# UNIVERSIDAD POLITÉCNICA DE VALENCIA

Departamento de Matemáticas Aplicadas
Instituto de Matemática Multidisciplinar


# Mathematical modeling of childhood obesity from a social epidemic point of view for the Spanish Region of Valencia: Numerical and analytical solutions

**PhD THESIS**


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Certify that the present thesis, *Mathematical modeling of childhood obesity from a social epidemic point of view for the Spanish Region of Valencia: Numerical and analytical solutions* has been directed under our supervision in the Department of Applied Mathematics of the Valencia Polytechnic University by Gilberto Carlos González Parra and makes up him thesis to obtain the doctorate in Applied Mathematics.

As stated in the report, in compliance with the current legislation, we authorize the presentation of the above Ph.D thesis before the doctoral commission of the Valencia Polytechnic University, signing the present certificate

Valencia, April 24th 2009

Lucas Antonio Jódar Sánchez    Rafael Jacinto Villanueva Micó

”*It is the mark of an educated mind to be able to entertain a thought without accepting it.*”

**Aristotle**, (384 BC - 322 BC).

”*Doubt is not a pleasant condition, but certainty is absurd.*”

**Voltaire**, (1694 - 1778).

”*Common sense is the collection of prejudices acquired by age eighteen.*”

**Albert Einstein**, (1879 - 1955).

”*Modesty is a shining light; it prepares the mind to receive knowledge, and the heart for truth.*”

**Madam Guizot**

*To my parents*
*Zaida, for your advise*
*Gilberto for your support*
*to my loved*
*daughter Carla*
*my son Carlos Daniel*
*and to my brother Enrique Daniel*†

# Thanks

Firstly, thanks to life for allowing me to write this dissertation.

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# Abstract

This thesis dissertation deals with the mathematical modeling of childhood obesity from a social epidemic point of view for the Spanish region of Valencia. Three mathematical models based on systems of nonlinear ordinary differential equations of first order were constructed. The first one is constructed for simulating childhood obesity for the $3-5$ years old population. For this model a nonstandard scheme based on the techniques developed by Ronald Mickens is constructed. This model is simulated with real data and the results show an increasing trend of obesity for the next years. The second model is an age-structured model developed in order to study the influence of age stages in the obesity population dynamics. This model considers overweight and obese in the groups $6-8$ and $9-12$ years old. Based on the numerical simulations of different scenarios it is shown that the prevention of children obesity in early years is of paramount importance. Therefore public health strategies should be designed as soon as possible to reduce the worldwide social obesity epidemic. The last model considers seasonal fluctuations of obesity prevalence using a nonautonomuos system of nonlinear of ordinary differential equations and we show that their solutions are periodic using a Jean Mawhin's Theorem of Coincidence. To corroborate the analytical results and perform numerical simulations, multistage Adomian and differential transformation methods are implemented. Numerical solutions using these methods are compared with those produced using Runge-Kutta type schemes. These implemented methods ensure good approximations using larger step sizes.

# Resumen

Esta memoria esta relacionada con la modelización matemática de la obesidad infantil en la Comunidad Valenciana de España desde un punto de vista epidemiológico social. Se construyen tres modelos matemáticos basados en sistemas de ecuaciones diferenciales ordinarias no lineales de primer orden. El primer modelo es construido para modelizar la obesidad infantil en la población con edades comprendidas entre 3 y 5 años. Para este modelo un esquema no-estándar es construido utilizando las técnicas desarrolladas por Mickens, donde se pueden usar tamaños de paso mayores a los usados en algunos métodos tradicionales. Las simulaciones numéricas utilizando datos reales indican un crecimiento de la obesidad en los próximos años. El segundo modelo es un modelo estructurado por edades para estudiar la influencia de las edades en los grupos 6-8 y 9-12 años. Basado en las simulaciones numéricas se encuentra que la prevención en tempranas edades es de fundamental importancia para reducir la epidemia mundial de la obesidad. El último modelo de esta memoria considera fluctuaciones estacionales de la prevalencia de la obesidad usando un modelo basado en un sistema de ecuaciones diferenciales ordinarias no lineales de primer orden no-autónomo y se muestra que sus soluciones son periódicas utilizando el teorema de coincidencia de Jean Mawhin. Para corroborar los resultados teóricos y realizar simulaciones numéricas se utilizan los métodos de Adomian con múltiple etapas y la transformada diferencial. Estos resultados son comparados con los esquemas de tipo Runge-Kutta ofreciendo así buenas aproximaciones con tamaños de paso mas grandes.

# Resum

Esta memòria està relacionada amb la modelització matemàtica de l'obesitat infantil a la Comunitat Valenciana (Espanya) des de un punt de vista de epidèmia social. Es contrueixen tres models matemàtics basats en sistemes d'equacions diferencials ordinàries no linials de primer ordre. El primer model modelitza la obesitat infantil en una població de xiquets entre 3 i 5 anys. Per a este model es contrueix un esquema numèric no estàndar utilitzant les tècniques desentrrollades per Mickens, on es poden utilitzar tamanys de pas majors que els que s'utilitzen en alguns métodes tradicionals. Les simulacions numèriques utilitzant dades reals indiquen un creiximent de l'obesitat en els pròxims anys.

El segón model és un model estructurat per edats per a estudiar l'influència de les edats en els grups 6-8 i 9-12 anys. Basat en les simulacions numèriques es troba que la prevención en edats primerenques é de fonamental importància per a reduïr l'epidèmia mundial de l'obesitat.

L'últim model d'esta memòria considera fluctiacions estacionals de la prevalència de l'obesitat utilitzant un model basat en un sistema d'equacions diferencials ordinàries no linials de primer ordre no autònom i es mostra que les seues solucions son periòdiques aplicant el teorema de coincidència de Jean Mawhin. Per a comprovar els resultats teòrics i realitzar simulacions numèriques s'han utilitzat els métodes numèrics d'Adomian amd múltiples etapes i transformada integral. Estos resultats s'han comparat amb esquemes clàssics de Runge-Kutta oferint bones aproximacions amb tamanys de pas mès grans.

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# Basic notation

| | |
|---|---|
| $\mathbb{R}$ | Set of real numbers |
| $\mathbb{C}$ | Set of real complex |
| $\mathbb{R}^n = \Big\{(x_1, ..., x_n)\Big\}$ | $x_i \in \mathbb{R}$ for all $i = 1, ..., n$ |
| $\mathbb{R}_+$ | Set of positive real numbers |
| $\mathbb{R}_-$ | Set of negative real numbers |
| $\mathbb{R}^n_+ = \Big\{(x_1, ..., x_n)\Big\}$ | $x_i \in \mathbb{R}_+$ for all $i = 1, ..., n$ |
| $\mathbb{R}^n_- = \Big\{(x_1, ..., x_n)\Big\}$ | $x_i \in \mathbb{R}^-$ for all $i = 1, ..., n$ |
| $B(x_0, R)$ | Set $\Big\{x \in \mathbb{R}^n / \lVert x - x_0 \rVert < R\Big\}$ |
| $\overline{B(x_0, R)}$ | Set $\Big\{x \in \mathbb{R}^n / \lVert x - x_0 \rVert \leq R\Big\}$ |
| $\partial B(x_0, R)$ | Set $\Big\{x \in \mathbb{R}^n / \lVert x - x_0 \rVert = R\Big\}$ |
| $:=$ | Defined as |
| $]a, b[$ | Open interval $a < t < b$ in $\mathbb{R}$ |
| $[a, b]$ | Closed interval $a \leq t \leq b$ in $\mathbb{R}$ |
| $C^k(I)$ | Set $\Big\{f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R} / \frac{d^i f(x)}{dx^i}$ exists and are continuous, for $i = 0, 1, ..., k\Big\}$ |
| $C^k(\mathbb{R}, \mathbb{R}^n)$ | Set $\Big\{f : \mathbb{R} \longrightarrow \mathbb{R}^n / f(t) = (f_1(t), ..., f_n(t))$ $f_i(t) \in C^k(\mathbb{R})$ for all $i = 0, 1, ..., k\Big\}$ |
| $f^u = \sup\limits_{t \in [0, \infty[} f(t)$ | $f : [0, \infty[ \longrightarrow \mathbb{R}$ is a bounded continuous function |
| $f^l = \inf\limits_{t \in [0, \infty[} f(t)$ | $f : [0, \infty[ \longrightarrow \mathbb{R}$ is a bounded continuous function |
| $\overline{f} = \frac{1}{T} \int_0^T f(t) dt$ | Where $f(t) \in C[0, T]$, and we call it mean value |

# Introduction

Obesity is growing at an important rate in developed and developing countries and it is becoming a serious disease not only from the individual health point of view but also from the public socioeconomic one, motivated by the high cost of the Health Public Care System due to the assistance expenditure of people with overweight and obesity.

Some related fatal diseases such as diabetes, heart attacks, blindness, renal failures and nonfatal related diseases such as respiratory difficulties, arthritis, infertility and psychological disorders are linked to overweight and obesity, see (CDC, 2007a,b; Ebbeling et al., 2002). One disease of particular concern is Type 2 diabetes, which has increased dramatically in children and adolescents (CDC, 2003). In addition the prevalence of gallbladder diseases in obese populations has been found to range as high as 60–95% when evaluated by gross and histologic examination after cholecystectomy (Liew et al., 2007). Obese patients not only have a high frequency of gallstones, but also a high proportion of abnormal histologic findings in the gallbladder mucosa (Liew et al., 2007).

Several studies correlate infant and adult obesity at the point that infant obesity is a powerful predictor of adult age obesity (Dietz, 1998; Krebs and Jacobson, 2003; Whitaker et al., 1997). For instance, according with (Krebs and Jacobson, 2003), an obese 4 years old child has a 20% increased probability to become an adult obese. Looking at the long-term consequences, overweight adolescents have a 70% chance of becoming overweight or obese adults, which increases to 80% if one or both parents are overweight or obese (Torgan, 2007). Additionally, it is important to consider that Wang (2001)(Wang, 2001) and Wang and Beydoun (2007)(Wang and Beydoun, 2007) associated obesity to socioeconomic status.

There are many factors that play a role in body weight, and therefore, in becoming obese. But the main factor is the excess intake of calories, higher than the daily expenditure of energy, which leads to weight gain and can eventually lead to obesity. Other factors include individual's environment, socioeconomic status, culture, metabolism, genes, etc. Obesity emerges partially from obesogenic environment. However, defining obesogenic environments remains problematic, especially in relation to sociocultural factors (Ulijaszek, 2007).

In this work, we study the obesity like a disease of social transmission, from an epidemiological point of view, which means the *study of the spread of the disease, in space and time, with the objective to trace factors that are responsible for, or contribute to, his occurrence* (Diekmann and Heesterbek, 2005). We treat obesity like a disease that spreads by social contact and this contact depends on the social environment around the people. Of course this environment consider media, time, accessibility of foods, economical status and others which have influence over the probability of transmission of the obesity.

To the best of our knowledge the only antecedent of obesity mathematical models for population dynamics appears in (Evangelista et al., 2004) where a fast-food obesity mathematical model for the USA population is proposed. The infinite-time behavior of the obesity study developed in (Evangelista et al., 2004) is based on the equilibrium points of the underlying system of differential equations. However, the model proposed there presents several drawbacks such as the invariance of parameters of the system in the infinite-time domain which is an unrealistic hypothesis, or the rough parameter estimation used that could be improved by means of the use of reliable data coming from local health institutions.

Note that our goal is to model and to predict future behavior of the childhood obesity, but the model also helps to the understanding of the mechanisms of the obesity spread. Mathematical models, simpler than the reality, allow us to understand the global dynamical behavior of the obesity in the population and to establish sustainable public health programs for the prevention of the childhood obesity.

Other studies have been used mathematical tools to investigate different type of issues regarding obesity. In (Ergün, 2009) an automated system to recognize and to follow-up obesity based on two different mathematical models, such as the traditional statistical method based on logistic regression and a multi-layer perception (MLP) neural network, was investigated.

Classical models of disease dynamics rely on systems of differential equations that represent the number of individuals in various categories through continuous variables, allowing for infinitesimal population densities. The origin of these models is commonly traced back to the well-known pioneer work of Kermack and McKendrick (Bailey, 1975; Hethcote, 2000). In their work, they obtained the epidemic threshold result that the density of susceptibles must exceed a critical value in order for an epidemic outbreak to occur (Bailey, 1975; Hethcote, 2000).

Other historical antecedents include the smallpox model formulated and solved by Daniel Bernoulli in 1760 in order to evaluate the effectiveness of variolation of healthy people with the smallpox virus, and the discrete time model proposed in 1906 by Hamer formulated in his attempt to understand the recurrence of measles epidemics, which may have been the first model to assume that the incidence (number of new cases per unit time) depends on the product of the densities of the susceptibles and infectives (Hethcote, 2000). In addition Ross developed differential equation models for malaria as a host-vector disease in 1911, and he won the second nobel prize in medicine. For more details readers can see the first edition in 1957 of Bailey's book which is one important book that helped the developing of mathematical epidemiology and the review made by Hethcote in 2000 (Hethcote, 2000).

Systems of ordinary differential equations (ODEs) are well-known tools that have been used to model different type of diseases (Brauer and Castillo-Chavez, 2001; Hethcote, 2000; Murray, 2002; Solis et al., 2005). The most discussed type of infection spread is the SIR-system, in which individuals are susceptible (S), infective (I) or removed/immune (R). The SIR epidemiology model is based on the flow of the individuals between the compartments S, I and R. In addition, several differential equations models have been proposed for modeling social behavior, such rumors, social behavior and ideologies (Brauer and Castillo-Chavez, 2001; Kawachi, 2008; Murray, 2002; Noymer, 2001; Santonja et al., 2008).

Mathematical models and computer simulations are useful experimental tools for building and testing theories, assessing quantitative conjectures, determining sensitivities to changes in parameter values, and estimating key parameters from data. Understanding the transmission characteristics of infectious diseases in communities, regions, and countries can lead to better approaches to decrease the transmission of these diseases (Hethcote, 2000). Mathematical models are used in planning, evaluating and optimizing various detection, prevention, therapy and control programs. Epidemiology modeling can contribute to the design and analysis of epidemiological surveys, suggest crucial data that should be collected, identify trends, make general forecasts and estimate the uncertainty in forecasts (Hethcote, 2000). Many of these models are based upon systems of ordinary differential equations (ODEs). In these models commonly the variables represent subpopulations of susceptibles ($S$), infected ($I$), recovered ($R$), latent ($E$), transmitted diseases vectors, and so forth. Thus, the ODE system describes the dynamics of the different classes of subpopulations in the model (Brauer and Castillo-Chavez, 2001; Murray, 2002; Hoppensteadt, 1975; Renshaw, 1991). In this way, acronyms for epidemiology models are often based on the flow patterns between the compartments such as $SI, SIS, SIRS, SEIR$, and $SEIRS$. All these models and most of the current models ones are extensions of the SIR model elaborated by W.O. Kermack and A.G. McKendrick in 1927 (Hethcote, 2000). Closed-form expressions for these functions in the mathematical models only are known for the $SI$ epidemic model and closely related variants (Bailey, 1975). One of the major difficulties of the analytical epidemic approach has been the rapid growth of the mathematical complexity of these models used to describe the various aspects of phenomena in sufficient detail and the difficulty in solving them in an analytical form. The main challenges arise from the presence of the nonlinear term $SI$ which comes from the law of mass action (Capasso, 2008).

In (Evangelista et al., 2004) and (Christakis and Fowler, 2007) the authors suggest that obesity spreads by social contact (Evangelista et al., 2004; Christakis and Fowler, 2007). Therefore, in this dissertation, we develop a statistical analysis of obesity influence factors focused in the target population of children between 3 and 5 years old. This analysis helps us to the construction of a finite-time childhood obesity mathematical model, somewhat different to the one considered in (Evangelista et al., 2004) and based on a more carefully adapted modeling to the real situation and to solving numerically systems of quadratic type ordinary differential equations in finite-time intervals. The results and simulations will lead us to present conclusions about the nearby future evolution of the obesity for a $3-5$ years old infant population. In addition, an age structured model is developed in order to study the influence of age stages in the obesity population dynamics. This proposed model considers the proportion of overweight and obese children populations in the groups $6-8$ and $9-12$ years old. Based on the numerical simulations of different scenarios we show that the prevention of children obesity in early years is of paramount importance. Therefore, public health strategies should be designed as soon as possible to reduce the worldwide social obesity epidemic. In this work, we modify the time-invariant parameter obesity model for the $3-5$ years old population to a nonautonomous model. The modification of the autonomous model is justified by the fact that the social and physical environment has fluctuations over the time. This seasonal effect has been studied in several works related to overweight and obesity of individuals (Plasqui and Westerterp, 2004; Westerterp, 2001; Van Staveren et al., 1986; Kobayashi, 2006; Katzmarzyk and Leonard, 1998; Tobe et al., 1994). In addition in (González et al., 2008) the dynamics for a $3-5$ years old obesity population under uncertainty in the initial condition and parameters of the model was investigated.

Generally the first approach to investigate the dynamics of nonlinear first order ordinary differential equations systems is the study of the eigenvalues of the associated Jacobian of the linearized system about the equilibrium points. Based in this approach, the local behavior of the solutions in time can be determined (Hirsch et al., 2004). In addition, in some cases using Lyapunov's functions the global stability can investigated (González-Parra et al., 2009c; Aranda et al., 2008; Zhang and Teng, 2008). On the other hand, if the system is nonautonomous, the above theory mentioned can not be applied because the equilibrium points depend on time, but other mathematical tools and notions such Lozinskii measure and uniform persistence can be applied (McCluskey, 2005).

The study of global existence of positive periodic solutions of models of dynamic populations in a periodic environment is an important problem. Several works have been presented with the assumption that the contact rate is a general continuous, bounded, positive and periodic function with period $T$ and the authors have shown the existence of positive periodic solution with the help of a continuation theorem based on coincidence degree (J.Hui and Zhu, 2005; Arenas et al., 2008a; Xia et al., 2007). In addition some authors have studied periodic solutions for others population mathematical models (Huoa et al., 2007; Kouche and Tatar, 2007). Since the global existence of positive periodic solutions plays a similar role as a globally stable equilibrium does in the autonomous model. In this thesis we show that our obesity seasonal model has periodic solutions using Jean Mawhin's continuation theorem which is based on coincidence degree theory (Gaines and Mawhin, 1977).

The existence of periodic solutions is important in the obesity population model since obesity is increasing worldwide and several studies are now being developed. Therefore, knowing that the behavior of obese and overweight populations present periodic oscillations is important in order to do more accurate studies. In this thesis we show that our obesity seasonal model has periodic solutions using a continuation theorem based on coincidence degree theory

To corroborate the analytical results and perform numerical simulations regarding the seasonal obesity model, Adomian multistage and differential transformation methods are implemented. On the other hand, the autonomous obesity model is simulated numerically using schemes constructed using the nonstandard finite difference techniques developed by Ronald Mickens (Mickens, 1994, 2000). All these methods are used to solve numerically the obesity mathematical model with parameters derived from data of the region of Valencia regarding overweight, obesity and diet. Numerical results are compared with those produced using Runge-Kutta type schemes. The new numerical methods ensure competitive approximations using time step sizes larger than those normally used by traditional schemes.

### A brief Outline of this Dissertation

This thesis dissertation is as follows:

In Chapter 1 we present a finite-time $3-5$ years old childhood obesity model to study the evolution of the obesity in the next future in the Spanish region of Valencia. After a statistical study, it can be seen that sociocultural characteristics determine the nutritional habits and the unhealthy ones as high frequency of consumption of bakery, fried meals and soft drinks (BFS), are prevalent factors in childhood obesity. This analysis allows us to consider obesity as a disease of social transmission caused by high frequency consumption of BFS and to build a mathematical model of epidemiological-type to study the childhood obesity evolution. The parameters of the model using data from surveys related to obesity in the Spanish region of Valencia are computed adjusting the model to data from year 1999 and 2002. Furthermore the simulation shows an increasing trend of childhood obesity in the coming years.

In chapter 2 we present an age-structured mathematical model for the dynamical evolution of childhood obesity at population level with the aim of study the influence of age stages in the obesity population dynamics. The proposed model is fitted to real data in order to estimate unknown parameters and then used to predict the proportion of overweight and obese children populations in the groups $6-8$ and $9-12$ years old in the region of Valencia, Spain for the coming years. Based on the fitting of the model and numerical simulations of different scenarios it is shown that the prevention of children obesity in early years is of paramount importance. Therefore public health strategies should be designed as soon as possible to reduce the worldwide social obesity epidemic.

