & \varphi_{13} & -\varphi_{12} \\ 0 & \varphi_{23} & -\varphi_{22} \\ 0 & \varphi_{33} & -\varphi_{32} \end{bmatrix} \tag{26}$$

where the notation $\mathbf{B}^{-1} = [\varphi_{ij}]$ is used for convenience. The following matrixes are defined with six linearly independent parameters in every grid point:

$$\mathbf{A}_{21} = \begin{bmatrix} 0 & 0 & \xi_1 \\ 0 & 0 & \xi_2 \\ 0 & 0 & \xi_3 \end{bmatrix} \qquad \mathbf{A}_{22} = \begin{bmatrix} 0 & \xi_1 & \zeta_1 \\ 0 & \xi_2 & \zeta_2 \\ 0 & \xi_3 & \zeta_3 \end{bmatrix} \tag{27}$$

Developing now eq. (24), we arrive easily to:

$$\mathbf{F}_{2i} = \overline{H}_1(\theta_{1i} + w_{2i}) \begin{Bmatrix} \varphi_{13} \\ 0 \\ \varphi_{33} \end{Bmatrix}_i - (\overline{H}_1\theta_{2i} + q) \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \\ 0 \end{Bmatrix}_i . \tag{28}$$

We develop also (25) as:

$$(\theta_{1i} + w_{2i}) \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix}_i + \theta_{2i} \begin{Bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{Bmatrix}_i = \mathbf{F}_{2i} \tag{29}$$

Comparing (28) and (29), the following scheme is taken:

$$\begin{aligned} \xi_{1i} &= \overline{H}_1 \varphi_{13i} \\ \xi_{2i} &= 0 \\ \xi_{3i} &= \overline{H}_1 \varphi_{33i} \end{aligned} \tag{30}$$

Consequently, it follows that:

$$\theta_{2i} \begin{Bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{Bmatrix}_i = -(\overline{H}_1 \theta_{2i} + q) \begin{Bmatrix} \varphi_{12} \\ \varphi_{22} \\ 0 \end{Bmatrix}_i \rightarrow \begin{aligned} \zeta_{1i} &= -\left(\overline{H}_1 + \tfrac{q}{\theta_{2i}}\right) \varphi_{12i} \\ \zeta_{2i} &= -\left(\overline{H}_1 + \tfrac{q}{\theta_{2i}}\right) \varphi_{22i} \\ \zeta_{3i} &= 0 \end{aligned} , \tag{31}$$

except if $\theta_{2i} = 0$. Such singularity will be avoided by taking:

$$\begin{aligned} \zeta_{1i} &= -\overline{H}_1 \varphi_{12i} \\ \zeta_{2i} &= -\overline{H}_1 \varphi_{22i} \\ \zeta_{3i} &= 0 \end{aligned} \tag{32}$$

Finally, the following coefficient matrixes are obtained:

$$\mathbf{A}_{21i} = \begin{bmatrix} 0 & 0 & \overline{H}_1 \varphi_{13i} \\ 0 & 0 & 0 \\ 0 & 0 & \overline{H}_1 \varphi_{33i} \end{bmatrix} \qquad \mathbf{A}_{22i} = \begin{bmatrix} 0 & \overline{H}_1 \varphi_{13i} & -\left(\overline{H}_1 + \tfrac{q}{\theta_{2i}}\right) \varphi_{12i} \\ 0 & 0 & -\left(\overline{H}_1 + \tfrac{q}{\theta_{2i}}\right) \varphi_{22i} \\ 0 & \overline{H}_1 \varphi_{33i} & 0 \end{bmatrix} \tag{33}$$

4. Multi-step Tangential Linearization Method (MTnL)

In this section the alternative Multi-step Tangential Linearization Method (MTnL) (Viswanath and Roy, 2007), is developed. The nonlinear system is replaced by a linearized system which tangent space would be the same of the first one, in such manner that both solution manifolds are tangent to each other at pre-selected points.

The approach to develop MTnL is analogue to MTrL in the fully implicit form. The left member in eq. (14) is replaced by:

$$\begin{Bmatrix} \overline{\mathbf{u}}_1' \\ \alpha_1 \mathbf{Z} \overline{\mathbf{u}}_2' + (1 - \alpha_1) \mathbf{F} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \begin{Bmatrix} \overline{\mathbf{u}}_1 \\ \overline{\mathbf{u}}_2 \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{q} \end{Bmatrix} \tag{34}$$

Tangential intersection of manifolds in grid points corresponds with $\overline{\mathbf{u}}_{1i}' = \mathbf{u}_{1i}'$ and $\overline{\mathbf{u}}_{2i}' = \mathbf{u}_{2i}'$. Equivalence between (14) and (34) in nodal points leads to:

$$\mathbf{B}_i \mathbf{u}_{2i}' = \alpha_1 \mathbf{Z}_i \overline{\mathbf{u}}_{2i}' + (1 - \alpha_1) \mathbf{F}_i . \tag{35}$$

Equating coefficients for α_1 gives:

$$\mathbf{B}_i \mathbf{u}'_{2i} = \mathbf{F}_i \tag{36a}$$

$$\mathbf{0} = \mathbf{Z}_i \overline{\mathbf{u}}'_{2i} - \mathbf{F}_i \tag{36b}$$

And equating both conditions:

$$\mathbf{Z}_i = \mathbf{B}_i \tag{37}$$

Obviously the MTnL admits other alternative linearizations, although expressions (36a) and (37) provide a very suitable formulation for its later numerical processing.