In chapter 3, a nonstandard finite difference scheme has been developed with the aim of solving numerically a mathematical model for obesity population dynamics. The construction of the proposed discrete scheme is such that it is dynamically consistent with the original differential equations model. Since the total population in this mathematical model is assumed constant, the proposed scheme has been constructed to satisfy the associated conservation law and positivity condition. Numerical comparisons between the competitive nonstandard scheme developed here and Euler method show the effectiveness of the proposed nonstandard numerical scheme. Numerical examples show that the nonstandard difference scheme methodology is a good option to solve numerically different mathematical models where essential properties of the populations need to be satisfied in order to simulate the real world.

In chapter 4 we study the periodic behavior of the solutions of a nonautonomous model for a obesity population. The mathematical model represented by a nonautonomous system of nonlinear ordinary differential equations is used to model the dynamics of obese populations. Numerical simulations suggest periodic behavior of the subpopulations solutions. Sufficient conditions which guarantee the existence of a periodic positive solution are obtained using a continuation theorem based on coincidence degree theory.

In chapter 5, we apply the multistage Adomian Decomposition Method $MADM$ to solve the seasonal obesity model that is based on a nonautonomous system of nonlinear differential equations. This seasonal obesity model has periodic behavior due to the periodic transmission parameter. Here the concept of the $MADM$ is introduced and then it is employed to obtain a piecewise finite series solution. The $MADM$ is used here as a hybrid analytical-numerical technique for approximating the solutions of the epidemic models. In order to show the efficiency of the method, the obtained numerical results are compared with the fourth-order Runge-Kutta method solutions. Numerical comparisons show that the $MADM$ is accurate, easy to apply and the calculated solutions preserve the periodic behavior of the continuous models. Moreover, the method has the advantage of giving a functional form of the solution for any time interval.

Finally, in chapter 6, the main aim is to apply the differential transformation method ($DTM$) to solve a system of nonautonomous nonlinear differential equations that describe the seasonal obesity in the population. The solution of this model exhibit periodic behavior due to the seasonal transmission rate. The dynamics of this model describe the evolution of the different classes of the population. Here the concept of $DTM$ is introduced and then it is employed to derive a set of difference equations for the seasonal obesity social epidemic model. The $DTM$ is used here as an algorithm for approximating the solutions of the seasonal obesity model in a sequence of time intervals. In order to show the efficiency of the method, the obtained numerical results are compared with the fourth-order Runge-Kutta method solutions. The numerical comparisons show that the $DTM$ is accurate, easy to apply and the calculated solutions preserve the properties of the continuous models, such as the periodic behavior. Furthermore, it is shown that the $DTM$ avoids large computational work and symbolic computation.

# Chapter 1

# Mathematical modeling of infant obesity population as a social transmission disease: the case of the Spanish region of Valencia †

In this chapter we present a finite-time $3-5$ years old childhood obesity model to study the evolution of the obesity in the next years in the Spanish region of Valencia. After a statistical study, it can be seen that sociocultural characteristics determine the nutritional habits and the unhealthy ones as high frequency of consumption of bakery products, fried meals and soft drinks (BFS), are prevalent factors in childhood obesity. This analysis allows us to consider obesity as a disease of social transmission caused by high frequency consumption of BFS and to build a mathematical model of epidemiological-type to study the childhood obesity evolution. The parameters of the model using data from surveys related to obesity in the

†This chapter is based on (Jódar et al., 2008)

Spanish region of Valencia are computed adjusting the model to data of year 1999 and 2002. Furthermore the simulation shows an increasing trend of childhood obesity in next years.

## Introduction

Obesity is growing at an important rate in developed and developing countries and it is becoming a serious disease not only from the individual health point of view but also from the public socioeconomic one, motivated by the high cost of the Health Public Care System due to the assistance expenditure of people suffering related fatal diseases such as diabetes, heart attacks, blindness, renal failures and nonfatal related diseases such as respiratory difficulties, arthritis, infertility and psychological disorders, see (CDC, 2007a,b). One disease of particular concern is Type 2 diabetes, which is linked to overweight and obesity and has increased dramatically in children and adolescents (CDC, 2003).

Several studies correlate infant and adult obesity at the point that infant obesity is a powerful predictor of adult age obesity (Dietz, 1998; Krebs and Jacobson, 2003; Whitaker et al., 1997). For instance, according with (Krebs and Jacobson, 2003), an obese 4 years old child has a 20% increased probability to become an adult obese. Looking at the long-term consequences, overweight adolescents have a 70% chance of becoming overweight or obese adults, which increases to 80% if one or both parents are overweight or obese (Torgan, 2007).

Although there are still some differences on criteria around the optimal measure for children obesity, for obesity measurement, we use here the Body Mass Index (BMI), which is a number calculated using individual's height and weight. Nevertheless, the change of the criterion only will produce minor changes in the conclusions.

There are many factors that play a role in body weight, and therefore, in becoming obese. But the main fact is the excess intake of calories, higher than the daily expenditure of energy, that leads to weight gain and can eventually lead to obesity. Other factors include individual's environment, socioeconomic status, culture, metabolism, genes, etc.

In this work we study the obesity like a disease of social transmission, from an epidemiological point of view, which means the *study of the spread of the disease, in space and time, with the objective to trace factors that are responsible for, or contribute to, his occurrence* (Diekmann and Heesterbek, 2005). We treat obesity like a disease that spread by social contact, this contact depends on the social environment around the people. Of course this environment consider media, time, accessibility of foods, economical status and others which have influence over the probability of transmission of the obesity.

To our knowledge the only antecedent of obesity mathematical models for populations appears in (Evangelista et al., 2004) where a fast-food obesity mathematical model for the USA population is proposed. The infinite-time behavior of the obesity study developed in (Evangelista et al., 2004) is based on the equilibrium points of the underlying system of differential equations. However, the model proposed in (Evangelista et al., 2004) presents several drawbacks, for example, the invariance of parameters of the system in the infinite-time domain is an unrealistic hypothesis and the rough parameter estimation used could be improved by means of the use of reliable data coming from local health institutions.

It is worth here to point out the difficulties to obtain reliable data. For instance, in the Spanish region of Valencia, a health survey is done every 5 years and data should be prepared, processed and stored in database before their availability. Moreover, the economic cost of this survey is very high. Therefore we had to search in several sources such the Health Survey of the Region of Valencia, survey about alimentary habits developed by the Nutritional Observatory or a report from Abbot laboratories to complete the needed data, but data corresponding to children is not easy to find.

In this chapter, we develop a statistical analysis of obesity influence factors focused in the target population of children between 3 and 5 years old. This analysis helps us to the construction of a finite-time childhood obesity mathematical model, somewhat different to the one considered in (Evangelista et al., 2004), based on a more carefully adapted modeling to the real situation and solving numerically systems of quadratic type ordinary differential equations in finite-time intervals. The results and simulations will lead us to present conclusions about the nearby future evolution of the obesity for a $3-5$ years old infant population.

Note that our goal is to model and to obtain future behavior of the childhood obesity, but also the model helps the understanding of the mechanisms of the obesity spread. Mathematical models, simpler than the reality, allow to understand the global dynamical behavior of the obesity in the population and to establish sustainable public health programs for the prevention of the childhood obesity.

The chapter is organized as follows. In Section 1.1 a statistical analysis of the data from *Encuesta de Salud de la Comunidad Valenciana 2000-2001* (Health Survey of the Region of Valencia 2000-2001) (ConselleriaSanitat, 2007) to identify the prevalent factors of childhood obesity is presented. Once identified the prevalent factors, we assume that the obesity is mainly transmitted by the quoted factors. Section 1.2 is addressed to the construction of the model, estimation of the model parameters, numerical simulation and sensitivity analysis. Section 1.6 is devoted to a short discussion about how the model analysis may suggest general health public strategies to avoid increasing of childhood obesity. Conclusions are presented in Section 1.7.

## 1.1 Significance analysis of influence factors in childhood obesity

The region of Valencia is located in eastern Mediterranean Spain, with an extension of $23,255$ $km^2$ and a population of $4,543,304$ inhabitants (2004), composed by three provinces, Castellón (north) with $527,345$ inhabitants, Alicante (south) with $1,657,040$, and Valencia (middle) with $2,358,919$.

In this section, we study the predictive influence factors in $3-5$ years old childhood obesity in the region of Valencia according to the logistic regression analysis (Hair et al., 1998, Chapter 5), founded on the database of $1,187$ children belonging to different families from the Health Survey of the Region of Valencia 2000-2001 (ConselleriaSanitat, 2007), where the dependent variable is a dummy variable, 0 means normal weight and 1 means existence of overweight or obesity. We consider that a 3 or 4 years old child is overweight or obese if its Body Mass Index (BMI) is greater than 1.75 where,

$$BMI = \frac{\text{Weight (Kg)}}{\text{Height (m)}^2},$$

and, a 5 years old child is overweight or obese if its BMI is greater than 1.80 (Fullana et al., 2004; Sobradillo et al., 2007).

The considered variables, as possible predictors of obesity in children between $3-5$ years old, are:

| **Gender** | **Age** | **Studies level** | **Residence** |
|---|---|---|---|
| Male | 3 years old | Illiterate | Alicante |
| Female | 4 years old | Primary education | Castellón |
| | 5 years old | Secondary education | Valencia |
| | | Higher education | |

The reference category considered for each variable is

| **Gender** | **Study level** | **Residence** |
|---|---|---|
| Female | Higher education | Valencia |

The results of the logistic regression (Hair et al., 1998, Chapter 5) are showed in the Table 1.1 (see (Jódar et al., 2006)). The level of significance of the $p-$ value is 0.05.

The reading of the $p-$values column of Table 1.1 reveals us that the combination of the parents study level and the residence have influence on

| **Variable** | $\beta$ | **Wald contrast** | **p−value** |
|---|---|---|---|
| **Study level × residence** | | 13.71 | **0.01** |
| Illiterate & Castellón | 21.81 | 0.00 | 1 |
| Primary & Alicante | **0.53** | 8.81 | **0.00** |
| Primary & Castellón | -0.11 | 0.11 | 0.73 |
| Secondary & Alicante | **0.62** | 5.94 | **0.01** |
| Secondary & Castellón | -0.10 | 0.06 | 0.80 |
| **Age** | | 0.28 | 0.86 |
| 3 years old | 0.09 | 0.26 | 0.60 |
| 4 years old | 0.07 | 0.15 | 0.69 |
| **Gender (male)** | 0.03 | 0.06 | 0.79 |
| Constant | -0.73 | 17.71 | 0.00 |

Table 1.1: Results of the logistic regression. The column $p-$value shows the variables with influence in obesity. The column $\beta$ measures the influence of each independent variable on the dependent variable (childhood overweight and obesity).

children obesity, for instance, if a child belongs to a family without higher studies and lives in Alicante increases the risk to be overweight or obese because, in this case, the $p-$value is less than 0.05 and $\beta = 0.53$, $\beta = 0.62$, positive values. On the other hand it can be seen that variables as gender or age have not influence on obesity since the associated $p-$value is greater than 0.05. Therefore, the logistic regression statistical study suggests that the obesity of $3 - 5$ years old children in the region of Valencia depends mainly on sociocultural characteristics where they live and grow. Other similar studies as (Welsh et al., 2005) support this idea. But, how sociocultural characteristics determine the children nutritional habits? And what is the type of food underlying of unhealthy nutritional habits? To answer these questions we carried out the cluster statistical analysis (Hair et al., 1998, Chapter 9) of Health Survey of the Region of Valencia 2000-2001 showed in Figure 1.1.

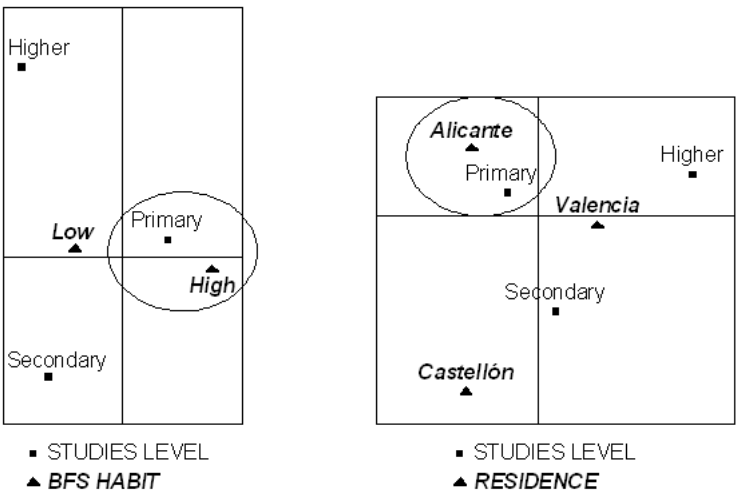



Figure 1.1: Correspondence analysis between variables *parents study level* and *children BFS consumption*. The proximity of *Primary* and *High* indicates a close relation (Left). Correspondence analysis between variables *parents study level* and *residence*. Analogously, the proximity of *Primary* and *Alicante* indicates a close relation (Right).

First, in Figure 1.1 (left) we consider the consumption frequency of bakery, fried meals and soft drinks (BFS). The analysis allows us to define two groups of people according to their nutritional habits, one composed by those children consuming a low frequency, less than 2 portions per week of each product, of bakery, fried meals and soft drinks, and a second group where the consumption frequency of each of the mentioned products is more than 3 portions of each type. Furthermore, the correspondence statistical analysis (Hair et al., 1998, Chapter 10) permits to correlate consumption frequency of bakery, fried meals and soft drinks, together. The proximity between the categories *Primary* and *High* indicates that the BFS consumption habit is greater in children which parents have only primary studies. These results are in well accordance with other Spanish studies, for instance (Bes-Rastrollo et al., 2006).

Second, in Figure 1.1(right) we find a statistical relationship between the residence and the parent level studies. To be precise, the parents with primary studies are concentrated mainly in Alicante. The proximity between the categories *Primary* and *Alicante* indicates this fact. Also, in Table 1.1 with a $p-$value less than 0.05. Hence we can consider residence as an additional sociocultural characteristic.

Summarizing, obesity can be considered as a disease of social transmission where the transmission is done by unhealthy nutritional habits, BFS consumption, that depends on the sociocultural characteristic known as *parents study level*. The non-parametric contrast carried out shows the significative statistic relation between the parents study level and the child inclusion in a certain group of BFS consumption ($p-$value less than 0.05).Therefore, in the rest of the chapter we will refer to BFS consumption as the prevalent factor in childhood obesity, where the BFS consumption frequency (nutritional habit) is determined by the study level of the parents (sociocultural characteristic).

## 1.2 The mathematical model

In this section, we build a mathematical model of the evolution of infant obesity, regarded as a social disease transmitted by social environment. The statistical study of previous section allows the hypothesis of childhood obesity as a social transmission epidemic disease produced by the BFS consumption frequency. These facts lead us to propose an epidemiological-type model.

For model building, childhood population is divided in six sub-populations: individuals with normal weight ($N(t)$), latent individuals, that is, people with habit of BFS consumption but are still normal weight ($L(t)$), people with overweight ($S(t)$), obese individuals ($O(t)$), people overweighted on diet ($D_S(t)$) and obese individuals on diet ($D_O(t)$). In addition we consider the following assumptions:

- Let us assume population homogeneous mixing (Murray, 2002, p. 320 and p. 328).
- From Section 1.1, we can assume that BFS consumption increases individual weight of children. Hence, the transitions between the different sub-populations are determined as follows:
  - Once a child starts having BFS consumption he/she becomes BFS addicted, $L(t)$, and starts a progression to overweight $S(t)$ due to continuous BFS consumption. We assume that once a child is in the latent sub-population he/she will progress, after a period (latency period), to overweight sub-population. If the child continues having BFS he/she can become an obese individual $O(t)$. Children in both classes can stop having BFS, and then move to diet classes $D_S(t)$ and $D_O(t)$, respectively.
  - An individual of class $D_S(t)$ becomes a member of class $N(t)$ if he/she gives up or reduces the BFS consumption in an appropriate rate, or return to $S(t)$ otherwise. Analogously, a child of class $D_O(t)$ becomes a member of class $D_S(t)$ if he/she gives

up or reduces the BFS consumption in an appropriate rate, or return to $O(t)$ otherwise.

- The transits between the sub-populations $N$, $L$, $S$, $O$, $D_S$, $D_O$, are governed by terms proportional to the sizes of these sub-populations. However since the transit from normal to latent occurs through the transmission of BFS consumption from latent, overweight and obese sub-populations to normal weight sub-population, which depends on the meetings between their parents and due to our assumption that these social encounters between parents of different sub-populations are proportional to the product of the children's sub-populations, the transit is modeled using the term,

$$\beta N(t)\left[L(t)+S(t)+O(t)\right].$$

- For this model, transition time-constant parameters are more suitable due that our goal is to model the evolution of child obesity over a short finite time.

- We also assume that the parent's nutritional habits and lifestyle determine the children's habits, for instance, a child is on diet if their parents are on diet (Carter et al., 1993). The values that determine the transition between sub-populations on diet are estimated using adult statistic surveys later explained.

- Their proportional sizes and their behavior with the time will determine the dynamic evolution of infant obesity population.

- The increase in excessive weight gain does not occur during infancy (Heinzer, 2005), then we also assume that the new 3 years old new recruited children are of normal weight.

Under the above assumptions, this dynamic obesity model for $3-5$ years old in the Spanish region of Valencia is given by the following nonlinear system of ordinary differential equations

$$\begin{aligned}
N'(t) &= \mu + \varepsilon D_S(t) - \mu N(t) - \beta N(t)\left[L(t) + S(t) + O(t)\right], \\
L'(t) &= \beta N(t)\left[L(t) + S(t) + O(t)\right] - \left[\mu + \gamma_L\right] L(t), \\
S'(t) &= \gamma_L L(t) + \varphi D_S(t) - \left[\mu + \gamma_S + \alpha\right] S(t), \\
O'(t) &= \gamma_S S(t) + \delta D_O(t) - \left[\mu + \sigma\right] O(t), \\
D_S'(t) &= \gamma_D D_O(t) + \alpha S(t) - \left[\mu + \varepsilon + \varphi\right] D_S(t), \\
D_O'(t) &= \sigma O(t) - \left[\mu + \gamma_D + \delta\right] D_O(t).
\end{aligned} \tag{1.1}$$

where the constant parameters of the model are:

- $\beta$, transmission rate due to social pressure to BFS consumption (family, friends, marketing, TV, ...),
- $\mu$, average stay time in the system of $3-5$ years old children,
- $\gamma_L$, rate at which a latent individual moves to the overweight subpopulation,
- $\gamma_S$, rate at which an overweight individual becomes an obese individual by continuous consumption of BFS,
- $\varepsilon$, rate at which an overweight individual on diet becomes a normal weight individual,
- $\alpha$, rate at which an overweight individual stops or reduces BFS consumption, i.e., the individual is on diet,
- $\varphi$, rate at which an overweight individual on diet fails, i.e., the individual resumes a high BFS consumption,
- $\sigma$, rate at which an obese individual stops or reduces BFS consumption,
- $\delta$, rate at which an obese individual on diet fails,

- $\gamma_D$, rate at which an obese individual on diet becomes an overweight individual on diet.

Throughout of this chapter, we focus on the dynamics of the model (1.1) in the following restricted region:

$$\omega = \{(N, L, S, O, D_S, D_O)^T \in \mathbb{R}_+ / \quad N + L + S + O + D_S + D_O = 1\},$$

where the basic results as usual local existence, uniqueness and continuation of solutions are valid for the Lipschitzian system (1.1). The dynamic of transits between subpopulations is depicted graphically in Figure 1.2.

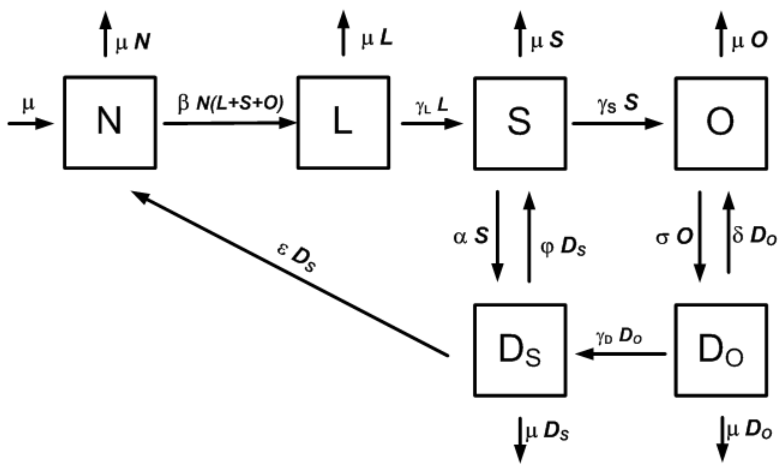



Figure 1.2: The diagram for $3-5$ years old children obesity model in the Spanish region of Valencia as defined in system (1.1). The boxes represent the sub-populations and the arrows represent the transitions between the sub-populations, labelled by the parameters of the model.

## 1.3 Estimation of parameters

The estimation of some of the parameters of the model is intrinsically difficult due to the strong influence of intangible variables as advertising, marketing, public education programs, health programs, etc. Moreover, the lack of the specific data for $3-5$ years old children leads us to consider data from adult surveys considering that adults are the parents of the children and the familiar habits are the same. In spite of these facts, we obtained all parameters of the model except $\beta$ and $k$ ($k$ is a parameter related to $\gamma_S$ and $\gamma_D$) using the following sources:

- the Health Survey of the Region of Valencia $2000-2001$ (ConselleriaSanitat, 2007),
- a technical report published by the Valencian Health Department where is described the present situation of infant obesity and data from 1999 to 2005, in regard to obesity and overweight, are available (Fullana et al., 2004),
- a survey about alimentary habits developed by the Nutritional Observatory of the company Nutricia (Nutricia, 2007),
- a report from Abbot laboratories about the success to get normal weight from overweight and obese people on diet (Abbot, 2007),
- and a survey we prepared with 4 questions about population diet habits to the members of the Valencian Society of Endocrinology and Nutrition (González-Parra et al., 2007).

Parameters $\beta$ and $k$ will be estimated by fitting the model with the data. The following parameters are computed for time $t$ in weeks as:

- $\mu = \frac{1}{156}\ weeks^{-1}$, the average stay of a child in the system is 3 years, that is, 156 weeks.
- $\gamma_L = 0.0089\ weeks^{-1}$, is estimated using the weekly growth of the average weight of a child in the region of Valencia (Fullana et al., 2004). This rate shows how many weeks take a latent child (normal weight)

to become an overweight individual by continuous consumption of BFS, that is,

$$\frac{1}{\gamma_L} = \frac{1}{0.0089} = 112.36 \; weeks.$$

- $\gamma_S$, we consider this parameter proportional to $\gamma_L$ because it is describing a phenomenon of the same characteristics as $\gamma_L$ with an increasing difficulty to become obese based on two main facts; one is that once the individual is overweight he/she realizes more his/her overweight problem and takes more care about his nutrition (from data about overweight and obese people on diet in (ConselleriaSanitat, 2007)), and the second fact is that the basal metabolic rate increases with the weight, therefore the bodies of heavier people consume more calories (de Luis et al., 2006; Ravussin et al., 1982). So,

$$\gamma_S = k\gamma_L = k \times 0.0089 \; weeks^{-1}, \; 0 < k < 1,$$

where $k$ will be determined by fitting model to the real data.

- $\varepsilon$, an individual with BFS consumption takes $\frac{1}{\gamma_L}$ weeks to transit from normal weight to overweight, then if he/she gives up BFS consumption, he/she will take $\frac{1}{\gamma_L}$ weeks multiplied by the success rate (González-Parra et al., 2007) to come back to normal weight, that is,

$$\varepsilon = 0.0089 \times 0.312 = 2.776\,8 \times 10^{-3} \; weeks^{-1}.$$

- $\alpha$, this parameter is estimated taking into account the average time that an individual finishes a diet and starts another, $1.56 \times 52$ weeks (Nutricia, 2007), and the percentage of overweight individuals who puts on diet (González-Parra et al., 2007), that is,

$$\alpha = \frac{1}{1.56 \times 52} \times 0.33 = 4.068 \times 10^{-3} \; weeks^{-1}.$$

- $\varphi$, this parameter is estimated using the average time that people stay on diet, 5.4 weeks (Abbot, 2007), and the percentage of failure of overweight individuals on diet (González-Parra et al., 2007),

$$\varphi = (1 - 0.312) \times \frac{1}{5.4} = 0.12735 \; weeks^{-1}.$$

- $\sigma$, this parameter is estimated taking into account the average time between an individual finishing a diet and starting another, $1.56 \times 52$ weeks (Nutricia, 2007), and the percentage of obese individuals who are on a diet (González-Parra et al., 2007), that is,

$$\sigma = \frac{1}{1.56 \times 52} \times 0.36 = 4.4379 \times 10^{-3} \; weeks^{-1}.$$

- $\delta$, this parameter is estimated using the average time that people stay on diet, 5.4 weeks (Abbot, 2007), and the percentage of failure of obese individuals on diet (González-Parra et al., 2007),

$$\delta = (1 - 0.137) \times \frac{1}{5.4} = 0.15974 \; weeks^{-1}.$$

- $\gamma_D$, this parameter measures the partial success of an obese individual in his/her goal of reaching a normal weight, specifically, it measures the flow from obese individuals on diet to overweight individuals on diet. $\gamma_D$ is estimated using the value of $\gamma_S$ and the percentage of obese individuals on diet who have success (González-Parra et al., 2007),

$$\gamma_D = \gamma_S \times 0.146 = k \times 1.2994 \times 10^{-3} \; weeks^{-1}.$$

We summarize the obtained parameters in Table 1.2.

| **Parameter** | **Value** |
|---|---|
| $\mu$ | $\frac{1}{156}$ |
| $\gamma_S$ | $k \times 0.0089$ |
| $\alpha$ | $4.068 \times 10^{-3}$ |
| $\sigma$ | $4.4379 \times 10^{-3}$ |
| $\gamma_D$ | $k \times 1.2994 \times 10^{-3}$ |

| **Parameter** | **Value** |
|---|---|
| $\gamma_L$ | 0.0089 |
| $\varepsilon$ | $2.7768 \times 10^{-3}$ |
| $\varphi$ | 0.12735 |
| $\delta$ | 0.15974 |

Table 1.2: Obtained parameters of the model given by the system of differential equations (1.1) for the region of Valencia.

**Remark 1.3.1** *Each one of the parameters $\mu$, $\gamma_L$, $\gamma_S$, $\varepsilon$, $\alpha$, $\varphi$, $\sigma$, $\delta$, $\gamma_D$ can be interpreted as the mean of the length of the transit period between two sub-populations. Length of the transit period for a sub-population is usually assumed to follow an exponential distribution (Brauer and Castillo-Chavez, 2001, p. 41 and p. 283). Therefore the above numerical values computed for each parameter should be considered as average length of transition periods between two sub-populations and does not be regarded as a fixed spent time after which each individual crosses to a new sub-population.*

## 1.4 Numerical simulations of the mathematical model

We obtain the initial conditions (year 1999, i.e., $t = 0$) and final conditions (year 2002, i.e., $t = 156$) for the model from (Fullana et al., 2004). We use this final condition (2002) because is the last data available for the sub-population proportions.

With the obtained parameters from surveys and data and the initial and final conditions, we performed several simulations for different values of $\beta$ and $k$ in an appropriate range ($\beta, k \in (0, 1)$) in order to find the value of $\beta$ and $k$ that minimize the mean square error of the difference between the solutions of the model for the proportions of sub-populations of normal weight (including latent), overweight and obese and the real data in year 2002 (final condition). The obtained values for $\beta$ and $k$ were

$$\begin{aligned} \beta &= 0.02, \\ k &= 0.32584. \end{aligned} \tag{1.2}$$

Then, we use the model with the parameters of Table 1.2 and the just obtained parameters (1.2) to extrapolate for the following 8 years (until 2010). The result is presented in Figure 1.3.

In Figure 1.3 it can be seen an increasing trend of obese and overweight $3-5$ years old children sub-populations until 2010, in well accordance with the tendency observed in several countries (Wang and Lobstein, 2006).

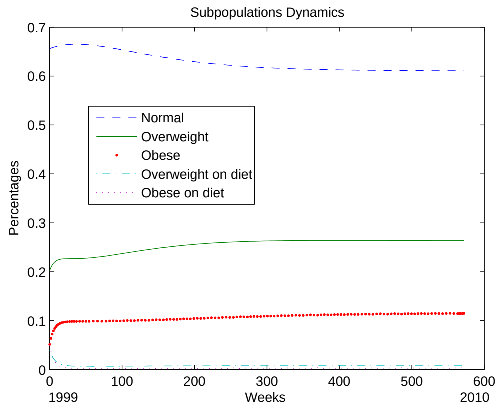



Figure 1.3: Evolution of the different sub-populations of $3-5$ years old children in the region of Valencia, $1999-2010$. Note a slight but sustained increasing of the overweight and obese sub-populations. Latent individuals are included in normal sub-population.

Also, there exists a decreasing in normal weight sub-population and the percentage of people on diet remain constant and low. In the Table 1.3, we present some of the numerical values represented in the Figure 1.3.

## 1.5 Sensitivity analysis of the mathematical model

We performed several simulations varying the parameters of the model in order to find out what is the influence of the changes on the final solution. We observed that the most sensitive parameters were $\beta$, $\gamma_L$ and $k$.

As we did in Figure 1.3, in the following Figures 1.4, 1.5 and 1.6 latent individuals are included in normal sub-population.

In Figure 1.4 we present a simulation of the proposed model where parameter $\beta = 0.02$ has been changed to $\beta = 0.04$. This change implies an increasing of the transmission rate of BFS consumption and, consequently, more people may become overweight by continued high BFS consumption.

| **Year** | **Normal+latent** | **Overweight** | **Obese** |
|---|---|---|---|
| 1999 | 0.656 | 0.2025 | 0.0515 |
| 2000 | 0.6639 | 0.227801 | 0.0987835 |
| 2001 | 0.6524 | 0.238055 | 0.0998146 |
| 2002 | 0.6386 | 0.249237 | 0.101979 |
| 2003 | 0.6274 | 0.256948 | 0.104708 |
| 2004 | 0.6207 | 0.261257 | 0.10739 |
| 2005 | 0.6162 | 0.26327 | 0.109685 |
| 2006 | 0.6138 | 0.263993 | 0.111485 |
| 2007 | 0.6122 | 0.264087 | 0.112818 |
| 2008 | 0.6114 | 0.263929 | 0.113765 |
| 2009 | 0.6109 | 0.263705 | 0.114417 |
| 2010 | 0.6107 | 0.263494 | 0.114857 |

Table 1.3: Evolution of proportion of normal weight (including latents), overweight and obese sub-populations for different years in the proposed model. This model predicts that the 61.07%, the 26.34% and the 11.48% of the $3-5$ years old children in the region of Valencia will be normal weight, overweight and obese, respectively, in 2010.

Comparing with Figure 1.3, it can be noted an increase in overweight sub-population and a decrease in normal weight sub-population. Therefore, the increasing of $\beta$ implies an increasing of overweight sub-population.

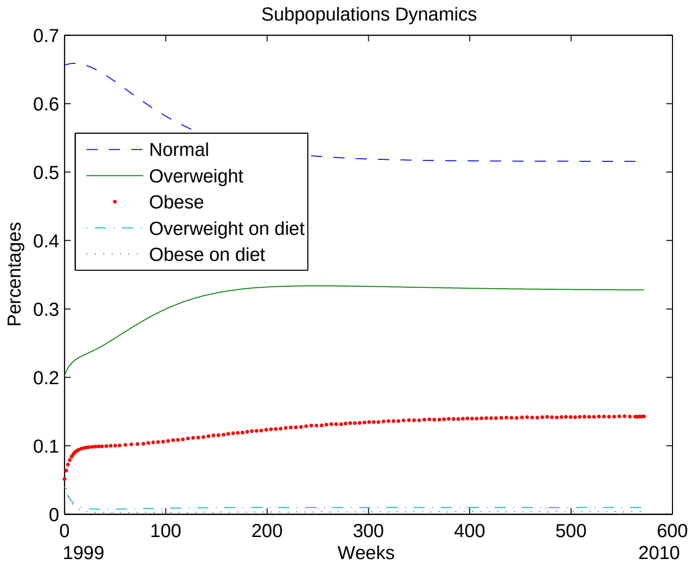



Figure 1.4: Simulation of the proposed model with $\beta = 0.04$. The increasing of $\beta$ implies an increasing of overweight sub-population.

In Figure 1.5 the parameter $\gamma_L = 0.0089$ has been changed to $\gamma_L = 0.02$. This change implies a faster transition to become an overweight individual by continued BFS consumption. In this case, at the beginning, there is an increasing of the overweight and obese sub-populations and finally a stabilization.

In Figure 1.6 the parameter $k = 0.32584$ has been changed to $k = 1$. This change affects to parameters $\gamma_S$ and $\gamma_D$ and implies a faster transit from overweight to obese individual and a faster transit from obese on diet to overweight on diet. Because $\gamma_S$ is much greater than $\gamma_D$, in this case, there is an increasing of the obese population, until to go above the overweight population.

Due to the sensitivity of the estimated parameter $\gamma_L$, we repeated the

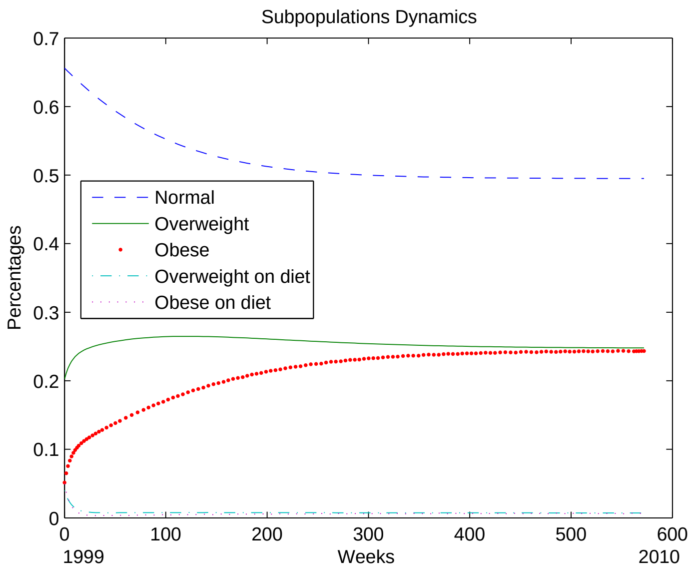



Figure 1.5: Simulation of the model with $\gamma_L = 0.02$. The increasing of $\gamma_L$ implies an increasing of overweight and obese sub-population at the beginning and a stabilization of both sub-populations at the end.

model fitting with the real data in year 2002 (final condition) to compute the parameters $\beta$, $k$ and moreover the parameter $\gamma_L$, in order to check the consistency of its previous estimation. The obtained values were

$$\begin{aligned} \beta &= 0.021, \\ k &= 0.30539, \\ \gamma_L &= 0.00907. \end{aligned}$$

Note that the new obtained values for these parameters of the model are very similar to the previous ones. In particular, the parameter $\gamma_L$ is almost equal to our estimation carried out in Section 1.3 using data in (Fullana et al., 2004).

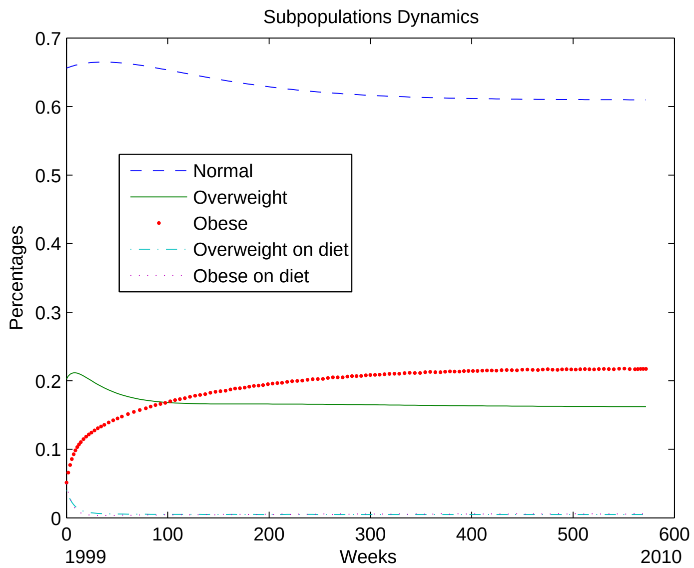



Figure 1.6: Simulation of the model with $k = 1$. The increasing of $k$ implies a fast transit from overweight to obese and consequently, an increasing of the obese sub-population, until to go above the overweight sub-population.

## 1.6 Model application to Health Public System strategies

Medicine covers the prevention and treatment of illnesses. The proposed obesity model considers both possibilities. With parameters $\beta$, $\gamma_L$ and $\gamma_S$ prevention can be controlled whereas treatment parameters are $\gamma_D$, $\varepsilon$, $\alpha$ and $\sigma$.

The simulations carried out suggest that the obesity prevention strategies should lead to the reduction of $\beta$, $\gamma_L$ and $\gamma_S$, that is, reducing the pressure to BFS consumption and the amount and the frequency of consumption. Two main strategies may be suggested to achieve this objective. In long term, from the statistical study carried out in Section 1.1, developing educative plans in order to increase the study level of families. In short term, reducing the BFS products advertising spots and designing health programs and advertising campaigns to show the population how to change to healthier nutritional habits.

On the other hand, for overweight and obese individuals, the objective is to increase the treatment parameters $\gamma_D$, $\varepsilon$, $\alpha$ and $\sigma$. It could be done if the Health System does a monitoring in primary attention to the people that decides to go to the physicist to put on diet. The monitoring may prevent the majority of the individuals give up the diet in the first stages of the process.

## 1.7 Conclusions

In this chapter we presented a finite-time $3-5$ years old childhood obesity model to study the evolution of the obesity in the next years in the Spanish region of Valencia. After a statistical study, high frequency of consumption of BFS (bakery, fried meals and soft drinks) is detected as prevalent factor in childhood obesity. This analysis allows us to consider obesity as a disease of social transmission caused by high frequency consumption of BFS and to build a mathematical model of epidemiological-type to study the childhood obesity evolution. Once the mathematical model is built and most of the parameters were obtained using several surveys of the Spanish region of Valencia, we find the best estimated values only for the parameters $\beta$ and $k$ fitting the model in order to minimize the mean square error between the model and the real data in year 2002.

The simulations carried out with this model indicated an increasing trend in the $3-5$ years old overweight and obese populations in the next future. The parameters $\beta$, $\gamma_L$ and $k$ are the most important in the proposed model, because, from the sensitivity analysis, we find that small changes in these parameters imply appreciable changes in the final results. Hence, childhood obesity should be faced up through public health programs in order to reduce the values of these parameters, to be precise, on the transmission rate due to social pressure to BFS consumption measured by $\beta$ (family, friends, marketing, TV), and on the frequency of BFS consumption measured by $\gamma_L$ and $\gamma_S$. Some possible general strategies are suggested.

Finally, as we noted in the introduction, this kind of models work well for a short time, since as we said it is difficult to believe that parameters remain constant for a long time periods and it is more suitable variable parameters that can be modeled through dependent time parameters or using stochastic white noise.

# Chapter 2

# An age-structured model for childhood obesity in the Spanish region of Valencia †

Obesity is a complex condition, one with serious social and psychological dimensions, that affects virtually all age and socioeconomic groups and threatens to overwhelm both developed and developing countries. Several studies have been dedicated to stop obesity epidemic. In this chapter obesity is considered as a health concern that is spread by social transmission of unhealthy habits. Here we present an age-structured mathematical model for the dynamical evolution of childhood obesity at population level with the aim of study the influence of age stages in the obesity population dynamics. The proposed model is fitted to real data in order to estimate unknown parameters and then used to predict the proportion of overweight and obese children populations in the groups $6-8$ and $9-12$ years old in the region of Valencia, Spain for the next future. Based on the fitting of the model and numerical simulations of different scenarios it is shown that the prevention of children obesity in early years is of paramount importance. Therefore public health strategies should be designed as soon as possible

†This chapter is based on González-Parra et al. (2009b)

to reduce the worldwide social obesity epidemic.