The tangential linearized expression of system (34) may be written as:

$$\begin{Bmatrix} \overline{\mathbf{u}}'_1 \\ \overline{\mathbf{u}}'_2 \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \alpha \mathbf{Z}^{-1}\mathbf{C}_{21} & \alpha \mathbf{Z}^{-1}\mathbf{C}_{22} \end{bmatrix} \begin{Bmatrix} \overline{\mathbf{u}}_1 \\ \overline{\mathbf{u}}_2 \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{Z}^{-1}[\alpha \mathbf{q} + (1-\alpha)\mathbf{F}] \end{Bmatrix}, \tag{38}$$

where we have taken a new *parameter of implicitness* $\alpha = 1/\alpha_1$.

5. Numerical solution

5.1. Description of the iterative strategy

Numerical results are obtained by linearization of the problem (15). An initial approach to the solution is evaluated in nodal points. Then, the multi-point finite difference method (section 5.2, Keller (1969)) is applied on every interval. Taking into account continuity restraints between intervals [3] ($\mathbf{u}^j_{p+1} = \mathbf{u}^{j+1}_1$ with $j = 1, 2, \cdots, k-1$) and the boundary conditions (17) a new solution is obtained and compared with the initial one in order to verify convergence.

The iterative strategy adopted is outlined in the flow diagram depicted in figure 3.

Convergence criteria is based on evaluating the differences for all components of nodal displacements in two consecutive steps and comparing the modulus of every difference vector with the absolute error $\varepsilon = 10^{-3}$.

5.2. Multi-point finite difference method

In order to obtain numerical values of the solution on grid points in every step, the multi-point finite difference method (Keller, 1969) was adopted. It solves boundary value problems with "boundary" conditions written as *linear constraints* of solution values in nodal points and end points. The fundamentals are outlined here.

[3] Subscripts indicate the current point inside the interval and superscripts the current interval inside the domain $[0, L]$

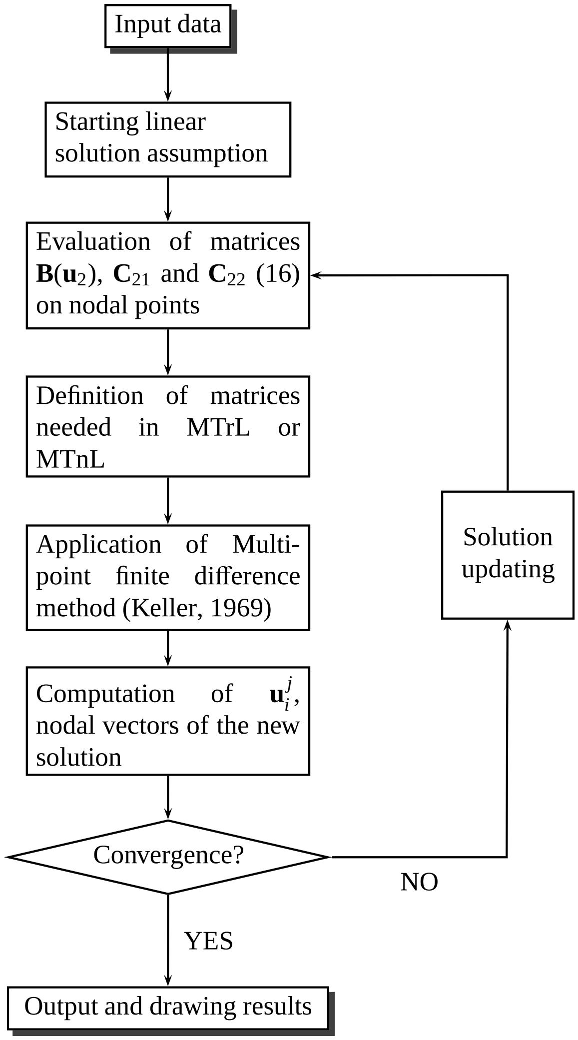


Figure 3: Iterative strategy employed in obtaining values

The discretization depicted in figure 2 is adopted. The initial approach to the solution in nodal points is denoted by ω_j ($j = 2, 3, \ldots, k$). On intermediate points

$$\mathbf{u}_j = \mathbf{u}(s_j) = \omega_j \quad j = 2, 3, \ldots, k. \tag{39}$$

Addend to both members in eq. (17), we obtain the *linear constraints*:

$$\begin{gathered} \mathbf{M}_1\mathbf{u}_1 + \sum_{j=2}^{k-1} \mathbf{u}_j + \mathbf{M}_k\mathbf{u}_k = \sum_{j=2}^{k-1} \omega_j \\ \sum_{j=1}^{k} \mathbf{M}_j\mathbf{u}_j = \sum_{j=2}^{k-1} \omega_j = \widehat{\mathbf{\Omega}} \end{gathered} \tag{40}$$

where:

$$\mathbf{M}_j = \mathbf{I} \quad j = 2, 3, \ldots, k-1 \tag{41}$$

Equations (40) can not guarantee that the boundary conditions (17) are kept within the iterative procedure. Therefore, the later equations had to be forced in every step.