## 2.1 Introduction

Obesity has become a serious health concern, with an increasing cost due to related diabetes, heart attacks, blindness, renal failures, respiratory difficulties, arthritis, infertility, and psychological disorders (CDC, 2007b; Ebbeling et al., 2002).

Infant and juvenile obesity is a powerful predictor of adult age obesity. We propose an age-structured model for obesity in the $6-8$ and $9-12$ year old classes in the region of Valencia, Spain. We study obesity such as a health concern of social transmission of unhealthy habits. Jódar et al. (2008) showed that obesity is associated to eating habits, in particular to the high consumption of bakery snacks, fried meals, and soda drinks, and that habits spread by social contact (Evangelista et al., 2004; Christakis and Fowler, 2007). Wang (2001) and Wang and Beydoun (2007) associated obesity to socioeconomic status. Jódar et al. (2008) modeled the dynamics of children between 3 and 5 years old as a group.

We construct an age-structured model, adapted to the real situation and fitted to real data provided by the Health Institution of the Government of Valencia. In the region of Valencia, a health survey is done every five years and data are prepared, processed, and stored in databases. We completed the data with the same sources of the $3-5$ years old obesity model (1.1).

In Section 2 a data analysis of the *Encuesta de Salud de la Comunidad Valenciana 2000-2001* (Health Survey of the Region of Valencia 2000-2001) (ConselleriaSanitat, 2007) is used to identify the prevalent factors of obesity for the $6-12$ year old age class. This statistical analysis allows us to divide the $6-12$ year old class into $6-8$ and $9-12$ year old classes. In Section 3, we develop a demographic model using data from the Instituto Valenciano de Estadística (IVE, 2007). The age-structured model is fitted to the available data. Numerical simulations illustrate some possible behaviors in the next few years of overweight and obese subpopulations in order to analyze the prophilaxy of obesity.

## 2.2 Data Analysis

The data set consists of 1,298 girls and 1,465 boys between 6 and 12 years from the Health Survey of the region of Valencia 2000 − 2001 (ConselleriaSanitat, 2007). A child is considered overweighted if its body mass index (BMI) is higher than the 85 percentile, and obese if higher than the 97 percentile (Sobradillo et al., 2007). Three categories are defined with respect to BMI; normal weight, overweight, and obese. The possible predictors of overweight and obesity in children between 6 and 12 years old are shown in Table 2.1.

Table 2.1: Variables considered as possible predictors of obesity in 6 − 12 year old children in the region of Valencia, Spain.

| Variables | Category 1 | Category 2 | Category 3 | Category 4 |
|---|---|---|---|---|
| Gender | Girl | Boy | | |
| Age | 6 − 8 year | 9 − 12 year | | |
| Parent´s education | Illiterate | Primary | Secondary | Higher |
| Residence | Alicante | Castellón | Valencia | |
| Snacks and soda | Less than 2 portions | 3 portions and over | | |
| Fruits and vegetables | Daily | No daily | | |
| Physical activity | None | Occasional | Usual | Intense |
| TV daily | $\leq$ 2 hours | 2 − 3 hours | $\geq$ 4 hours. | |

The $\chi^2$ test shown in Table 2.2 reveal that obesity in children of the region of Valencia is not significantly independent of age, education of their parents, consumption of snacks and soda drinks, and their physical activity. The incidence of obesity differs in the age groups 6 − 8 and 9 − 12 years old, this is why we model these two groups separately. The results are in accordance to the statistics obtained in (Christakis and Fowler, 2007; Evangelista et al., 2004; Jódar et al., 2008), where obesity is considered a health concern developed by unhealthy eating habits transmitted by social contact.

Table 2.2: Non parametric $\chi^2$ tests showing lack of independence with obesity in children between 6 and 12 years old in the region of Valencia, Spain.

| Variable | p-value | $\chi^2$ | Degrees of freedom |
|---|---|---|---|
| Gender | 0.066 | 5.44 | 2 |
| Age ($6-8$ year and $9-12$ year) | 0.000 | 65.34 | 2 |
| Parents education | 0.004 | 19.42 | 6 |
| Residence | 0.025 | 11.16 | 4 |
| Snacks and soda drinks | 0.009 | 9.33 | 2 |
| Fruits and vegetables | 0.045 | 6.22 | 2 |
| Physical activity | 0.007 | 17.77 | 6 |
| TV daily | 0.113 | 10.30 | 6 |

## 2.3 The Demography model

We develop an age-structured model in order to take into account the effect of the age variable on the incidence of obesity. Age-structured models have been developed for other health issues, see (Brauer and Castillo-Chavez, 2001; Diekmann and Heesterbek, 2005; Hethcote, 2000; Thieme, 2003). The first part of the model consists of using the balance equation (Thieme, 2003) for a group $G_1$ of children aged between 6 and 8 years old class, and group $G_2$ of children aged between 9 and 12 years old. The demographic model is given by the linear differential system (Hethcote, 2000):

$$\begin{cases} P_1'(t) = \pi P_{\text{total}}(t) - c_1 P_1(t) - d_1 P_1(t), \\ P_2'(t) = c_1 P_1(t) - c_2 P_2(t) - d_2 P_2(t). \end{cases} \tag{2.1}$$

where $P_{\text{total}}(t)$ is the total population size in the $6-12$ year old class, $P_i(t)$ is the population size of the group $G_i$, $\pi$ is the recruitment rate and $d_i$ and $c_i$ are respectively the death and transfer rates between the successive age groups. The real demographic data show that the death rates are almost zero in these age classes (IVE, 2007), we assume them null. In addition, from demographic data we can assume that the inflow population $\pi$ is balanced approximately by the outflow of our age group (IVE, 2007). Thus, we assume that the total number of children is constant (Anderson and May, 1991; Brauer and Castillo-Chavez, 2001). Eq. (2.1) simplifies to:

$$\begin{cases} \pi P_{\text{total}} - c_1 P_1 = 0, \\ c_1 P_1 - c_2 P_2 = 0. \end{cases} \tag{2.2}$$

As $P_{\text{total}} = P_1 + P_2$ and normalizing the total number of children to unity, one gets

$$n_1 + n_2 = 1,$$

where $n_i(t)$ is the proportion of children in the group $G_i$. Eq. (2.2) simplifies to:

$$\begin{cases} \pi - c_1 n_1 = 0, \\ c_1 n_1 - c_2 n_2 = 0. \end{cases} \tag{2.3}$$

Using $P_{\text{total}} = 1$ one gets

$$1 = \int_6^{13} \rho(a,t)\ da,$$

where $\rho(a,t)$ denotes the density of individuals of age a (measured in years) at time $t$. As $\rho$ is taken constant one obtains $\rho = \frac{1}{7}\text{year}^{-1}$, then the inflow rate is $\pi = \frac{1}{7}\text{year}^{-1}$. Using Eq. (2.3) and

$$n_1 = \int_6^{9} \rho\ da = \frac{3}{7}, \tag{2.4}$$

the parameter values of Eq. (2.2) and (2.3) are

$$c_1 = \tfrac{1}{3}, \quad c_2 = \tfrac{1}{4}, \quad n_1 = \tfrac{3}{7}, \quad n_2 = \tfrac{4}{7}. \tag{2.5}$$

The transfer rate $c_i$ is inversely proportional to the length of the age class $G_i$, the inflow population rate $\pi$ is inversely proportional to the length of the $6 - 12$ year old class. All these properties come from the assumptions of constant population and null death rate. From (IVE, 2007), the demographic data for each age class from 1990 to 2006 is available. The inflow population is approximately balanced by the outflow in our system.

## 2.4 Age-structured Obesity Model

For each age class, the population is divided into the proportion $N_i(t)$ of children with normal weight, for each age class $i$; the proportion $L_i(t)$ of children with unhealthy habits but still, normal weight; the proportion $S_i(t)$ of overweighted children; the proportion $O_i(t)$ of obese children, the proportion $D_{Si}(t)$ of overweighted children on diet, and the proportion $D_{Oi}(t)$ of obese children on diet. $N_i(t)$, $L_i(t)$, $S_i(t)$, $O_i(t)$, $D_{Si}(t)$, and $D_{Oi}(t)$, $i = 1, 2$, correspond to proportions of normal weight, latent, overweight, obese, overweight on diet and obese on diet of each age class $G_i$, $i = 1, 2$. By definition,

$$N_i(t), L_i(t), S_i(t), O_i(t), D_{Si}(t), D_{Oi}(t), \in [0, 1], i = 1, 2. \tag{2.6}$$

For the sake of clarity we obviate the explicit dependence on time. The age-structured model is depicted graphically in Figure 2.1 and analytically as follows:

$$\begin{cases}
N_1' = \pi N_0 + \varepsilon_1 D_{S1} - c_1 N_1 - \beta_1 N_1 (L_1 + S_1 + O_1 + L_2 + S_2 + O_2) \\
N_2' = \varepsilon_2 D_{S2} + c_1 N_1 - c_2 N_2 - \beta_2 N_2 (L_1 + S_1 + O_1 + L_2 + S_2 + O_2) \\
L_1' = \pi L_0 + \beta_1 N_1 (L_1 + S_1 + O_1 + L_2 + S_2 + O_2) - (c_1 + \gamma_{L1}) L_1 \\
L_2' = \beta_2 N_2 (L_1 + S_1 + O_1 + L_2 + S_2 + O_2) - (c_2 + \gamma_{L2}) L_2 + c_1 L_1 \\
S_1' = \pi S_0 + \gamma_{L1} L_1 + \varphi D_S - (c_1 + \gamma_{S1} + \alpha) S_1 \\
S_2' = \gamma_{L2} L_2 + \varphi D_S - (c_2 + \gamma_{S2} + \alpha) S_2 + c_1 S_1 \\
O_1' = \pi O_0 + \gamma_{S1} S_1 + \delta D_{O1} - (c_1 + \sigma) O_1 \\
O_2' = \gamma_{S2} S_2 + \delta D_{O2} - (c_2 + \sigma) O_2 + c_1 O_1 \\
D_{S1}' = \pi D_{S0} + \gamma_{D1} D_{O1} + \alpha S_1 - (c_1 + \varepsilon_1 + \varphi) D_{S1} \\
D_{S2}' = \gamma_{D2} D_{O2} + \alpha S_2 - (c_2 + \varepsilon_2 + \varphi) D_{S2} + c_1 D_{S1} \\
D_{O1}' = \pi D_{O0} + \sigma O_1 - (c_1 + \gamma_{D1} + \delta) D_{O1} \\
D_{O2}' = \sigma O_2 - (c_2 + \gamma_{D2} + \delta) D_{O2} + c_1 D_{O1},
\end{cases} \tag{2.7}$$

where the constant parameters for age class $G_i$ are the transmission rate $\beta_i$ to unhealthy habits due to social pressure; the children inflow rate $\pi$

for the system inversely proportional to the mean time spent by a child in the system, from 6 to 12 years old; the transfer rate $c_i$ between successive age classes; the rate $\gamma_{Li}$ at which latent children become overweighted; the rate $\gamma_{Si}$ at which overweighted children become obese by unhealthy habits; the rate $\varepsilon_i$ at which overweighted children on healthy habits become normally weighted; the rate $\alpha$ at which overweighted children stop or reduce unhealthy habits; the rate $\varphi$ at which overweighted children relapse to unhealthy habits; the rate $\sigma$ at which obese children stop or reduce unhealthy habits; the rate $\delta$ at which obese children relapse to unhealthy habits; the rate $\gamma_{Di}$ at which obese children with healthy habits become overweighted with healthy habits. The initial distribution $(N_0, L_0, S_0, O_0, D_{S0}, D_{O0})$ comes from the previous age group (5 year old) entering into the system and assumed constant. The values of the parameters $\alpha$, $\sigma$, $\varphi$, and $\delta$ are related to the diet rates of the parents, and we assume that these parameters are independent of age. Some of these pa-

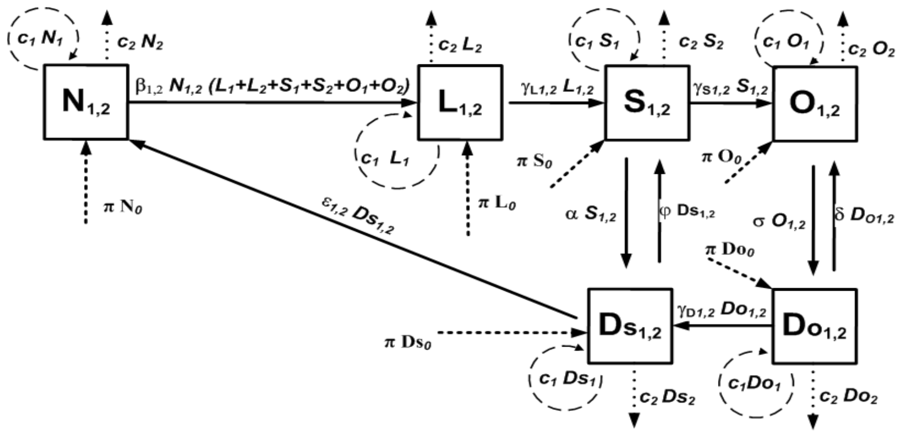



Figure 2.1: The diagram for obesity as defined in Eq. (2.7), $i = 1, 2$. Long dashed arrows represent flows between subpopulations of class $G_1$ and class $G_2$, point arrows represent outflows from subpopulations of $G_2$ and the small dashed arrows represent inflows to $G_1$.

rameters were estimated in chapter 1. Fitting the model to real data is necessary to estimate the other parameters. Each parameter is the mean of the length of the transit period between two subpopulations. The length of the transit period for a subpopulation is usually implicitly assumed to follow an exponential distribution (Brauer and Castillo-Chavez, 2001). The parameters should be considered as the average lengths of transition periods between two subpopulations.These estimates are presented in Table 2.3. Parameters $\beta_i$, $\gamma_{Li}$, and $\gamma_{Si}$ for $i = 1, 2$ still need to be estimated by the model fitting.

## 2.5 Model Fitting

The model is fitted to data provided by the Health Institution of the Government of Valencia (Fullana et al., 2004), varying parameters $\beta_i$, $\gamma_{Li}$, and $\gamma_{Si}$ $(i = 1, 2)$. An objective function $E$ from $\mathbb{R}^6$ to $\mathbb{R}$ is minimized.

We solve numerically the system of differential Eq. (2.7) with initial values of the subpopulations provided by (Fullana et al., 2004) for 1999; then we compute the mean square error between the values obtained and the data. We minimize this objective function $E$ using the Nelder-Mead algorithm (Nelder and Mead, 1964), which needs no computation of any derivative or gradient. Initial conditions are obtained from data provided by (Fullana et al., 2004) and data from the *Encuesta de Salud de la Comunidad Valenciana 2000-2001* (Health Survey of the Region of Valencia 2000-2001) (ConselleriaSanitat, 2007). The estimates of the unknown parameters values resulting from the minimization process gives $\beta_1 = 0.016$, $\beta_2 = 0.0008$, $\gamma_{L1} = 0.013$, $\gamma_{L2} = 0.0011$, $\gamma_{S1} = 0.003$ and $\gamma_{S2} = 0.0002$. These values give the global minimum of the objective function $E$.

Though the two age classes $G_1$ and $G_2$ have the same increasing trend, the estimated transmission parameter value $\beta_2$ corresponding to age class $G_2$ is smaller than $\beta_1$ of age class $G_1$. Similarly with the other parameters. This fact means that once the children of class $G_2$ are overweighted and obese it is easier to maintain the same increasing trend of overweight and obesity of the previous age class $G_1$.

## 2.6 Simulation of Scenarios

Simulations illustrate some possible behaviors in the next few years of overweighted and obese subpopulations among the $6-8$ and $9-12$ years old in the region of Valencia, Spain. First, a numerical simulation from 1999 to 2010 uses the model with the estimated parameter values shown in Table 2.3 and obtained from the model fitting. As shown in Figure 2.2, obese and overweighted subpopulations in $G_1$ and $G_2$ present an increasing trend in accordance with the one observed in other countries (Bes-Rastrollo et al., 2006; Wang and Lobstein, 2006). The subpopulation of normal weight decreases and the percentage of children on diets remains very low.

Table 2.3: Parameter values in the model (2.7) for the region of Valencia.

| Parameters | Symbol | Values (weeks$^{-1}$) |
|---|---|---|
| Children inflow rate | $\pi$ | 0.0027 |
| Transfer rate from $G_1$ to $G_2$ | $c_1$ | 0.0064 |
| Children outflow rate | $c_2$ | 0.0048 |
| Population flow rate from $D_{S_i}$ to $N_i$ | $\varepsilon_i$ | $0.31 \times \gamma_{Li}$ |
| Population flow rate from $S_i$ to $D_{S_i}$ | $\alpha$ | 0.0041 |
| Population flow rate from $D_{S_i}$ to $S_i$ | $\varphi$ | 0.1273 |
| Population flow rate from $O_i$ to $D_{O_i}$ | $\sigma$ | 0.0044 |
| Population flow rate from $D_{O_i}$ to $O_i$ | $\delta$ | 0.1597 |
| Population flow rate from $D_{O_i}$ to $D_{S_i}$ | $\gamma_{Di}$ | $0.14 \times \gamma_{Si}$ |

In addition, in order to analyze the prophilaxy of obesity, the parameters $\beta_1, \gamma_{L1}$, and $\gamma_{S1}$, are reduced of 50%. The simulation of this case is shown in Figure 2.3, where the population size of normal weight increases and the population size of obese children decreases. A second case is shown in Figure 2.4, where the parameters $\beta_2, \gamma_{L2}$, and $\gamma_{S2}$ corresponding to $G_2$ are reduced of 50%, in this case the increasing trend of overweight and obesity varies only slightly. From these facts, reducing the parameters $\beta_1, \gamma_{L1}$, and $\gamma_{S1}$ in $G_1$ is more efficient than reducing their counterparts parameters $\beta_2, \gamma_{L2}$, and $\gamma_{S2}$ in $G_2$. Moreover, the control parameters of diets $\alpha_i$

and $\sigma_i$ $(i = 1, 2)$ were modified but their effects on overweight and obese populations are very small. Therefore, it is important to design strategies to reduce obesity in children´s early years.

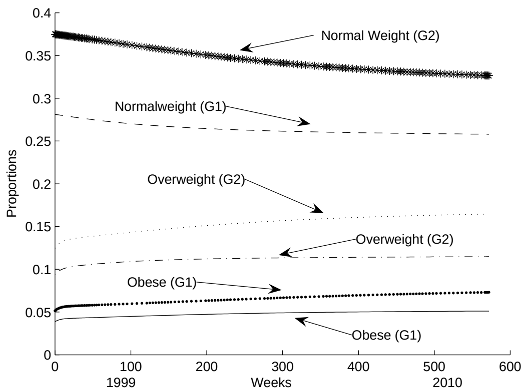



Figure 2.2: Variation of the different subpopulations of $G_1$ and $G_2$ in the period $1999 - 2010$.

## 2.7 Conclusion

Our age-structured model took advantage of a statistical study to include the fact that obesity among the $6 - 12$ year old children of the region of Valencia, Spain, depends on the age and education of their parents, consumption of snacks and soda drinks, and physical activity. Then incidence of obesity is different in classes $6 - 8$ and $9 - 12$, therefore the model distinguishes these two groups.

The model helps to understand the spread of obesity and predict it for these age classes. The simulations showed an increasing number of overweighted and obese in the next few years. This tendency ought to be modified by social educative campaigns. If two consecutive age classes $G_1$ and $G_2$ have the same increasing trend of obesity, children of group $G_2$ can have healthier habits than those of $G_1$ and maintain the same increasing

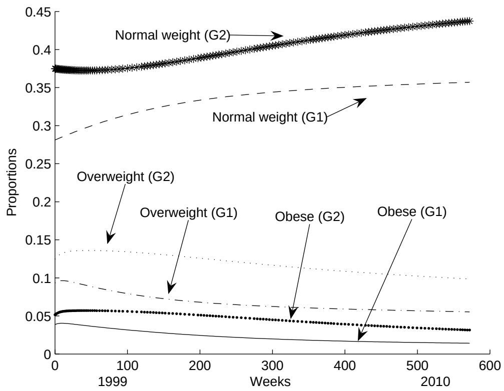



Figure 2.3: Variation of the different subpopulations of $G_1$ and $G_2$ when $\beta_1, \gamma_{L1}$ and $\gamma_{S1}$ related to obesity growth of $G_1$ are reduced of 50%.