The first order MTrL (19) or MTnL-linearized (34) ODE-s system is approached through the centered finite difference scheme [4]:

$$\frac{\mathbf{u}_j - \mathbf{u}_{j-1}}{h} - \frac{1}{2}\mathbf{B}(s_{j-1/2})(\mathbf{u}_j + \mathbf{u}_{j-1}) = \mathbf{F}(s_{j-1/2}) \quad j = 2, 3, \ldots, k \tag{42}$$

where

$$s_{j-1/2} = s_j - \frac{h}{2} = s_{j-1} + \frac{h}{2} \quad j = 2, 3, \ldots, k.$$

Taking now the definitions

$$\mathbf{R}_j = \mathbf{I} - \frac{h}{2}\mathbf{B}(s_{j-1/2}) \tag{43}$$

$$\mathbf{P}_j = [\mathbf{I} - \frac{h}{2}\mathbf{B}(s_{j-1/2})]^{-1}[\mathbf{I} + \frac{h}{2}\mathbf{B}(s_{j-1/2})] = \mathbf{R}_j^{-1}[2\mathbf{I} - \mathbf{R}_j], \tag{44}$$

eq. (42) becomes finally

$$\boxed{\mathbf{u}_j = \mathbf{P}_j\mathbf{u}_{j-1} + h\mathbf{R}_j^{-1}\mathbf{F}(s_{j-1/2})} \quad j = 2, 3, \ldots, k. \tag{45}$$

To compute the components in $\mathbf{u}_1$ which are not included in boundary conditions (17), the next k matrices are recursively defined:

$$\begin{cases} \mathbf{S}_k &= \widehat{\mathbf{B}}_k \\ \mathbf{S}_{j-1} &= \widehat{\mathbf{B}}_{j-1} + \mathbf{S}_j\mathbf{P}_j \quad j = k, k-1, \ldots, 3, 2 \end{cases} \tag{46}$$

Handling suitably definitions (46), we may write

$$\mathbf{S}_1 = \sum_{j=1}^{k} \widehat{\mathbf{B}}_j\mathbf{Z}_j \tag{47}$$

where

$$\begin{cases} \mathbf{Z}_1 &= \mathbf{I} \\ \mathbf{Z}_j &= \mathbf{P}_j\mathbf{Z}_{j-1} \quad j = 2, 3, \ldots, k \end{cases} \tag{48}$$

Finally

$$\boxed{\mathbf{u}_1 = \mathbf{S}_1^{-1}[\widehat{\mathbf{\Omega}} - \sum_{j=2}^{k} h\mathbf{S}_j\mathbf{R}_j^{-1}\mathbf{F}(s_{j-1/2})]} \tag{49}$$

Expressions (49) and (45) lead to the new solution vectors on nodal points in every step.

[4] All values $\mathbf{B}(s_{j-1/2})$ and $\mathbf{F}(s_{j-1/2})$ needed on intermediate points were evaluated by means of Lagrange interpolation (see Viswanath and Roy (2007), Roy and Kumar (2005))

6. The analytical solution

6.1. Boundary value problem approach. Mixed formulation

Writing the total potential of the problem (4) in the form:

$$\mathcal{L} = \int_0^L \widetilde{\mathcal{F}} dx - [\widetilde{\mathbf{f}}_A^T \mathbf{u}_A + \widetilde{\mathbf{f}}_B^T \mathbf{u}_B] \tag{50}$$

where $\widetilde{\mathcal{F}}$ is the potential linear density along the neutral axis, the general form of the equilibrium equations (Euler-Lagrange form, see Monleón (1999)) can be written as a second order n ODE-s system:

$$\widetilde{\mathcal{F}}_{\mathbf{u}} - \frac{\partial}{\partial x} \widetilde{\mathcal{F}}_{\mathbf{u}'} = \mathbf{0} \tag{51}$$

subject to the $2n$ boundary conditions:

$$\left[\widetilde{\mathcal{F}}_{\mathbf{u}'}|_{x=x_B} - \widetilde{\mathbf{f}}_B\right]^T \delta\mathbf{u}_B - \left[\widetilde{\mathcal{F}}_{\mathbf{u}'}|_{x=x_A} + \widetilde{\mathbf{f}}_A\right]^T \delta\mathbf{u}_A = \mathbf{0} \tag{52}$$

where $\widetilde{\mathcal{F}}_{\mathbf{u}}$ is a vector with components $\frac{\partial \widetilde{\mathcal{F}}}{\partial u_i}$.

The energy definition of generalized stresses, eq. (51), can be transformed in a first order system of $2n$ ODE-s:

$$\frac{\partial \widetilde{\mathcal{F}}}{\partial \mathbf{u}} - \mathbf{f}' = \mathbf{0} \qquad \frac{\partial \widetilde{\mathcal{F}}}{\partial \mathbf{u}'} - \mathbf{f} = \mathbf{0} \tag{53}$$

By using the simplified expressions (2) of the Green strain tensor, the system becomes:

$$EA\left(u' + \frac{1}{2}w'^2\right) - N_x = 0 \tag{54a}$$

$$EA\left(u' + \frac{1}{2}w'^2\right)w' + GA_Q(w' + \theta_y) - Q_z = 0 \tag{54b}$$

$$EI_y\theta_y' - M_y = 0 \tag{54c}$$

$$q_x + N_x' = 0 \tag{54d}$$

$$q_z + Q_z' = 0 \tag{54e}$$

$$GA_Q(w' + \theta_y) - M_y' = 0 \tag{54f}$$

Similarly, boundary conditions (52) become:

$$\begin{array}{ll} u(0) = w(0) = 0 & u(L) = w(L) = 0 \\ M_y(0) = 0 & M_y(L) = 0 \end{array} \tag{55}$$

It is convenient to point out that in system (54) displacements u, w, θ_y and section forces N_x, Q_z, M_y are unknown functions. This equation form will be referred to as the *mixed formulation of the BVP* of a simply supported beam pinned at both ends. The analytical solution of this formulation has been adopted as a reference for comparing results for the generalized displacements u_1, w_1, θ_1.

6.2. *Obtaining the analytical solution*

Most of the analytical solutions for the pinned beam problem in the moderately large displacements range (Ohtsuki, 1986a, 1986b) are based in the non-extensible rod assumption, and they get as reference the solution formulated in Love (1944) by means of Jacobi elliptic integrals. In this paper, we will adopt as starting point the solution of Monleón et al. (2008) for eqs. (54) which considers extension of the rod, and will extend the results to the uniformly distributed load case.

Replacing eq. (54a) in (54b) and regarding that q_x vanishes in (54d), the bending problem represented by eqs. (54b), (54c), (54e) and (54f) (which form a linear system) may be decoupled from the axial one and solved independently. Conversely, decoupling is not possible due to the setting of eq. (54a). We will say then, the bending problem is *partially decoupled* of the axial problem.

The fundamental system matrix (FSM) of the bending problem is obtained by direct integration:

$$\mathbf{E}_h(\xi) = \mathbf{G}(\xi)\mathbf{E}_0 = \begin{bmatrix} 1 & -\frac{L}{\rho\omega}\sinh\omega\xi & \frac{L}{N_0\rho\omega}(\rho\omega\xi - \sinh\omega\xi) & \frac{1}{N_0}(1-\cosh\omega\xi) \\ 0 & \cosh\omega\xi & \frac{1}{N_0}(\cosh\omega\xi - 1) & \frac{\rho\omega}{N_0 L}\sinh\omega\xi \\ 0 & 0 & 1 & 0 \\ 0 & \frac{N_0 L}{\rho\omega}\sinh\omega\xi & \frac{L}{\rho\omega}\sinh\omega\xi & \cosh\omega\xi \end{bmatrix} \begin{Bmatrix} w_0 \\ \theta_{y0} \\ Q_{z0} \\ M_{y0} \end{Bmatrix} \tag{56}$$

with the following meaning for the variables:

$\mathbf{E}_h(\xi)$	state vector of the homogeneous bending problem, $\{w(\xi)\ \ \theta_y(\xi)\ \ Q_z(\xi)\ \ M_y(\xi)\}^t$
$\mathbf{E}_0$	starting value for the state vector of the homogeneous bending problem
$\mathbf{G}(\xi)$	fundamental system matrix of the bending problem
$\xi = \frac{x}{L}$	dimensionless variable

$$\rho = 1 + \frac{N_0}{GA_Q} \qquad \omega = L\sqrt{\frac{N_0}{\rho E I_y}} \tag{57}$$

The particular solution of the ODE system can be expressed as

$$\mathbf{E}_p(\xi) = -\int_{\tau=0}^{\xi} \mathbf{G}(\xi-\tau)\mathbf{F}(\tau)L\,d\tau \tag{58}$$

where $\mathbf{F}(\tau) = \{0\ 0\ q_z\ 0\}^T$ is the independent term of the bending problem, which contains the distributed loads applied on the beam. Operating:

$$\mathbf{E}_p(\xi) = -q_z L \begin{Bmatrix} \frac{L}{\rho\omega N_0}\left(\rho\omega\frac{\xi^2}{2} + \frac{1-\cosh\omega\xi}{\omega}\right) \\ \frac{1}{N_0}\left(\frac{\sinh\omega\xi}{\omega} - \xi\right) \\ \xi \\ -\frac{L}{\rho\omega^2}(1-\cosh\omega\xi) \end{Bmatrix} \tag{59}$$