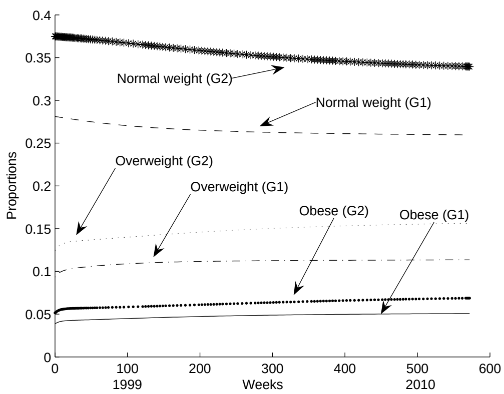



Figure 2.4: Variation of the different subpopulations of $G_1$ and $G_2$ when $\beta_2, \gamma_{L2}$ and $\gamma_{S2}$ related to obesity growth of $G_2$ are reduced of 50%.

structural trend. Failing to prevent obesity in children´s early years makes it more difficult to reverse this trend.

# Chapter 3

# A nonstandard dynamically consistent numerical scheme applied to obesity dynamics †

In this chapter a nonstandard finite difference scheme has been developed with the aim to solve numerically a mathematical model for obesity population dynamics. This interacting population model represented as a system of coupled nonlinear ordinary differential equations is used to analyze, understand and predict the dynamics of obesity populations. The construction of the proposed discrete scheme is developed such that it is dynamically consistent with the original differential equations model. Since the total population in this mathematical model is assumed constant, the proposed scheme has been constructed to satisfy the associated conservation law and positivity condition. Numerical comparisons between the competitive nonstandard scheme developed here and Euler method show the effectiveness of the proposed nonstandard numerical scheme. Numerical examples show

† This chapter is based on (Villanueva et al., 2008)

that the nonstandard difference scheme methodology is a good option to solve numerically different mathematical models where essential properties of the populations need to be satisfied in order to simulate the real world.

## 3.1 Introduction

Systems of nonlinear ordinary differential equations are used to model different kind of diseases in mathematical epidemiology (see (Brauer and Castillo-Chavez, 2001)). In these models the variables commonly represent populations of susceptibles, infected, recovered, transmitted disease vectors, and so forth. Since the system describes the evolution of different classes of populations a solution over time is required. The ideal scenario is to obtain analytical solutions, but in most of the cases this is not possible. Therefore it is necessary to turn to numerical methods to obtain approximate solutions.

Traditional schemes like forward Euler, Runge-Kutta and other numerical methods used to solve nonlinear initial value problems, sometimes fail by generating oscillations, bifurcations, chaos and steady states not present in the exact solutions (Lambert, 1973). Methods that use adaptive step size, may also fail (Moghadas et al., 2003). One approach to avoid this class of numerical instabilities is the construction of nonstandard finite difference schemes. This technique, developed by Mickens (1994, 2000) has brought a lot of applications where the nonstandard methods have been applied to various problems in science and engineering in which the numerical solutions preserve properties of the solution of the continuous model and additionally, in some cases, it is possible to use large time step sizes (Moghadas et al., 2004; Anguelov and Lubuma, 2003b,a; Piyawong et al., 2003; Gumel et al., 2003; Dimitrov and Kojouharov, 2005; Jansen and Twizell, 2002; Patidar, 2005; Chen et al., 1996). Furthermore, it has been shown that the nonstandard finite difference schemes, generated using the rules created by Mickens (1994), generally provide accurate numerical solutions to differential equations.

The aim of this chapter is to develop a nonstandard finite difference (NSFD) scheme free of numerical instabilities in order to obtain the numerical solution of a mathematical model of infant obesity with constant population size presented in chapter 1. This interacting population model represented as a system of coupled nonlinear ordinary differential equations exhibits two steady states; a trivial steady state called obesity free equilibrium (OFE) and a non-trivial steady state called obesity endemic equilibrium (OEE). The general philosophy for constructing NSFD schemes is to obtain numerical solutions dynamically consistent with the underlying continuous model. This means that all of the critical, qualitative properties of the solutions to the system of differential equations should also be satisfied by the solutions of the discrete scheme (Mickens, 2005).

The design of a nonstandard scheme is not a straightforward task, in fact many schemes may be developed for one model and several can fail to converge. Therefore the innovative part in this work is the construction of the NSFD scheme. All the numerical simulations with different parameter values suggest that the NSFD scheme developed here preserves numerical stability in larger regions in comparison to the smaller regions of other standard numerical methods. However, theoretical justification of this fact is not possible due to unmanageable analytic expression of the eigenvalues of the Jacobian matrix corresponding to the linearization at the equilibrium points. It is important to remark that most of the previous applications of nonstandard methods to ODE's have been done to one, two and three equations, where theoretical analysis is easier.

Since subpopulations must never take on negative values, the proposed NSFD scheme is designed to satisfy the positivity condition. Furthermore, due to the fact that in the mathematical model the population is assumed constant, the NSFD scheme has been developed in order to always exactly satisfy the associated conservation law (Mickens, 2007).

The organization of this chapter is as follows. In Section 2 the mathematical model for the evolution of overweight and obesity population is presented. The construction of the proposed NSFD numerical scheme is carried out in Section 3. In Section 4 a basic analysis of the stability of the mathematical model is developed using data of the region of Valencia, Spain. Additionally, numerical analysis of the NSFD numerical scheme is done. Numerical simulations for the NSFD scheme are reported also in this section in order to show its computational advantages. Discussion and conclusions are presented in Section 5.

## 3.2 Mathematical model

In chapter 1 a dynamic obesity model for the evolution of infant overweight and obesity population was introduced. This continuous model is based on the partition of the infant population into six subpopulations. The model is represented by the nonlinear system of ordinary differential equations (1.1), where the parameters are time-invariant. In this model since the constant population is normalized to unity one gets for all time $t$,

$$N(t) + L(t) + S(t) + O(t) + D_S(t) + D_O(t) = 1. \qquad (3.1)$$

We define equation (3.1) as the conservation law associated with the system (1.1) and this equation must be satisfied by the NSFD numerical scheme for any time $t$.

## 3.3 Nonstandard finite difference discretization

In order to avoid dynamic inconsistencies such as oscillations and numerical instabilities in this section a NSFD scheme for solving numerically system (1.1) is constructed. A numerical scheme is called NSFD if at least one of the following conditions is satisfied (Anguelov and Lubuma, 2003a; Lubuma and Patidar, 2003):

1. Nonlocal approximation is used (Mickens, 1994, 2000; Anguelov and Lubuma, 2003b).

2. Discretization of derivative is not traditional and a nonnegative function,

$$\psi(h) = h + \mathcal{O}(h^2), \tag{3.2}$$

called a denominator function is used (Mickens, 2006).

The discretization is based on the approximations of the temporal derivatives by a first order generalized forward scheme. If $f(t) \in C^1(\mathbb{R})$, we define an equivalent derivative as

$$\frac{df(t)}{dt} = \frac{f(t+h) - f(t)}{\psi(h)} + \mathcal{O}(\psi(h)) \text{ as } h \longrightarrow 0. \tag{3.3}$$

where $\psi(h)$ is a nonnegative real valued function defined in (3.2). Note that the above definition is consistent with the traditional definition of the derivative. Indeed

$$\begin{aligned}\frac{df(t)}{dt} &= \lim_{h\longrightarrow 0}\left\{\frac{f(t+h)-f(t)}{\psi(h)} + \mathcal{O}(\varphi(h))\right\} \\ &= \lim_{h\longrightarrow 0}\frac{f(t+h)-f(t)}{h}\lim_{h\longrightarrow 0}\frac{h}{\psi(h)} + \lim_{h\longrightarrow 0}\mathcal{O}(\psi(h)) \\ &= \dot{f}(t).\end{aligned}$$

An example of a function $\psi(h)$ that satisfies the above conditions is $\psi(h) = 1 - e^{-h}$, see (Mickens, 1994, 2006).

### 3.3.1 Construction of the NSFD scheme

In this subsection the main goal is to construct a dynamically consistent numerical discrete scheme for solving system (1.1). Notice that all the model variables (subpopulations) and parameters are positive. Therefore, in order to obtain a dynamically consistent discrete scheme, we must ensure that the resulting discrete solutions are all positive which is necessary to avoid scheme dependent instabilities. Furthermore, since the population is assumed constant, the NSFD scheme should satisfy the associated conservation law (Mickens, 2007). These aspects will be taken into account in the construction of the numerical scheme below.

Let us denote by $N^n$, $L^n$, $S^n$, $O^n$, $D_S^n$ and $D_O^n$ the approximations of $N(nh)$, $L(nh)$, $S(nh)$, $O(nh)$, $D_S(nh)$ and $D_O(nh)$, respectively, for $n = 0, 1, 2...$, and by $h$ the time step size of the scheme. Using (3.3), the first order numerical scheme used to obtain solutions $N$, $L$, $S$, $O$, $D_S$ and $D_O$ of the model (1.1) is given by the following semi-implicit scheme

$$
\begin{aligned}
\frac{D_O^{n+1} - D_O^n}{\psi(h)} &= \sigma O^n - (\delta + \mu + \gamma_D) D_O^{n+1}, \\
\frac{D_S^{n+1} - D_S^n}{\psi(h)} &= -(\mu + \epsilon + \varphi) D_S^{n+1} + \alpha S^n + \gamma_D D_O^{n+1}, \\
\frac{N^{n+1} - N^n}{\psi(h)} &= \mu - \beta N^{n+1}(S^n + O^n + L^n) + \epsilon D_S^{n+1} - \mu N^{n+1}, \\
\frac{L^{n+1} - L^n}{\psi(h)} &= \beta N^{n+1}(S^n + O^n + L^n) - (\gamma_L + \mu) L^{n+1}, \\
\frac{S^{n+1} - S^n}{\psi(h)} &= \gamma_L L^{n+1} + \varphi D_S^{n+1} - \alpha S^n - (\gamma_S + \mu) S^{n+1}, \\
\frac{O^{n+1} - O^n}{\psi(h)} &= \gamma_S S^{n+1} - \sigma O^n - \mu O^{n+1} + \delta D_O^{n+1},
\end{aligned}
\tag{3.4}
$$

and after rearranging, we obtain the following explicit form

$$D_O^{n+1} = \frac{D_O^n + \psi(h)\sigma O^n}{1 + \psi(h)(\delta + \mu + \gamma_D)}, \tag{3.5}$$

$$D_S^{n+1} = \frac{D_S^n + \psi(h)(\alpha S^n + \gamma_D D_O^{n+1})}{1 + \psi(h)(\epsilon + \mu + \varphi)}, \tag{3.6}$$

$$N^{n+1} = \frac{N^n + \psi(h)(\mu + \epsilon D_S^{n+1})}{1 + \psi(h)(\mu + \beta(S^n + O^n + L^n))}, \tag{3.7}$$

$$L^{n+1} = \frac{L^n + \psi(h)\beta N^{n+1}(S^n + O^n + L^n)}{1 + \psi(h)(\gamma_L + \mu)}, \tag{3.8}$$

$$S^{n+1} = \frac{S^n(1 - \psi(h)\alpha) + \psi(h)(\gamma_L L^{n+1} + \varphi D_S^{n+1})}{1 + \psi(h)(\gamma_S + \mu)}, \tag{3.9}$$

$$O^{n+1} = \frac{O^n(1-\psi(h)\sigma) + \psi(h)(\gamma_S S^{n+1} + \delta D_O^{n+1})}{1+\psi(h)\mu}, \tag{3.10}$$

where the denominator function is chosen as:

$$\psi(h) = \frac{1-e^{-hM}}{M},$$

with $M \geq \max\{\alpha, \sigma\}$ in order to guarantee the positivity condition. However, positivity can also be guaranteed taking the traditional $\psi(h) = h$ if $hM \leq 1$. Therefore, throughout this chapter when $hM \leq 1$ we use $\psi(h) = h$.

**Remark 1** Notice that the populations must be calculated in a particular order. This sequential form of calculation is a generic feature of NSFD methods (Mickens, 1994, 2005, 2006).

**Remark 2** The incorporation of equation (3.1) to system (1.1) introduces more complexity in order to satisfy that, if the same term occurs in more than one differential equation, then it must be modeled discretely the same way in all of the equations (Mickens, 1994, 2007).

### 3.3.2 Properties and computation in the NSFD scheme

It can be seen from system (3.5-3.10) that we have constructed a $NSFD$ scheme for the model (1.1) having the following properties:

1. Positivity. If $D_O^n > 0$, $D_S^n > 0$, $N^n > 0$, $L^n > 0$, $S^n > 0$, $O^n > 0$ then $D_O^{n+1} > 0$, $D_S^{n+1} > 0$, $N^{n+1} > 0$, $L^{n+1} > 0$, $S^{n+1} > 0$, $O^{n+1} > 0$, for all $n = 0, 1, 2, \ldots$, since that $1 - \psi(h)M > 0$,

2. Satisfy conservation law i.e. constant population. From system (3.4) one gets that

$$D_O^{n+1} + D_S^{n+1} + N^{n+1} + L^{n+1} + S^{n+1} + O^{n+1} = \frac{D_O^n + D_S^n + N^n + L^n}{1+\mu\psi(h)} + \frac{S^n + O^n + \mu\psi(h)}{1+\mu\psi(h)}.$$

It follows by induction that if $D_O^n + D_S^n + N^n + L^n + S^n + O^n = 1$, then

$$D_O^{n+1} + D_S^{n+1} + N^{n+1} + L^{n+1} + S^{n+1} + O^{n+1} = 1, \text{ for all}\, n = 0, 1, \ldots.$$

Now, observe that the subpopulations must be calculated from scheme (3.5-3.10) in the following order:

1. Select values $D_O^n > 0$, $D_S^n > 0$, $N^n > 0$, $L^n > 0$, $S^n > 0$, $O^n > 0$, such that $D_O^n + D_S^n + N^n + L^n + S^n + O^n = 1$.
2. Determine $D_O^{n+1}$ from (3.5) using $D_O^n$ and $O^n$.
3. Calculate $D_S^{n+1}$ from (3.6) with $D_O^{n+1}$, $D_S^n$ and $O^n$.
4. Obtain $N^{n+1}$ from (3.7) using $D_S^{n+1}$, $N^n$, $S^n$, $O^n$ and $L^n$.
5. Found $L^{n+1}$ from (3.8) of knowledge $N^{n+1}$, $S^n$, $O^n$ and $L^n$.
6. Get $S^{n+1}$ from (3.9) using $S^n$, $L^{n+1}$ and $D_S^{n+1}$.
7. Compute $O^{n+1}$ from (3.10) with the values of $O^n$, $S^{n+1}$ and $D_O^{n+1}$.

Note that the sequential form of calculation is a generic feature of $NSFD$ schemes (Mickens, 2007). In addition, it can be seen that the main part of the local truncation error associated with system (3.5-3.10) is of order $\mathcal{O}(h^2)$, confirming that the constructed NSFD scheme is first order accurate.

## 3.4 Numerical results and dynamic consistency

In this section, some theoretical properties of the mathematical model of overweight and obesity population dynamics described by the nonlinear system of ordinary differential equations (1.1) for a particular set of values of the parameters are studied. These theoretical properties allow us know a priori the correct behavior that need to come out from the numerical solution corresponding to the NSFD numerical scheme (3.4). Furthermore, this section is devoted to show numerically the dynamic consistency and numerical advantages of the developed NSFD scheme and some comparisons with other well known numerical methods. The chosen values for the set of parameters of model (1.1), are the ones used in chapter 1 and are shown in Table 1.2 in a week scale.

**Numerical stability of the mathematical model**

A linear stability analysis of system (1.1) for a particular set of values of the parameters is developed here in order to check numerically dynamic consistency between the continuous model and the NSFD discrete scheme. A more general stability analysis has not been performed since general parameters gives unmanageable analytic expressions. Nevertheless, several set of values of the parameters are used to suggest numerically the aforementioned dynamic consistency.

It is well known that an equilibrium point is an asymptotically stable node if and only if the real part of the eigenvalues of the Jacobian associated system evaluated at the equilibrium points are negative.

The system (1.1) has two steady states; a trivial steady state called obesity free equilibrium (OFE) and a non-trivial steady state called obesity endemic equilibrium (OEE). For the particular set of values of the parameters presented in Table 1.2 for system (1.1), the following equilibrium points are obtained:

$$OFE = (1, 0, 0, 0, 0, 0), OEE \approx (0.294, 0.297, 0.271, 0.126, 0.008, 0.003).$$

The eigenvalues of the Jacobian evaluated at the OFE point and at the positive OEE point are shown in Table 3.1. The OFE point is unstable, since the eigenvalue $\lambda_4$ is positive. On the other hand the OEE point is locally asymptotically stable, due to the fact that all real parts of the eigenvalues of the Jacobian of system (1.1) evaluated at the OEE point are negative. Therefore, a dynamic consistent numerical scheme should reproduce this numerical behavior of the continuous model. In next subsection this fact will be checked.

## 3.5 Numerical simulations

This subsection is devoted to show numerically the dynamic consistency and numerical advantages of the developed NSFD scheme. One basic property that should satisfy the NSFD numerical scheme (3.5-3.10) in order

| Eigenvalue | At OFE point | At the OEE point |
|---|---|---|
| $\lambda_1$ | -0.170976 | -0.170977 |
| $\lambda_2$ | -0.140644 | -0.140622 |
| $\lambda_3$ | -0.0151898 | -0.0143034 + 0.00134272i |
| $\lambda_4$ | 0.0134083 | -0.0143034-0.00134272i |
| $\lambda_5$ | -0.00921949 | -0.00909934 |
| $\lambda_6$ | -0.00641026 | -0.00641026 |

Table 3.1: The eigenvalues of the Jacobian of system (1.1) evaluated at the OFE point and at the OEE point.

to be dynamically consistent is that their fixed points should be the same equilibrium points as the continuous model (1.1). The equilibrium points of the NSFD numerical scheme (3.4) are obtained by setting to zero the left-hand sides of system (3.4). It is easy to prove that the NSFD numerical scheme (3.4) have the same OFE point that the continuous model for any set of parameter values.

Numerical simulation for different set of values of the parameters were performed to investigate the dynamic consistency of the developed NSFD numerical scheme (3.4). In Figure 3.1 it can be seen that the NSFD scheme (3.4) converges to the same OEE point of the continuous model using the particular set of values of the parameters shown in Table 1.2.

Additionally in Figure 3.2 it is observed that despite the initial condition is close to the OFE point, the numerical solution move further away from this equilibrium point, suggesting numerically the instability of the OFE point and the consistency of the NSFD numerical scheme with the continuous model (1.1).

**Numerical comparisons of the NSFD scheme**

In order to study the effect of the time step size in the NSFD numerical scheme and dynamic consistency related to local stability, the spectral radius $\rho$ of the Jacobian corresponding to the linearization of NSFD scheme and Euler schemes are computed for different time steps for system (1.1). Using general parameters values one gets a general Jacobian. However, the

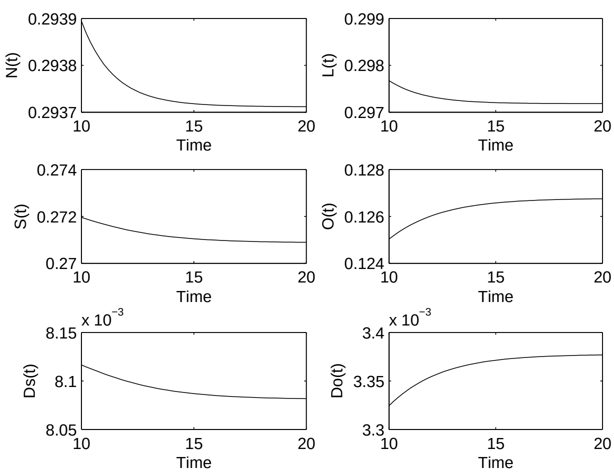



Figure 3.1: Profiles of subpopulations $N(t)$, $L(t)$, $S(t)$, $O(t)$, $D_S(t)$ and $D_O(t)$. Numerical solution is obtained by the proposed NSFD scheme with $\psi(h) = h$, where $h = 0.2$ and initial conditions are $N(0) = 0.462$, $L(0) = 0.194$, $S(0) = 0.2176$, $O(0) = 0.09$, $D_S(0) = 0.0249$ and $D_O(0) = 0.0115$.