Addition of (56) and (59) gives the analytical solution of the problem. In order to obtain values of all unknown variables at one end, boundary conditions (55) will be applied as follows:

$$\mathbf{E}(1) = \mathbf{G}(1)\mathbf{E}_0 + \mathbf{E}_p(1) \tag{60}$$

obtaining the initial values:

$$Q_{z0} = \frac{q_z L}{2} \qquad \theta_{x0} = -\frac{q_z L}{N_0}\left[\frac{1}{2} + \frac{1-\cosh\omega}{\omega\sinh\omega}\right] \tag{61}$$

For solving the extension problem, we replace $w_0(\xi)$ in (54a). After changing the variable $\xi = \frac{x}{L}$ and integrating, the following expression of horizontal displacements is obtained:

$$\begin{aligned} u(\xi) = & \frac{N_0 L}{EA}\xi - \frac{1}{2L}\left[\frac{Q_{z0}^2 L^2}{N_0^2}\xi + \frac{q_z^2 L^4}{N_0^2}\frac{\xi^3}{3} + \frac{A_0^2}{2}\left(\xi + \frac{\sinh 2\omega\xi}{2\omega}\right) + \frac{B_0^2}{2}\left(\frac{\sinh 2\omega\xi}{2\omega} - \xi\right) - \right. \\ & -\frac{Q_{z0} q_z L^3}{N_0^2}\xi^2 + \frac{2A_0 Q_0 L}{N_0}\frac{\sinh\omega\xi}{\omega} + \frac{2B_0 Q_0 L}{N_0}\frac{\cosh\omega\xi}{\omega} - \frac{2A_0 q_z L^2}{N_0}\left(\frac{\xi\sinh\omega\xi}{\omega} - \frac{\cosh\omega\xi}{\omega^2}\right) - \\ & \left. -\frac{2B_0 q_z L^2}{N_0}\left(\frac{\xi\cosh\omega\xi}{\omega} - \frac{\sinh\omega\xi}{\omega^2}\right) + A_0 B_0 \frac{\sinh^2\omega\xi}{\omega}\right] + \widehat{C}_u \end{aligned} \tag{62}$$

where:

$$A_0 = -\frac{L}{\rho}\left(\theta_{x0} + \frac{Q_{z0}}{N_0}\right) \qquad B_0 = \left(\frac{q_z L^2}{\rho\omega N_0} - \frac{M_{y0}\omega}{N_0}\right) \tag{63}$$

The constant $\widehat{C}_u$ is determined by the condition $u_0(0) = 0$:

$$\widehat{C}_u = \frac{1}{2L}\left[\frac{2B_0 Q_0 L}{\omega N_0} + \frac{2A_0 q_z L^2}{\omega^2 N_0}\right] \tag{64}$$

The final obtention of N_0 requires $u_0(L) = 0$ in (62). This condition drives to a complicated implicit equation in N_0 which cannot be symbolically solved.

A numerical strategy is adopted to obtain N_0. The following iterative scheme is used:

Step 1. Input value of N_0

Step 2. Calculus of $u_0(L)$ by using (62)

Step 3. Updating of sorts of $N_0' = N_0 - \frac{EA}{L}u_0(L)$ until convergence ($|u_0(L)| < \varepsilon = 10^{-7}$)

Obtention of axial stress completes the definition of current problem analytical solution.

7. Numerical results

For obtaining results, the following numerical values have been adopted:

$L = 12m$	Length of beam
$E = 2.1 \cdot 10^8$	Young's Modulus
$\nu = 0.3$	Poisson's Coefficient
$A = 0.1m^2$	Cross-section area
$A_Q = 0.083333m^2$	Cross-section shear area
$I = 2.08333 \cdot 10^{-3} m^4$	Cross-section moment of inertia

The non-dimensional control parameter represents the load level:

$$\rho = \frac{qL^3}{EI} \tag{65}$$

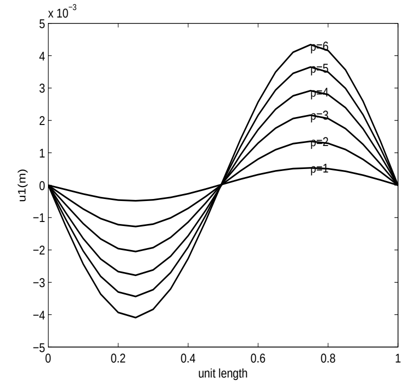
Load levels which produce integer values of the control parameter (65) are adopted. The MTnL obtained kinematic response of the beam is depicted in figures 4(a), 4(c) and 4(e) and the MTrL results are represented in figures 4(b), 4(d) and 4(f). In the last case only load levels which produce suitable results are represented.

For this first comparison a discretization of $k = 6$ and $p = 1$ has been adopted. It is the finest discretization supported by MTrL. Finer discretizations lead to divergence of the solutions. On the other hand, values of $\alpha = 1$ in MTnL and $\alpha = 0.6$ in MTrL were taken. In both cases (MTrL y MTnL) iterations were interrupted when the modulus of the difference vector of nodal displacements between two consecutive steps was smaller than $\varepsilon = 10^{-3}$.

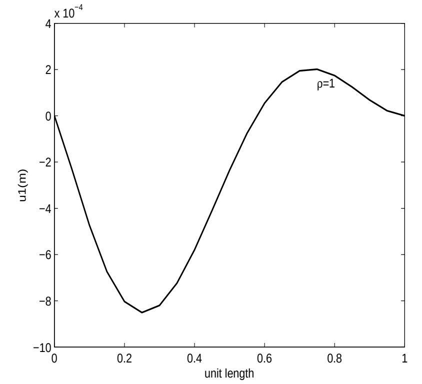
We outline next the representative values of each displacement referring to figure 4:

	MTrL		MTnL		Analytical
ρ	$u(L/6)$	ε_r	$u(L/6)$	ε_r	Solution
1	-0.0007	0.4000	-0.0004	0.2000	-0.0005
2	–	–	-0.0011	0.1538	-0.0013
3	–	–	-0.0018	0.1000	-0.0020
4	–	–	-0.0024	0.1429	-0.0028
5	–	–	-0.0030	0.1176	-0.0034
6	–	–	-0.0036	0.1220	-0.0041

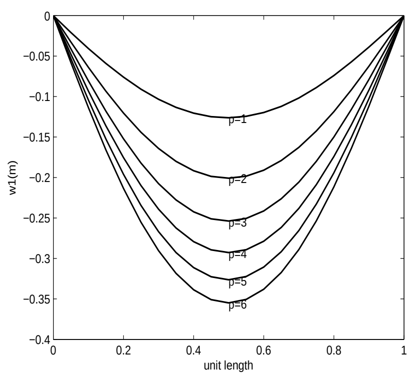
Table 1: Horizontal displacements $u(L/6)$ in m. ($k = 6$, $p = 1$)



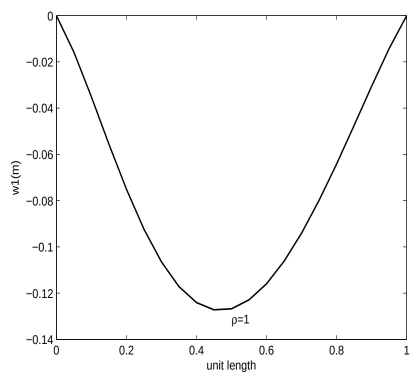
(a) Horizontal displ. for MTnL



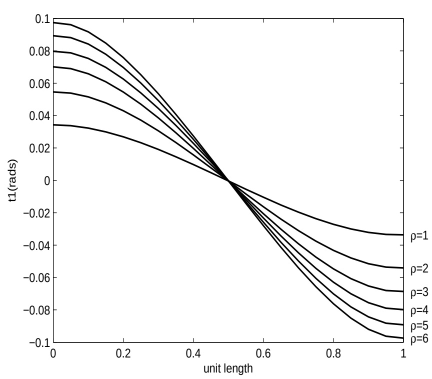
(b) Horizontal displ. for MTrL



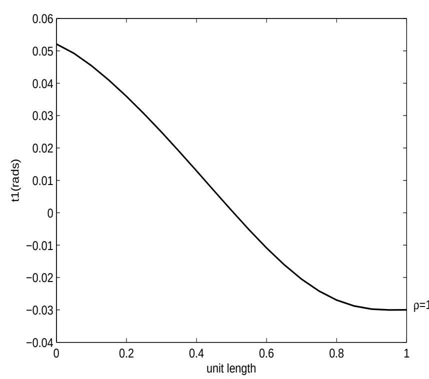
(c) Deflections for MTnL



(d) Deflections for MTrL



(e) Slopes for MTnL



(f) Slopes for MTrL

Figure 4: Comparison of both linearization methods in simply supported beam problem ($k = 6$, $p = 1$)

	MTrL		MTnL		Analytical
ρ	$w(L/2)$	ε_r	$w(L/2)$	ε_r	Solution
1	-0.1267	0.0261	-0.1262	0.0300	-0.1301
2	–	–	-0.2009	0.0262	-0.2063
3	–	–	-0.2538	0.0182	-0.2585
4	–	–	-0.2927	0.0207	-0.2989
5	–	–	-0.3263	0.0178	-0.3322
6	–	–	-0.3549	0.0164	-0.3608

Table 2: Deflections $w(L/2)$ in m. ($k = 6$, $p = 1$)

	MTrL		MTnL		Analytical
ρ	$\theta(0)$	ε_r	$\theta(0)$	ε_r	Solution
1	0.0521	0.5058	0.0343	0.0087	0.0346
2	–	–	0.0546	0.0460	0.0552
3	–	–	0.0701	0.0086	0.0695
4	–	–	0.0797	0.0112	0.0806
5	–	–	0.0893	0.0067	0.0899
6	–	–	0.0975	0.0051	0.0980

Table 3: Slopes $\theta(0)$ in rads. ($k = 6$, $p = 1$)

Figure 5 shows good agreement between the MTnL solution displacements ($k = 10$, $p = 1$ y $\alpha = 1$) and the analytical ones.