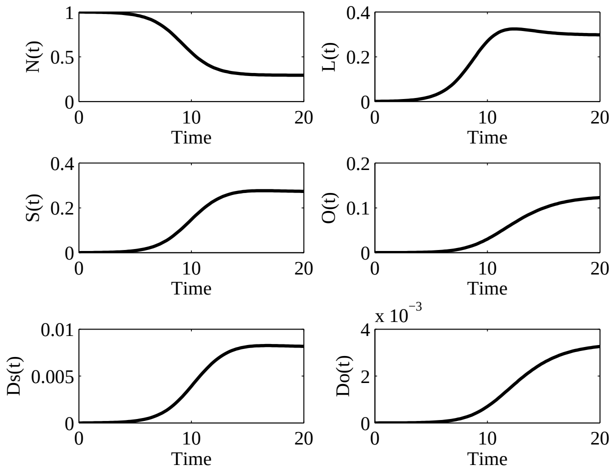



Figure 3.2: Profiles of subpopulations $N(t)$, $L(t)$, $S(t)$, $O(t)$, $D_S(t)$ and $D_O(t)$ obtained with the NSFD scheme with $\psi(h) = h$, $h = 0.2$ and initial conditions are $N(0) = 0.999, L(0) = 0.001, S(0) = 0, O(0) = 0, D_S(0) = 0$, and $D_O(0) = 0$.

functional relationship between the spectral radius $\rho$ and the time step size $h$ gives an unmanageable complex analytic expression. Therefore, in order to investigate numerically the dynamic consistency of the NSFD scheme, the particular set of values of the parameters presented in Table 1.2 (but in year scale) is used for numerical results. However, other sets of values of the parameters have been used to confirm the previous numerical results.

It is well known that a necessary and sufficient condition for the convergence of a fixed point scheme is that the spectral radius of the Jacobian satisfies $\rho < 1$. In Table 3.2 the spectral radius $\rho$ of the Jacobian at the OEE point of the NSFD and Euler schemes for different time steps $h$ are presented. Table 3.2 compares the convergence properties of the Euler scheme with that of NSFD scheme when used to integrate the system (1.1) subject to the same initial conditions and parameter values. It is clear from Table 3.2 that the NSFD scheme is more competitive in terms of numerical stability. In all of these simulations, the NSFD scheme is seen to be monotonically convergent to the correct endemic equilibrium point. However, Euler fails to converge for a time step size of $h = 0.25$. The Runge-Kutta fourth order scheme improves the result obtained by Euler scheme, but fails to converge for time step sizes $h > 0.33$ when the particular set of values of the parameters presented in Table 1.2 is used. As expected in other numerical simulations with several different values of the parameters the Runge-Kutta fourth order method outperform Euler scheme, since convergence is achieved for larger time step sizes.

Table 3.2 compares the convergence properties of the Euler scheme with that of the NSFD scheme for a specific set of values of the parameters obtained from a real world application. However, several different values of the parameters have been used and numerical results suggest that the NSFD scheme developed here preserves numerical stability in larger regions in comparison to the smaller regions of the Euler numerical scheme. However, theoretical justification of this fact is not possible due to unmanageable analytic expression of the eigenvalues of the Jacobian matrix corresponding to the linearization at the equilibrium points. It is important to remark that most of the previous applications of nonstandard methods to ODE's have been done to one, two and three equations, where theoretical analysis is easier. In Figure 3.3 it can be seen that the NSFD scheme converges and

| Time step $h$ | $\rho$(Euler) | Euler | $\rho$(NSFD) | NSFD($\psi(h) = h$) |
|---|---|---|---|---|
| 0.01 | 0.996667 | Convergence | 0.996678 | Convergence |
| 0.1 | 0.9935 | Convergence | 0.967742 | Convergence |
| 0.2 | 0.933333 | Convergence | 0.9375 | Convergence |
| 0.25 | 1.22302 | Divergence | 0.923077 | Convergence |
| 0.5 | 3.44604 | Divergence | 0.857143 | Convergence |
| 1 | 7.89207 | Divergence | 0.75 | Convergence |
| 10 | 87.920 | Divergence | 0.230769 | Convergence |

Table 3.2: Spectral radius for different time step sizes $h$ of the Euler and NSFD numerical schemes.

the Euler scheme oscillates incorrectly but converges to the OEE point for a time step size $h = 0.2$. In Figure 3.4 it is depicted that the NSFD scheme converges to the correct OEE point for a time step size $h = 0.25$, but in this case Euler fails to converge. Finally Figure 3.5 shows that the NSFD scheme converges to the OEE point taking only positive values. Nevertheless, it can be observed that the several routines from Matlab package produce incorrect negative values.

## 3.6 Discussion and conclusions

In this chapter we applied the NSFD methodology to develop a scheme to solve numerically a mathematical model for obesity population dynamics. This interacting population model represented as a system of coupled nonlinear ordinary differential equations can be used to analyze, understand, and predict the dynamics of overweight and obese populations.

The advantages of the NSFD scheme developed here are that they preserve numerical stability in larger regions for the time step size in comparison to the smaller regions of numerical stability of the approximations

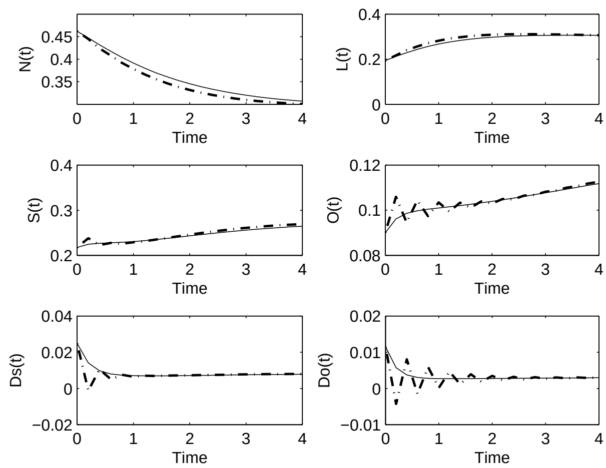



Figure 3.3: From top to right, profiles of subpopulations $N(t), L(t), S(t)$, $O(t)$, $D_S(t)$ and $D_O(t)$. Solutions are obtained by Euler(dashed) and the proposed NSFD scheme(line) with $\psi(h) = h$ and $h = 0.2$ for system (1.1).

obtained by the other standard numerical methods. Furthermore, the proposed scheme satisfies the positivity condition and the conservation law that any good scheme for mathematical models of population dynamics must fulfill. The construction of these nonstandard schemes is not a straightforward task, in fact many schemes may be developed for one model and several can fail to converge. Nevertheless, the principle of dynamic consistency can be used with great efficiency to place restrictions on the design of NSFD schemes in order to obtain efficient schemes, as has been shown in this work.

The NSFD scheme proposed here was analyzed and tested in several numerical simulations. All the numerical simulations performed in this work show that the NSFD scheme converges to the OEE point for any time step size and satisfies a condition of positivity required for populations. Finally, it is important to remark that the developed NSFD scheme is easy to use and convergence is achieved for larger time step sizes than those

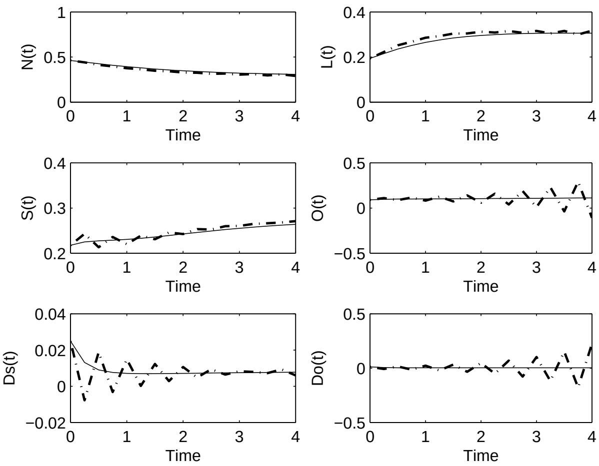



Figure 3.4: From top to right, profiles of subpopulations $N(t), L(t), S(t)$, $O(t)$, $D_S(t)$ and $D_O(t)$. Solutions are obtained by Euler scheme (dashed) and developed NSFD scheme(line) with $\psi(h) = h$ and $h = 0.25$ for system (1.1).

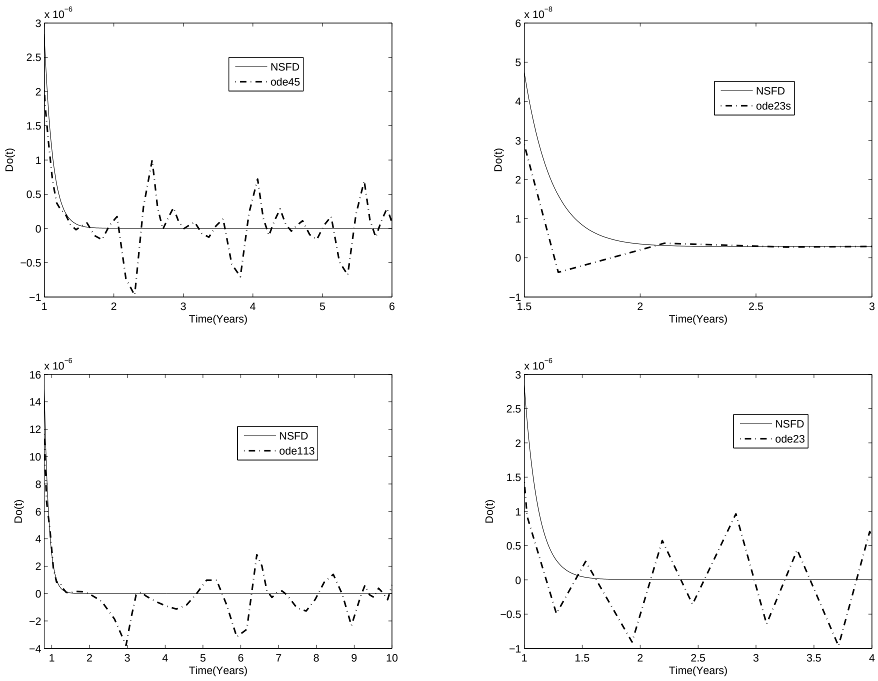



Figure 3.5: Solutions for subpopulation $D_O$ obtained by NSFD scheme and routines ode45, ode23s, ode113 and ode23 from Matlab package(dashed), using $\sigma = 0.0044379 \times 10^{-5}$.

for the Euler or the Runge-Kutta fourth order schemes. The NSFD scheme also outperforms Matlab package routines, since sometimes negatives values for populations are obtained with these routines. Therefore, for real world applications in nature and society, where numerical values are required it is important to be cautious about which numerical scheme is more suitable to solve the mathematical model, since unrealistic results can be obtained.

# Chapter 4

# Periodic solutions of nonautonomous differential systems modeling obesity population

†

In this chapter we study the periodic behavior of the solutions of a nonautonomous model for obesity population. The mathematical model represented by a nonautonomous system of nonlinear ordinary differential equations is used to model the dynamics of obese populations. Numerical simulations suggest periodic behavior of subpopulations solutions. Sufficient conditions which guarantee the existence of a periodic positive solution are obtained using a continuation theorem based on coincidence degree theory.

†This chapter is based on Arenas et al. (2009b)

## 4.1 Introduction

Obesity is growing at an important rate in developed and developing countries and it is becoming a serious disease not only from the individual health point of view but also from the public socioeconomic one, motivated by the high cost of the Health Public Care System due to the expense of treating people suffering related fatal diseases such as diabetes, heart attacks, blindness, renal failures and nonfatal related diseases such as respiratory difficulties, arthritis, infertility and psychological disorders CDC (2007b).

Mathematical models have been revealed as a powerful tool to analyze the epidemiology of the infectious illness, to understand its behavior, to predict its social impact and to find out how external factors change the impact. A classical technique to model the behavior of communicable diseases in the population is by means of systems of ordinary differential equations, where the variables represent different subpopulations such infected, susceptible, vaccinated and others. Once the model is constructed, it is fitted to clinical data related to infected individuals in order to find the values of the unknown parameters.

In chapter 1 an autonomous mathematical model of obesity with constant size population was presented for study the dynamical evolution of obesity in the childhood population. However the same model can be extended to a whole population modifying the parameters values to adapt them to the studied population. This mathematical model is structured with a set of six ordinary differential equations (ODE) giving rise to an autonomous system of ODE.

In this chapter we modify the time-invariant parameter obesity model introduced in chapter 1, to a nonautonomous model. The modification of the previous autonomous model is justified by the fact that it is more realistic to assume that the social and physical environment has fluctuations over the time. This seasonal effect has been studied in several works related to overweight and obesity of individuals Plasqui and Westerterp (2004); Westerterp (2001); Van Staveren et al. (1986); Kobayashi (2006); Katzmarzyk and Leonard (1998); Tobe et al. (1994). For instance in Plasqui and Westerterp (2004) a seasonal variation in the physical activity level in Dutch young adults is found. Additionally, in Westerterp (2001) it is concluded that body weight shows a clear seasonal variation triggered by ambient temperature. Furthermore, in Van Staveren et al. (1986) longitudinal surveys shows that mean bodyweight in populations consuming self-selected diets consistently show an increase of about 0.5 kg around Christmas time, and a decrease to a minimum value in July, although there is no corresponding change in reported energy intake, and little change in mean bodyweight from one year to the next. The aforementioned fluctuation can be modeled using the assumption of periodicity for the parameter $\beta(t)$, which is a standard way to incorporate the seasonality of the spread of communicable diseases in the environment Keeling et al. (2001).

The study of global existence of positive periodic solutions of models of dynamic populations in a periodic environment is an important problem. Several works have presented the assumption that the contact rate is a general continuous, bounded, positive and periodic function with period $T$ and the authors have shown the existence of positive periodic solution with the help of the continuation theorem based on coincidence degree J.Hui and Zhu (2005); Arenas et al. (2008a); Xia et al. (2007). In addition some authors have been studied periodic solutions for other population mathematical models, see Huoa et al. (2007); Kouche and Tatar (2007).

Since the global existence of positive periodic solutions plays a similar role as a globally stable equilibrium does in the autonomous model, in this work we seek conditions for the existence of periodic solutions for the time variant mathematical model introduced here. In particular this study is done using the continuation theorem based on coincidence degree theory Gaines and Mawhin (1977).

The existence of periodic solutions is important in this obesity population dynamic model since obesity is increasing worldwide and several studies are now being developed. Therefore, knowing that the behavior of obese and overweight populations present periodic oscillations is important in order to do more accurate studies.

The chapter is organized as follows. Mathematical preliminaries regarding continuous and differentiable functions, Brouwer degree, definitions of Fredholm mappings, coincidence degree theory and Jean Mawhin's Continuation Theorem are presented in Section 2. In Section 3, the mathematical model is presented. In Section 4, numerical simulations of the mathematical model are presented and the periodic behavior of the subpopulations are observed. Finally in Section 5, using Mawhin's Continuation Theorem, it is proved that the nonautonomous model introduced in this chapter has at least one positive periodic solution.

## 4.2 Preliminaries

In this section we provide a brief introduction to continuous and differentiable functions, Brouwer degree, definitions of Fredholm mappings, coincidence degree theory and Jean Mawhin's Continuation Theorem. The aim of this section is to provide the reader an adequate framework for the understanding of this section. More information can be found in $O'$Regan et al. (2006), Gaines and Mawhin (1977), Dieudonne (1969), Ward (2008).

### 4.2.1 Normed spaces

In this section we give some definitions on normed space.

**Definition 4.2.1** *A (real) complex normed space is a (real) complex vector space $X$ together with a map* $: X \longrightarrow \mathbb{R}$, *called the norm and denoted* $\|\cdot\|$, *such that*

1. $\|x\| \geq 0$, *for all* $x \in X$, *and* $\|x\| = 0$ *if and only if* $x = 0$.

2. $\|\alpha x\| = |\alpha|\|x\|$, *for all* $x \in X$ *and all* $\alpha \in \mathbb{C}$ *(or* $\mathbb{R}$*).*

3. $\|x + y\| \leq \|x\| + \|y\|$, *for all* $x, y \in X$.

**Remark 4.2.2** *If in (1) we only require that* $\|x\| \geq 0$, *for all* $x \in X$, *then* $\|\cdot\|$ *is called a seminorm.*

**Remark 4.2.3** *If $X$ is a normed space with norm $\|\cdot\|$, it is readily checked that the formula $d(x,y) = \|x-y\|$, for $x, y \in X$, defines a metric $d$ on $X$. Thus a normed space is naturally a metric space and all metric space concepts are meaningful. For example, convergence of sequences in X means convergence with respect to the above metric.*

**Definition 4.2.4** *A complete normed space is called a Banach space.*

Thus, a normed space $X$ is a Banach space if every Cauchy sequence in $X$ converges (where $X$ is given the metric space structure as outlined above). One may consider real or complex Banach spaces depending, of course, on whether $X$ is a real or complex linear space.

**Definition 4.2.5** *Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent if only if there are positive constants $\mu$, $\mu'$ such that*

$$\mu\|x\|_a \leq \|x\|_b \leq \mu'\|x\|_a$$

*for all $x \in X$.*

As a consequence of the above definition, we have the following result

**Proposition 4.2.6** *Let $X$ a normed space of finite dimension. Then all norms are equivalent.*

### 4.2.2 Continuous and Differentiable Functions

Here, we introduce some basic concepts of continuous and differentiable functions.

**Definition 4.2.7** *Let $\Omega \subset \mathbb{R}^n$ be an open subset and $f$ a function $f : \Omega \longrightarrow \mathbb{R}$. We say that $f$ has a local maximum at a point $x_0 \in \Omega$ if there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in \Omega$ with $\|x - x_0\| < \delta$. Local minima are defined likewise.*

**Definition 4.2.8** *Let $\Omega \subset \mathbb{R}^n$ be an open subset. We recall that a function $f : \Omega \longrightarrow \mathbb{R}^n$ is differentiable at $x_0 \in \Omega$ if there is a matrix $f'(x_0)$ such that $f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)$, where $x_0 + h \in \Omega$ and $\frac{\|o(h)\|}{\|h\|}$ tends to zero as $\|h\| \longrightarrow 0$.*

**Definition 4.2.9** *If f is differentiable at $x_0 \in \Omega$, we call $J_f(x_0) = det f'(x_0)$ the Jacobian of $f$ at $x_0$. If $J_f(x_0) = 0$, then $x_0$ is said to be a critical point of $f$ and we use $S_f(\Omega) = \Big\{x \in \Omega : J_f(x) = 0\Big\}$ to denote the set of critical points of $f$, in $\Omega$. If $f^{-1}(y) \cap S_f(\Omega) = \emptyset$, then $y$ is said to be a regular value of $f$. Otherwise, $y$ is said to be a singular value of $f$.*

### 4.2.3 Some properties of the Riemann integral

Some properties of the Riemann integral used in this work are taken from Rudin (1976) and Wade (2000). Let $f$ be a real function on interval $I \subseteq \mathbb{R}$. If $f$ is Riemann Integrable on $I$, then $\int_I f(x)dx < \infty$, and we write $f \in \mathcal{R}_I$.

**Proposition 4.2.10** *If $f$ is a continuous function on $I = [a, b]$, then $f \in \mathcal{R}_I$.*

Next, let $f_1, f_2$ be two functions on $I = [a, b]$. If $f_1, f_2 \in \mathcal{R}_I$, then the following properties it holds

1. $\int_I \big(c_1 f_1(x) + c_2 f_2(x)\big)dx = c_1 \int_I f_1(x)dx + c_2 \int_I f_2(x)dx$,
2. If $f_1(x) \leq f_2(x)$ on $I$, then $\int_I f_1(x)dx \leq \int_I f_2(x)dx$,
3. $\int_I f_1(x)f_2(x)dx = \int_I f_1(x)dx \int_I f_2(x)dx$.

As a consequence of the above it is easy to prove the following statements:

**Theorem 4.2.11** *[Mean value theorem for integrals] Suppose that $f, g \in \mathcal{R}_I$ where $I = [a, b]$ with $g(x) \geq 0$ for all $x \in I$. If*

$$m = \inf_{x \in I} f(x) \text{ and } M = \sup_{x \in I} f(x),$$

*then there is* $c \in [m, M]$ *such that*

$$\int_I f(x)g(x)dx = c \int_I g(x)dx.$$

*In particular, if* $f \in C(I)$*, then there is* $x_0 \in I$ *which satisfies*

$$\int_I f(x)g(x)dx = f(x_0) \int_I g(x)dx.$$

**Proposition 4.2.12** *Suppose* $f \geq 0$*,* $f \in C(I)$*, and* $\int_I f(x)dx = 0$*, then* $f(x) = 0$ *for all* $x \in I$*.*

### 4.2.4 Construction of Brouwer Degree

Now, we give the necessary elements for the construction of Brouwer degree.