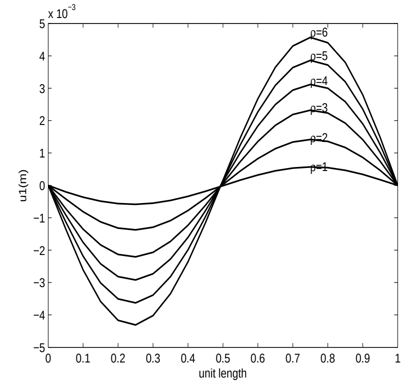
As we can see, MTnL produces much smaller relative error ε_r with respect to analytical solution that MTrL. The accuracy of MTnL with $k = 10$ (see figure 5) may be verified in the following table:

	Horizontal MTnL		Deflections MTnL		Slopes MTnL		Analytical solution		
ρ	$u(L/5)$	ε_r	$w(L/2)$	ε_r	$\theta(0)$	ε_r	$u(L/5)$	$w(L/2)$	$\theta(0)$
1	-0.0006	0.0000	-0.1295	0.0046	0.0347	0.0029	-0.0006	-0.1301	0.0346
2	-0.0013	0.0714	-0.2042	0.0102	0.0550	0.0036	-0.0014	-0.2063	0.0552
3	-0.0021	0.0454	-0.2579	0.0023	0.0691	0.0058	-0.0022	-0.2585	0.0695
4	-0.0028	0.0667	-0.2984	0.0017	0.0804	0.0025	-0.0030	-0.2989	0.0806
5	-0.0035	0.0541	-0.3317	0.0015	0.0900	0.0011	-0.0037	-0.3322	0.0899
6	-0.0042	0.0454	-0.3602	0.0017	0.0981	0.0010	-0.0044	-0.3608	0.0980

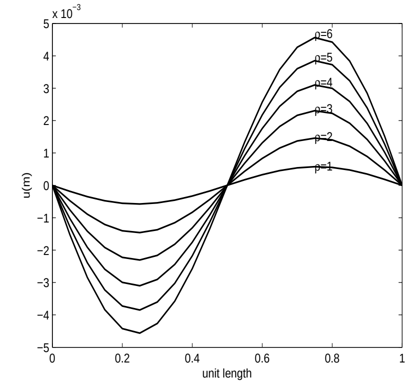
Table 4: Comparison displacements MTnL (m and rads), ($k = 10$, $p = 1$)

7.1. *Estimation of the order of error*

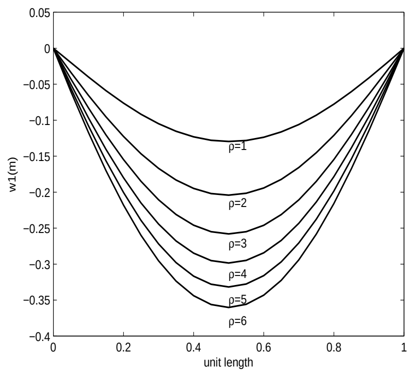
By successive evaluations of the numerical solution with different step sizes for the first load level ($\rho = 1$) with both kinds of linearization (MTrL and MTnL), the values for the deflection at the central point of the rod expressed in table 5 are obtained. The relative errors of the numerical solution are evaluated using the analytical solution for the deflection obtained from eqs. (56) and (59), $w(1/2) = -0.1301$, and are shown in table 5.



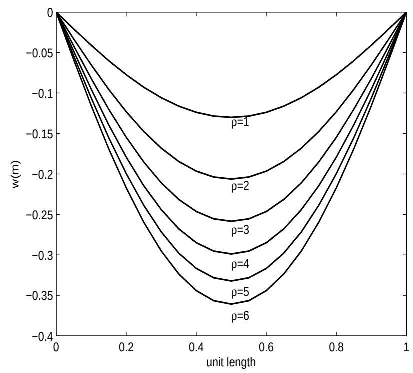
(a) Horizontal displ. MTnL



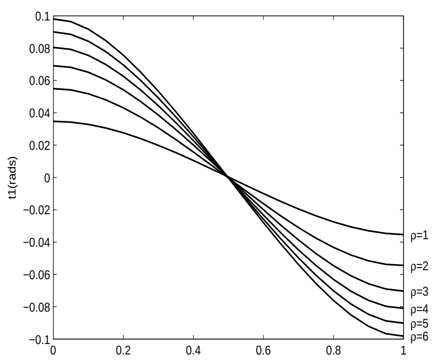
(b) Horizontal displ. analytic



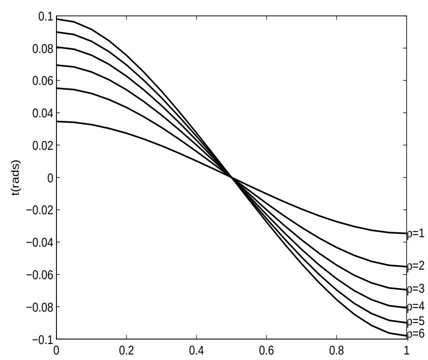
(c) Deflections for MTnL



(d) Deflections analytic



(e) Slopes for MTnL



(f) Slopes analytic

Figure 5: Comparison of MTnL ($k = 10$, $p = 1$) and analytical solution of simply supported beam problem

k	MTrL $w(L/2)$	MTrL ε_r	MTnL $w(L/2)$	MTnL ε_r
1	0.0000	1.0000	0.0000	1.0000
2	-0.0380	0.7079	-0.0902	0.3067
4	-0.1105	0.1507	-0.1206	0.0730
6	-0.1267	0.0260	-0.1262	0.0299
10	—	—	-0.1295	0.0046
20	—	—	-0.1303	0.0015