**Definition 4.2.13** *Let* $\Omega \subset \mathbb{R}^n$ *be an open subset and* $f \in C^1(\overline{\Omega})$*. If* $p \notin f(\partial\Omega)$ *and* $J_f(p) \neq 0$*, then we define*

$$deg(f, \Omega, p) = \sum_{x \in f^{-1}(p)} sgn J_f(x),$$

*with the agreement that the above sum is zero if* $f^{-1}(y) = \emptyset$*.*

**Definition 4.2.14** *Let* $\Omega \subset \mathbb{R}^n$ *be an open subset and* $f \in C^2(\overline{\Omega})$*. If* $p \notin f(\partial\Omega)$ *and* $J_f(p) \neq 0$*, then we define*

$$deg(f, \Omega, p) = deg(f, \Omega, p'),$$

*where* $p'$ *is any regular value of* $f$ *such that* $\|p' - p\| < d(p, \partial\Omega)$*.*

Finally, we are ready to introduce the following definition:

**Definition 4.2.15** *Let* $\Omega \subset \mathbb{R}^n$ *be an open subset and* $f \in C^1(\overline{\Omega})$*. If* $p \notin f(\partial\Omega)$ *and* $J_f(p) \neq 0$*, then we define*

$$deg(f, \Omega, p) = deg(g, \Omega, p'),$$

*where* $g \in C^2(\overline{\Omega})$ *and* $f$ *is such that* $\|f - g\| < d(p, \partial\Omega)$*.*

Now, one may check the following properties by a reduction to the regular case.

**Theorem 4.2.16** *Let $\Omega \subset \mathbb{R}^n$ be an open subset and $f : \overline{\Omega} \longrightarrow \mathbb{R}^n$ be a continuous mapping. If $p \notin f(\partial\Omega)$, then there exists an integer $deg(f, \Omega, p)$ satisfying the following properties:*

1. *(Normality) $deg(I, \Omega, p) = 1$ if and only if $p \in \Omega$, where $I$ denotes the identity mapping,*
2. *(Solvability) If $deg(f, \Omega, p) = 1$, then $f(x) = p$ has a solution in $\Omega$,*
3. *(Homotopy) If $f_t(x) : [0,1] \times \overline{\Omega} \longrightarrow \mathbb{R}^n$ is continuous and $p \notin \bigcup\limits_{t\in[0,1]} f_t(\partial\Omega)$, then $deg(f_t, \Omega, p)$ does not depend on $t \in [0,1]$,*
4. *(Additivity) Suppose that $\Omega_1, \Omega_2$ are two disjoint open subsets of $\Omega$ and $p \notin f(\partial\Omega - \Omega_1 \cup \Omega_2)$. Then $deg(f, \Omega, p) = deg(f, \Omega_1, p) + deg(f, \Omega_2, p)$,*
5. *$deg(f, \Omega, p)$ is a constant on any connected component of $\mathbb{R}^n - f(\partial\Omega)$.*

### 4.2.5 Coincidence degree theory

In the 1970$s$, Jean Mawhin systematically studied a class of mappings of the form $L + T$, where $L$ is a Fredholm mapping of index zero and $T$ is a nonlinear mapping, which he called a $L$-compact mapping. Based on the Lyapunov-Schmidt method, he was able to construct a degree theory for such mapping. Here, we introduce Mawhin's degree theory for $L$-compact mappings. In addition some introductory material on Fredholm mappings is presented.

### 4.2.6 Fredholm Mappings

Next, we define the concepts of linear mapping in normed spaces.

**Definition 4.2.17** *Let $X$ and $Y$ be normed spaces. A linear mapping $L : D(L) \subset X \longrightarrow Y$ is a mapping such that*

1. $L(x_1 + x_2) = L(x_1) + L(x_2),$ *for all* $x_1, x_2 \in D(L)$,

2. $L(\alpha x_1) = \alpha L(x_1),$ *for all* $\alpha$ *scalar,*

*where* $D(L) = Dom(L) = \Big\{x \in X : L(x) = y, \text{ for some } y \in Y\Big\}$.

**Definition 4.2.18** *Let* $X$ *and* $Y$ *be normed spaces and a linear mapping* $L : D(L) \subset X \longrightarrow Y$, *it defines*

1. $Ker(L) = \Big\{x \in D(L) : L(x) = 0\Big\}$,

2. $Im(L) = \Big\{y \in Y : L(x) = y, \text{ for } x \in D(L)\Big\}$.

**Definition 4.2.19** *Let* $X$ *and* $Y$ *be normed spaces. A linear mapping* $L : D(L) \subset X \longrightarrow Y$ *is called a Fredholm mapping if*

1. $Ker(L)$ *has finite dimension,*

2. $Im(L)$ *is closed and has finite codimension.*

**Definition 4.2.20** *If* $W$ *is a linear subspace of a finite-dimensional vector space* $V$, *then the codimension of* $W$ *in* $V$ *is the difference between the dimensions:*

$$\operatorname{codim}(W) = \dim(V) - \dim(W).$$

**Definition 4.2.21** *It is the complement of the dimension of* $W$, *in that, with the dimension of* $W$, *it adds up to the dimension of the ambient space* $V$*:*

$$\dim(W) + \operatorname{codim}(W) = \dim(V).$$

*Similarly, if* $N$ *is a submanifold or subvariety in* $M$, *then the codimension of* $N$ *in* $M$ *is*

$$\operatorname{codim}(N) = \dim(M) - \dim(N).$$

*Just as the dimension of a manifold is the dimension of the tangent bundle (the number of dimensions that you can move on the submanifold), the* *codimension is the dimension of the normal bundle (the number of dimensions you can move off the submanifold). More generally, if $W$ is a linear subspace of a (possibly infinite dimensional) vector space $V$ then the codimension of $W$ in $V$ is the dimension (possibly infinite) of the quotient space $V/W$, which is more abstractly known as the* cokernel *of the inclusion. For finite-dimensional vector spaces, this agrees with the previous definition*

$$\operatorname{codim}(W) = \dim(V/W) = \dim\operatorname{coker}(W \to V) = \dim(V) - \dim(W),$$

*and is dual to the relative dimension as the dimension of the kernel.*

**Definition 4.2.22** *Let $X$ and $Y$ be normed spaces. A linear mapping $L : D(L) \subset X \longrightarrow Y$ is said to be bounded if there is some $k > 0$ such that*

$$\|Lx\| \leq k\|x\|$$

*for all $x \in D(L)$. If $L$ is bounded, we define $\|L\|$ to be*

$$\|L\| = \inf\Big\{k : \|Lx\| \leq k\|x\|, x \in D(L)\Big\}.$$

**Proposition 4.2.23** *Let $X$ be a Banach space and $T : X \longrightarrow X$ be a linear bounded mapping. Then $dim(Ker(T)) < \infty$ and $Im(T)$ is closed if and only if, for $x_n \in \overline{B(0,1)}$ such that $Tx_n \longrightarrow y$, thus $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence.*

**Proposition 4.2.24** *Let $X$ be a Banach space, $T : X \longrightarrow X$ be a linear bounded Fredholm operator and $K : X \longrightarrow X$ be a linear continuous compact mapping. Then $T + K$ is a Fredholm mapping.*

### 4.2.7 Jean Mawhin's Continuation Theorem

**Definition 4.2.25** *Let $X$ and $Y$ be normed vector spaces, let $L : DomL \subset X \longrightarrow Y$ be a linear mapping, and $N : X \longrightarrow Y$ be a continuous mapping. The mapping $L$ is called a Fredholm mapping of index zero if $IndexL = dimKerL - codimImL = 0$ and $ImL$ is closed in $Y$.*

**Definition 4.2.26** *If $L$ is a Fredholm mapping of index zero, there exist continuous projectors $P : X \longrightarrow X$ and $Q : Y \longrightarrow Y$ such that $ImP = KerL$, $KerQ = ImL = Im(I - Q)$ and $X = KerL \oplus KerP$, $Y = ImL \oplus ImQ$. It follows that $L \mid_{DomL \bigcap KerP}: (I - P)X \longrightarrow ImL$ is invertible. We denote the inverse of that map by $K_P$.*

**Definition 4.2.27** *If $\Omega$ is an open bounded subset of $X$, the mapping $N$ is called $L$-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \longrightarrow X$ is compact. Since $ImQ$ is isomorphic to $KerL$ there exists an isomorphism $J : ImQ \longrightarrow KerL$.*

Let us recall the Continuation Theorem that will help us to prove the existence of positive periodic solutions.

**Theorem 4.2.28** *(Gaines and Mawhin, 1977, p.40) Let $\Omega \subset X$ be an open bounded set. Let $L$ be a Fredholm mapping of index zero and $N$ be $L$-compact on $X$. Assume that:*

1. *for each $\lambda \in ]0, 1[$, $x \in \partial\Omega \cap DomL$, $Lx \neq \lambda Nx$,*
2. *for each $x \in \partial\Omega \cap KerL$, $QNx \neq 0$,*
3. *$deg\Big\{JQN, \Omega \cap KerL, 0\Big\} \neq 0$.*

*Then the equation $Lx = Nx$ has at least one solution in $DomL \cap \overline{\Omega}$.*

## 4.3 The seasonal obesity mathematical model

In this section, the seasonal obesity mathematical model for obesity population dynamics is introduced. In the model the periodicity is included using the parameter $\beta(t)$, which is a continuous periodic function such that

$$0 < \beta^l := \min_{t \in R} \beta(t) \leq \beta(t) \leq \beta^u := \max_{t \in R} \beta(t).$$

Assumptions considered in this model are the same of model (1.1) addressed in Section 1.

The mathematical model with constant population size is given analytically by the following nonlinear system of ordinary differential equations,

$$\begin{aligned}
N'(t) &= \mu + \varepsilon D_S(t) - \mu N(t) - \beta(t)N(t)\left[L(t)+S(t)+O(t)\right], \\
L'(t) &= \beta(t)N(t)\left[L(t)+S(t)+O(t)\right] - \left[\mu+\gamma_L\right]L(t), \\
S'(t) &= \gamma_L L(t) + \varphi D_S(t) - \left[\mu+\gamma_S+\alpha\right]S(t), \\
O'(t) &= \gamma_S S(t) + \delta D_O(t) - \left[\mu+\sigma\right]O(t), \\
D'_S(t) &= \gamma_D D_O(t) + \alpha S(t) - \left[\mu+\varepsilon+\varphi\right]D_S(t), \\
D'_O(t) &= \sigma O(t) - \left[\mu+\gamma_D+\delta\right]D_O(t).
\end{aligned} \tag{4.1}$$

As in the previous model (1.1) the whole population is without loss of generality normalized to unity, i.e., $N(t) + L(t) + S(t) + O(t) + D_S(t) + D_O(t) = 1$.

## 4.4 Existence of Positive Periodic Solutions

The objective of this section is to derive sufficient condition for the existence of positive periodic solutions to system (4.1) by using Mawhin's Continuation Theorem (Gaines and Mawhin, 1977, p.40). To this end and for the sake of clarity in the presentation we shall use the following notation:

- For simplicity let $x_1(t) = N(t)$, $x_2(t) = L(t)$, $x_3(t) = S(t)$, $x_4(t) = O(t)$, $x_5(t) = D_S(t)$ and $x_6(t) = D_O(t)$.
- Let $\overline{\beta}$ satisfy the mean value property.
- The dynamical behavior of the solutions of this model will be analyzed on the set $D \subset \mathbb{R}^6_+$ where

$$D = \left\{ (x_1, x_2, x_3, x_4, x_5, x_6) \in R^6_+ \, ; \, \sum_{i=1}^{6} x_i = 1 \right\}$$

  and the set $D$ is invariant for system (4.1), i.e.,$\forall t \geq 0, \sum_{i=1}^{6} x_i(t) = 1$.

Moreover, we suppose that the following conditions for the system (4.1) are satisfied:

$(H_1)$ The parameters $(\mu, \gamma_L, \gamma_S, \varepsilon, \alpha, \varphi, \sigma, \delta, \gamma_D)$ are positive and bounded.

$(H_2)$ $\beta(t)$ is a continuous $T$-periodic function such that

$$0 < \beta^l := \min_{t\in R} \beta(t) \leq \beta(t) \leq \beta^u := \max_{t\in R} \beta(t).$$

We are now in a position to prove our main result on the existence of periodic solutions of system (4.1).

**Theorem 4.4.1** *Assume that the conditions $H_1$ and $H_2$ are satisfied, then the system (4.1) has at least one positive T-periodic solution.*

**Proof**. In order to prove the existence of positive periodic solutions of system (4.1), first we consider the following change of variables:

$$x_i(t) = e^{u_i(t)}. \tag{4.2}$$

Thus system (4.1) can be written in the form

$$\begin{aligned}
\dot{u_1}(t) &= \mu e^{-u_1(t)} + \varepsilon e^{u_5(t)-u_1(t)} - \mu - \beta(t)\left[e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)}\right], \\
\dot{u_2}(t) &= \beta(t) e^{u_1(t)-u_2(t)}\left[e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)}\right] - (\mu+\gamma_L), \\
\dot{u_3}(t) &= \gamma_L e^{u_2(t)-u_3(t)} + \varphi e^{u_5(t)-u_3(t)} - [\mu+\gamma_S+\alpha], \\
\dot{u_4}(t) &= \gamma_S e^{u_3(t)-u_4(t)} + \delta e^{u_6(t)-u_4(t)} - [\mu+\sigma], \\
\dot{u_5}(t) &= \gamma_D e^{u_6(t)-u_5(t)} + \alpha e^{u_3(t)-u_5(t)} - [\mu+\varepsilon+\varphi], \\
\dot{u_6}(t) &= \sigma e^{u_4(t)-u_6(t)} - [\mu+\gamma_D+\delta],
\end{aligned} \tag{4.3}$$

and since that $x_i(t) \leq 1$, for all $t \geq 0$, then one gets $u_i(t) \leq 0$, for all $t \geq 0$.

It is easy to see from (4.2), that if system (4.3) has one T-periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t), u_5^*(t), u_6^*(t))^T$, then $(x_1^*(t), x_2^*(t), x_3^*(t), x_4^*(t), x_5^*(t), x_6^*(t))^T$ is a positive T-periodic solution of the system (4.3), and consequently the system (4.1) has at least one positive T-periodic solution. Therefore, to complete the proof, it suffices to show that the system (4.3) has at least one T-periodic solution.

Let us introduce the space

$$\begin{aligned}
X = Y = \big\{u(t) =&(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t))^T \in C(\mathbb{R}, \mathbb{R}^6_-): \\
&u(T+t) = u(t),\ T > 0\big\},
\end{aligned}$$

and the following norm

$$||u|| = ||(u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t))^T|| = \sum_{i=1}^{6} \max_{t\in[0,T]} |u_i(t)|,$$

for any $u \in X$, where $|\cdot|$ is the Euclidean norm. Then $X$ and $Y$ are both Banach spaces with the norm $||\cdot||$. Let $u \in X$, and define

$$\begin{aligned}
\delta_1(u(t),t) &= \mu e^{-u_1(t)} + \varepsilon e^{u_5(t)-u_1(t)} - \mu - \beta(t)\left[e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)}\right],\\
\delta_2(u(t),t) &= \beta(t) e^{u_1(t)-u_2(t)}\left[e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)}\right] - (\mu+\gamma_L),\\
\delta_3(u(t),t) &= \gamma_L e^{u_2(t)-u_3(t)} + \varphi e^{u_5(t)-u_3(t)} - \left[\mu+\gamma_S+\alpha\right],\\
\delta_4(u(t),t) &= \gamma_S e^{u_3(t)-u_4(t)} + \delta e^{u_6(t)-u_4(t)} - \left[\mu+\sigma\right],\\
\delta_5(u(t),t) &= \gamma_D e^{u_6(t)-u_5(t)} + \alpha e^{u_3(t)-u_5(t)} - \left[\mu+\varepsilon+\varphi\right],\\
\delta_6(u(t),t) &= \sigma e^{u_4(t)-u_6(t)} - \left[\mu+\gamma_D+\delta\right].
\end{aligned}$$

It is clear that $\delta_i(u(t),t) \in C(\mathbb{R},\mathbb{R})$ for $i = 1, ..., 6$ and are all $T$-periodic.

Let

$$L : DomL \cap X \longrightarrow X, \text{ such that } L(u(t)) = \dot{u}(t) = \frac{du(t)}{dt},$$

where

$$DomL = \left\{u(t) \in C^1(\mathbb{R},\mathbb{R}^6_-); u(T+t) = u(t)\right\} \subseteq X \text{ and } N : X \longrightarrow X$$

such that

$$Nu(t) = (\delta_1(u(t),t), \delta_2(u(t),t), \delta_3(u(t),t), \delta_4(u(t),t), \delta_5(u(t),t), \delta_6(u(t),t))^T.$$

Let $P : X \longrightarrow X$ and $Q : Y \longrightarrow Y$ are continuous projectors such that

$$Pu(t) = Qu(t) = \frac{1}{T}\int_0^T u(t)dt.$$

Then

$$KerL = \mathbb{R}^6_-, ImL = KerQ = Im(I-Q) = \left\{u \in X \; ; \frac{1}{T}\int_0^T u(t)dt = 0 \right\}$$

is closed in $X$ and $IndiceL = dimKerL - CodimImL = 0$, thus $L$ is a Fredholm mapping of index zero. Therefore, the mapping

$$L_p = L|_{DomL\cap KerP} : (I-P)X \longrightarrow ImL$$

is invertible. Furthermore, the inverse (to $L_p$), $K_p : ImL \longrightarrow DomL \cap KerP$, exists and has the form

$$K_p(u) = \int_0^t u(s)ds - \frac{1}{T}\int_0^T \int_0^t u(s)dsdt,\ t \in [0,T].$$

Thus, $QN : X \longrightarrow X$ is $QNu(t) = (q_1, q_2, q_3, q_4, q_5, q_6)^T$ where

$$q_i = \frac{1}{T}\int_0^T \delta_i(u(\tau),\tau)d\tau, \quad \text{for } i = 1, ..., 6.$$

Now $K_P(I-Q)N : X \longrightarrow X$ is given by

$$\big(\varphi_1(u(t),t), \varphi_2(u(t),t), \varphi_3(u(t),t), \varphi_4(u(t),t), \varphi_5(u(t),t), \varphi_6(u(t),t)\big)^T,$$

where

$$\begin{aligned}\varphi_i(u(t),t) = & \int_0^t \delta_i(u(s),s)ds - \frac{1}{T}\int_0^T \int_0^t \delta_i(u(s),s)dsdt \\ & - \left(\frac{t}{T} - \frac{1}{2}\right)\int_0^T \delta_i(u(s),s)ds,\ \text{for } i = 1, ..., 6.\end{aligned}$$

It is clear that $QN$ and $K_P(I-Q)N$ are continuous. By the Arzela-Ascoli theorem, Dieudonne (1969), it is not difficult to show that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, $N$ is $L-$compact under $\overline{\Omega}$ with any open bounded set $\Omega \subset X$. The isomorphism $J$ from $ImQ$ under $KerL$ can be the identity mapping, since that $ImQ = KerL$.