Table 5: Comparison of relative errors between MTrL and MTnL, (ρ = 1)

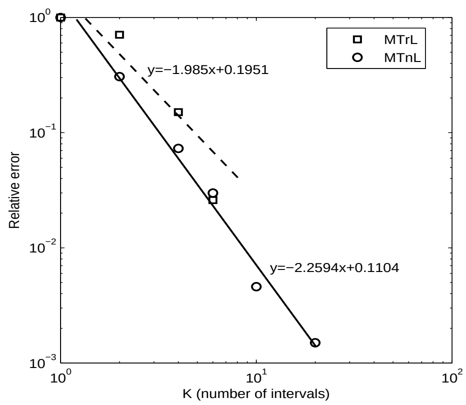


Figure 6: Comparison of relative errors between MTrL and MTnL, (ρ = 1)

Double logarithmic plots of the errors are shown in figure 6. The corresponding one to the MTnL solution is nearly a straight line with a slope around 2.

On the other hand, MTrL is unable to support more than $k = 6$ steps of discretization, and it seems clear that the average slope for this method is smaller than for MTnL. Therefore, it can be stated that MTnL with the proposed iterative scheme has an order of error of $O(h^2)$.

8. Discussion and concluding remarks

In this paper some new insights are provided in order to establish the best way of linearization for the nonlinear bending problem of a simply supported beam pinned at both ends. The numerical integration of the governing BVP of the problem is carried out by means of two kinds of linearization (MTnL - Multi step Tangential Linearization and MTrL - Multi step Transversal Linearization). Numerical treatment has been done by a multi-point difference method (Keller, 1969) inside of an iterative procedure. Finally, numerical results have been compared with the analytical solution (Monleón et al., 2008).

The adopted iterative strategy introduces some differences compared to the usual treatment (Ghosh and Roy, 2007; Ramachandra and Roy, 2001a, 2001b; Roy and Kumar, 2005; Viswanath and Roy, 2007), which is based in converting the governing BVP in a nonlinear algebraic equations system and solving it through Newton-Raphson or fix point method. The proposed strategy uses a centered finite difference method with multi-point linear constraints (Keller, 1969), avoiding the evaluation of the Magnus series expansion and the use of the Newton-Raphson method (Viswanath and Roy, 2007).

A further improvement of the proposed method in comparison with the one by Viswanath and Roy (2007) is found in the treatment of the equations proposed in section 2.1: the ODE system (12) has been arranged in a way which groups all nonlinear terms in the sub-matrix $\mathbf{B}(\mathbf{u}_2)$. Therefore, the linearization is only necessary for the lower equation block–refer to expression (34).

The low performance of MTrL (LTL-zeroth level) has been proved. It is only able to reproduce the actual solution suitably for $\rho = 1$ level. Values of $\rho > 1.5$ cause divergence of the iterative process. On the other hand, great agreement between results with MTnL (or LTL-first level) and the analytical ones for all load levels (including values until $\rho = 10$) has been found. Furthermore, we found the MTrL becomes unstable against slight variations of the parameter α, being $\alpha = 0.6$ almost the unique value which enables the convergence of the process. Nonetheless, the MTnL is able to produce acceptable solutions for α included in the range of values $[0.9, 1]$. This result contradicts Viswanath and Roy (2007), where the better convergence by MTrL compared to MTnL, for all the possible values of α is stated.

Although not included in this paper the integration has been also solved by means of other available techniques, different from the multi-point finite difference method, as simple shooting or parallel shooting. They lead to similar results, proving that the deficiencies observed in MTrL are not due to mismatch of the numerical technique.

It may be concluded that MTnL is able to reproduce the analytical solution for all load levels and works better than MTrL for this particular problem. This result is opposed to the conclusion reached in Viswanath and Roy (2007), where the relative errors produced by MTrL are said to be smaller than the MTnL ones for the simply supported beam and the tip-loaded cantilever beam problems.

The different behavior between MTnL and MTrL can be explained by the higher closeness between solution manifolds on intermediate points in MTnL compared to MTrL, because equating derivatives allows to achieve better accuracy. Such conclusion in consistent with Ramachandra and Roy (2001a), where it is suggested that higher levels of LTL systems can be deduced by means of successive derivation of the ODE-s system and correspondingly equating the solution and its derivatives with the linearized ones in every grid point.

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