In order to apply the theorem 4.2.28, we need to search an appropriate open bounded subset $\Omega$. In order to do it, we use the operator equation

$Lu = \lambda Nu$ with $\lambda \in (0,1)$, therefore

$$\begin{aligned}
\dot{u_1}(t) &= \lambda \left( \mu e^{-u_1(t)} + \varepsilon e^{u_5(t)-u_1(t)} - \mu - \beta(t) \left[ e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)} \right] \right), \\
\dot{u_2}(t) &= \lambda \left( \beta(t) e^{u_1(t)-u_2(t)} \left[ e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)} \right] - (\mu + \gamma_L) \right), \\
\dot{u_3}(t) &= \lambda \left( \gamma_L e^{u_2(t)-u_3(t)} + \varphi e^{u_5(t)-u_3(t)} - [\mu + \gamma_S + \alpha] \right), \\
\dot{u_4}(t) &= \lambda \big( \gamma_S e^{u_3(t)-u_4(t)} + \delta e^{u_6(t)-u_4(t)} - [\mu + \sigma] \big), \\
\dot{u_5}(t) &= \lambda \big( \gamma_D e^{u_6(t)-u_5(t)} + \alpha e^{u_3(t)-u_5(t)} - [\mu + \varepsilon + \varphi] \big), \\
\dot{u_6}(t) &= \lambda \big( \sigma e^{u_4(t)-u_6(t)} - [\mu + \gamma_D + \delta] \big).
\end{aligned} \tag{4.4}$$

Suppose that $u(t) = (u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t))^T \in X$ is any solution of system (4.4) for a certain $\lambda \in (0,1)$. Next, multiplying the first equation of system (4.4) by $e^{u_1(t)}$, the second equation by $e^{u_2(t)}$, the third equation by $e^{u_3(t)}$, the fourth equation by $e^{u_4(t)}$ , the fifth equation by $e^{u_5(t)}$ and the sixth equation by $e^{u_6(t)}$ we get

$$\begin{aligned}
\dot{u_1}(t) e^{u_1(t)} &= \lambda \left( \mu + \varepsilon e^{u_5(t)} - \mu e^{u_1(t)} - \beta(t) e^{u_1(t)} \left[ e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)} \right] \right), \\
\dot{u_2}(t) e^{u_2(t)} &= \lambda \left( \beta(t) e^{u_1(t)} \left[ e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)} \right] - (\mu + \gamma_L) e^{u_2(t)} \right), \\
\dot{u_3}(t) e^{u_3(t)} &= \lambda \left( \gamma_L e^{u_2(t)} + \varphi e^{u_5(t)} - e^{u_3(t)} [\mu + \gamma_S + \alpha] \right), \\
\dot{u_4}(t) e^{u_4(t)} &= \lambda \big( \gamma_S e^{u_3(t)} + \delta e^{u_6(t)} - e^{u_4(t)} [\mu + \sigma] \big), \\
\dot{u_5}(t) e^{u_5(t)} &= \lambda \big( \gamma_D e^{u_6(t)} + \alpha e^{u_3(t)} - e^{u_5(t)} [\mu + \varepsilon + \varphi] \big), \\
\dot{u_6}(t) e^{u_6(t)} &= \lambda \big( \sigma e^{u_4(t)} - e^{u_6(t)} [\mu + \gamma_D + \delta] \big).
\end{aligned} \tag{4.5}$$

Now, integrating (4.5) on both sides from 0 to $T$ with respect to $t$, one obtains that

$$\begin{aligned}
&\mu \int_0^T e^{u_1(t)} dt + \int_0^T \beta(t) e^{u_1(t)} \left( e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)} \right) dt = \mu T \\
&+ \varepsilon \int_0^T e^{u_5(t)} dt, \\
&(\mu + \gamma_L) \int_0^T e^{u_2(t)} dt = \int_0^T \beta(t) e^{u_1(t)} \left( e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)} \right) dt, \\
&(\mu + \gamma_S + \alpha) \int_0^T e^{u_3(t)} dt = \gamma_L \int_0^T e^{u_2(t)} dt + \varphi \int_0^T e^{u_5(t)} dt, \qquad (4.6) \\
&(\mu + \sigma) \int_0^T e^{u_4(t)} dt = \gamma_S \int_0^T e^{u_3(t)} dt + \delta \int_0^T e^{u_6(t)} dt, \\
&(\mu + \varepsilon + \varphi) \int_0^T e^{u_5(t)} dt = \gamma_D \int_0^T e^{u_6(t)} dt + \alpha \int_0^T e^{u_3(t)} dt, \\
&(\mu + \gamma_D + \delta) \int_0^T e^{u_6(t)} dt = \sigma \int_0^T e^{u_4(t)} dt.
\end{aligned}$$

From (4.6) it is easy to see that

$$\int_0^T e^{u_1(t)} dt + \cdots + \int_0^T e^{u_6(t)} dt = T, \qquad (4.7)$$

and hence

$$\int_0^T e^{u_i(t)} dt + \int_0^T e^{u_j(t)} dt < T, \ i \neq j, \ \ i, j = 1, ..., 6. \qquad (4.8)$$

Since we are considering the solution $u(t) \in X$ in the interval $[0, T]$, there exist $\xi_i, \eta_i \in [0, T]$ for $i = 1, ..., 6$, such that

$$u_i(\xi_i) = \min_{t \in [0,T]} u_i(t), \qquad u_i(\eta_i) = \max_{t \in [0,T]} u_i(t).$$

Therefore, from (4.8) and the mean value theorem for integrals, there are $\theta_j \in [0, T]$ such that

$$e^{u_i(\xi_i)} < 1 - e^{u_j(\theta_j)} = m_i, \quad 0 < m_i < 1, \quad i \neq j, \ i, j = 1, ..., 6. \qquad (4.9)$$

On the other hand, adding the first and second equation of (4.6), one gets that

$$(\mu + \gamma_L) \int_0^T e^{u_2(t)} dt + \mu \int_0^T e^{u_1(t)} dt = \mu T + \varepsilon \int_0^T e^{u_5(t)} dt,$$

and from (4.7) it follows that

$$(\mu + \gamma_L) \int_0^T e^{u_2(t)} dt > \varepsilon \int_0^T e^{u_5(t)} dt. \tag{4.10}$$

From (4.10) there exist $\theta_5 \in [0, T]$ such that

$$\varepsilon e^{u_5(\theta_5)} < (\mu + \gamma_L + \varepsilon e^{u_5(\theta_5)}) e^{u_2(\eta_2)},$$

hence

$$e^{u_2(\eta_2)} > \frac{\varepsilon e^{u_5(\theta_5)}}{\left(\mu + \gamma_L + \varepsilon e^{u_5(\theta_5)}\right)} = M_2. \tag{4.11}$$

In the same way, from the third, fourth, fifth and sixth equations of (4.5) we have respectively the following inequalities:

$$e^{u_3(\eta_3)} > \frac{\gamma_L e^{u_2(\theta_2)}}{\left(\gamma_L e^{u_2(\theta_2)} + \mu + \gamma_S + \alpha\right)} = M_3, \tag{4.12}$$

$$e^{u_4(\eta_4)} > \frac{\gamma_S e^{u_3(\theta_3)}}{\left(\mu + \sigma + \gamma_S e^{u_3(\theta_3)}\right)} = M_4, \tag{4.13}$$

$$e^{u_5(\eta_5)} > \frac{\gamma_D e^{u_6(\theta_6)}}{\left(\mu + \varepsilon + \varphi + \gamma_D e^{u_6(\theta_6)}\right)} = M_5, \tag{4.14}$$

$$e^{u_6(\eta_6)} > \frac{\sigma e^{u_4(\theta_4)}}{\left(\mu + \gamma_D + \delta + \sigma e^{u_4(\theta_4)}\right)} = M_6. \tag{4.15}$$

Now, using (4.7) and the first equation of (4.6) one gets that

$$\mu T < \mu T e^{u_1(\eta_1)} + \beta^u T e^{u_1(\eta_1)},$$

thus

$$e^{u_1(\eta_1)} > \frac{\mu}{\mu + \beta^u} = M_1. \tag{4.16}$$

If we integrate (4.4) on both sides from 0 to $T$ with respect to $t$, one gets

$$\begin{aligned}
&\mu \int_0^T e^{-u_1(t)}dt + \varepsilon \int_0^T e^{u_5(t)-u_1(t)}dt = \mu T + \int_0^T \beta(t)e^{u_2(t)}dt \\
&+ \int_0^T \beta(t)\left(e^{u_3(t)} + e^{u_4(t)}\right)dt, \\
&\int_0^T \beta(t)e^{u_1(t)-u_2(t)}\left(e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)}\right)dt = (\mu + \gamma_L)T, \\
&\gamma_L \int_0^T e^{u_2(t)-u_3(t)}dt + \varphi \int_0^T e^{u_5(t)-u_3(t)}dt = (\mu + \gamma_S + \alpha)\,T, \\
&\gamma_S \int_0^T e^{u_3(t)-u_4(t)}dt + \delta \int_0^T e^{u_6(t)-u_4(t)}dt = (\mu + \sigma)\,T, \\
&\gamma_D \int_0^T e^{u_6(t)-u_5(t)}dt + \alpha \int_0^T e^{u_3(t)-u_5(t)}dt = (\mu + \varepsilon + \varphi)\,T, \\
&\sigma \int_0^T e^{u_4(t)-u_6(t)}dt = (\mu + \gamma_D + \delta)\,T,
\end{aligned} \tag{4.17}$$

and again from (4.4) and using (4.17) together with (4.7), we have the following bounds

$$\begin{aligned}
\int_0^T |\dot{u}_1(t)|dt \le \lambda\Big(&\mu \int_0^T e^{-u_1(t)}dt + \varepsilon \int_0^T e^{u_5(t)-u_1(t)}dt + \mu T \\
&+ \int_0^T \beta(t)\left(e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)}\right)dt\Big) \\
&< 2\,(\mu + \beta^u)T = K_1,
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
\int_0^T |\dot{u}_2(t)|dt \le \lambda\Big(&\int_0^T \beta(t)e^{u_1(t)-u_2(t)}\left(e^{u_2(t)} + e^{u_3(t)} + e^{u_4(t)}\right)dt \\
&+ (\mu + \gamma_L)T\Big) < 2\,(\mu + \gamma_L)T = K_2,
\end{aligned} \tag{4.19}$$

$$\int_0^T |\dot{u_3}(t)|dt \leq \lambda\Big(\gamma_L \int_0^T e^{u_2(t)-u_3(t)}dt + \varphi \int_0^T e^{u_5(t)-u_3(t)}dt + (\mu+\gamma_S+\alpha)T\Big) < 2\,(\mu+\gamma_S+\alpha)T = K_3, \tag{4.20}$$

$$\int_0^T |\dot{u_4}(t)|dt \leq \lambda\Big(\gamma_S \int_0^T e^{u_3(t)-u_4(t)}dt + \delta \int_0^T e^{u_6(t)-u_4(t)}dt + (\mu+\sigma)T\Big) < 2\,(\mu+\sigma)T = K_4, \tag{4.21}$$

$$\int_0^T |\dot{u_5}(t)|dt \leq \lambda\Big(\gamma_D \int_0^T e^{u_6(t)-u_5(t)}dt + \alpha \int_0^T e^{u_3(t)-u_5(t)}dt + [\mu+\varepsilon+\varphi]T\Big) < 2\,[\mu+\varepsilon+\varphi]T = K_5, \tag{4.22}$$

$$\int_0^T |\dot{u_6}(t)|dt \leq \lambda\big(\sigma \int_0^T e^{u_4(t)-u_6(t)}dt + [\mu+\gamma_D+\delta]\,T\big) < 2\,[\mu+\gamma_D+\delta]\,T = K_6. \tag{4.23}$$

Then, for $t \in [0,T]$, from (4.9) and (4.18)-(4.23) one gets

$$u_i(t) \leq u_i(\xi_i) + \int_{\xi_i}^T |\dot{u_i}(t)|dt \leq u_i(\xi_i) + \int_0^T |\dot{u_i}(t)|dt < \ln(m_i) + K_i, \text{for } i = 1, ..., 6. \tag{4.24}$$

Using the same argument, with (4.11)-(4.16) we obtain

$$u_i(t) \geq u_i(\eta_i) - \int_0^T |\dot{u_i}(t)|dt > \ln(M_i) - K_i, \text{ for } i = 1, ..., 6. \tag{4.25}$$

Thus, from (4.24) and (4.25) leads to

$$\max_{t\in[0,T]} |u_i(t)| < \max\{|\ln(m_i)+K_i|, |\ln(M_i)-K_i|\} = R_i, \text{ for } i = 1, ..., 6. \tag{4.26}$$

Clearly, the $R_i$ $(i = 1, ..., 6)$ are independent of $\lambda$. On the other hand, for $\mu_0 \in [0, 1]$, we consider the following algebraic equations

$$\mu e^{-u_1} - \overline{\beta} e^{u_3} + \mu_0 \big( \varepsilon e^{u_5 - u_1} - \mu - \overline{\beta}(e^{u_2} + e^{u_4} \big) = 0, \tag{4.27i}$$
$$\overline{\beta} e^{u_1 - u_2} e^{u_3} - (\mu + \gamma_L) + \mu_0 \left( \overline{\beta} e^{u_1 - u_2}(e^{u_2} + e^{u_4}) \right) = 0, \tag{4.27ii}$$
$$\gamma_L e^{u_2 - u_3} - (\mu + \gamma_S + \alpha) + \mu_0 \varphi e^{u_5 - u_3} = 0, \tag{4.27iii}$$
$$\gamma_S e^{u_3 - u_4} - (\mu + \sigma) + \mu_0 \delta e^{u_6 - u_4} = 0, \tag{4.27iv}$$
$$\gamma_D e^{u_6 - u_5} - (\mu + \varepsilon + \varphi) + \mu_0 \alpha e^{u_3 - u_5} = 0, \tag{4.27v}$$
$$\sigma e^{u_4 - u_6} - (\mu + \gamma_D + \delta) = 0, \tag{4.27vi}$$

where $(u_1, u_2, u_3, u_4, u_5, u_6)^T \in \mathbb{R}^6$. From (4.27) we can deduce the following inequalities

$$\begin{aligned}
&\mu + \varepsilon e^{u_5} \geq (\mu + \gamma_L) e^{u_2}, \\
&\gamma_L e^{u_2} + \varphi e^{u_5} \geq (\mu + \gamma_S + \alpha)\, e^{u_3}, \\
&\gamma_S e^{u_3} + \delta e^{u_6} \geq (\mu + \sigma)\, e^{u_4}, \\
&\gamma_D e^{u_6} + \alpha e^{u_3} \geq (\mu + \varepsilon + \varphi)\, e^{u_5}, \\
&\sigma e^{u_4} = (\mu + \gamma_D + \delta)\, e^{u_6},
\end{aligned} \tag{4.28}$$

and summing one gets that

$$\sum_{i=2}^{6} e^{u_i} \leq 1. \tag{4.29}$$

Therefore, from (4.29) and (4.27i)-(4.27vi) we get respectively that

$$\begin{aligned}
&\mu \leq \overline{\beta} e^{u_3} e^{u_1} + e^{u_1}\mu + \overline{\beta} e^{u_1}(e^{u_2} + e^{u_4}\big) < (\overline{\beta} + \mu + 2\gamma_L) e^{u_1}, \\
&\overline{\beta} e^{u_1} e^{u_3} \leq (\mu + \gamma_L) e^{u_2} \leq (\mu + \gamma_L), \\
&\frac{\gamma_L e^{u_2}}{\mu + \gamma_S + \alpha + \gamma_L} + e^{u_2} < e^{u_2} + e^{u_3} \leq 1, \\
&\frac{\gamma_S e^{u_3}}{\mu + \sigma + \gamma_S} + e^{u_3} < e^{u_3} + e^{u_4} \leq 1, \\
&\frac{\gamma_D e^{u_6}}{\mu + \epsilon + \varphi + \gamma_D} + e^{u_6} < e^{u_5} + e^{u_6} \leq 1, \\
&\frac{\sigma e^{u_4}}{\mu + \gamma_D + \delta + \sigma} + e^{u_4} < e^{u_6} + e^{u_4} \leq 1,
\end{aligned}$$

and also

$$e^{u_1} > \frac{\mu}{\mu + \overline{\beta} + 2\gamma_L} = c_1, \tag{4.30}$$

$$e^{u_1} e^{u_3} \leq \frac{\mu + \gamma_L}{\overline{\beta}}, \tag{4.31}$$

$$e^{u_2} < \frac{\mu + \gamma_S + \alpha + \gamma_L}{\mu + \gamma_S + \alpha + \gamma_L + \gamma_L} = C_2, \tag{4.32}$$

$$e^{u_3} < \frac{\mu + \sigma + \gamma_S}{\mu + \gamma_S + \sigma + \gamma_S} = C_3, \tag{4.33}$$

$$e^{u_6} < \frac{\mu + \epsilon + \varphi + \gamma_D}{\mu + \epsilon + \varphi + \gamma_D + \gamma_D} = C_6, \tag{4.34}$$

$$e^{u_4} < \frac{\mu + \gamma_D + \delta + \sigma}{\mu + \gamma_D + \delta + \sigma + \sigma} = C_4. \tag{4.35}$$

It is clear that

$$e^{u_1} < \frac{\mu}{\overline{\beta} + \mu + \gamma_L} = C_1, \tag{4.36}$$

satisfy (4.31).

Now, we reduce (4.27i)-(4.27ii) to

$$\mu + \mu_0 \epsilon e^{u_5} = \mu_0 \mu e^{u_1} + (\mu + \gamma_L) e^{u_2}, \text{ it implies that}$$

$$\frac{\mu}{\mu + \gamma_L} < e^{u_1} + e^{u_2}. \tag{4.37}$$

From (4.36) and (4.37) we obtain

$$\frac{\mu}{\mu + \gamma_L} < e^{u_1} + e^{u_2} < e^{u_2} + \frac{\mu}{\overline{\beta} + \mu + \gamma_L}, \text{ which implies that}$$

$$e^{u_2} > \frac{\overline{\beta}\mu}{(\mu + \gamma_L)(\overline{\beta} + \mu + \gamma_L)} = c_2. \tag{4.38}$$

With (4.27iii)-(4.27vi) we obtain respectively that

$$\gamma_L e^{u_2} < (\mu + \gamma_S + \alpha + \gamma_L)\, e^{u_3},$$
$$\gamma_S e^{u_3} < (\mu + \sigma + \gamma_S)\, e^{u_4},$$
$$\sigma e^{u_4} < (\mu + \gamma_D + \delta + \sigma)\, e^{u_6},$$
$$\gamma_D e^{u_6} < (\mu + \varepsilon + \varphi + \gamma)\, e^{u_5},$$

and using (4.38) we conclude that

$$e^{u_3} > \frac{\gamma_L c_2}{(\mu + \gamma_S + \alpha + \gamma_L)} = c_3, \tag{4.39}$$

$$e^{u_4} > \frac{\gamma_S c_3}{(\mu + \sigma + \gamma_S)} = c_4, \tag{4.40}$$

$$e^{u_6} > \frac{\sigma c_4}{(\mu + \gamma_D + \delta + \sigma)} = c_6, \tag{4.41}$$

$$e^{u_5} > \frac{\gamma_D c_6}{(\mu + \varepsilon + \varphi + \gamma)} = c_5. \tag{4.42}$$

Finally since $e^{u_2} + e^{u_5} < 1$ and from (4.38) we have that

$$e^{u_5} < \frac{\gamma_L \overline{\beta} + (\mu + \gamma_L)^2}{(\mu + \gamma_L)(\overline{\beta} + \mu + \gamma_L)} = C_5. \tag{4.43}$$

If we select $R = \max_{i \in \{1,\ldots,6\}} \{|\ln(C_i)|, |\ln(c_i)|\}$, then

$$|u_1| + |u_2| + |u_3| + |u_4| + |u_5| + |u_6| < 6R = R_0. \tag{4.44}$$

Taken $R_T = \sum_{i=0}^{6} R_i$, and

$$\Omega = \left\{u(t) = (u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t))^T \in X : \|u\| < R_T\right\}, \tag{4.45}$$

since, for each $\lambda \in (0, 1)$, $u \in \partial\Omega \cap DomL$, $Lu \neq \lambda Nu$, then $\Omega$ verifies requirement (1) of the Theorem 4.2.28 is satisfied.
When $u = (u_1, u_2, u_3, u_4, u_5, u_6)^T \in \partial\Omega \cap KerL = \partial\Omega \cap \mathbb{R}^6_-$, $u$ is a constant vector in $\mathbb{R}^6_-$ with $\|u\| = R_T$. If $QNu = 0$, then $(u_1, u_2, u_3, u_4, u_5, u_6)^T$ is a constant solution of the system

$$\begin{aligned}
&\mu e^{-u_1} - \overline{\beta} e^{u_3} + \mu_0\big(\varepsilon e^{u_5 - u_1} - \mu - \overline{\beta}(e^{u_2} + e^{u_4})\big) = 0,\\
&\overline{\beta} e^{u_1 - u_2} e^{u_3} - (\mu + \gamma_L) + \mu_0\left(\overline{\beta} e^{u_1 - u_2}(e^{u_2} + e^{u_4})\right) = 0,\\
&\gamma_L e^{u_2 - u_3} - (\mu + \gamma_S + \alpha) + \mu_0 \varphi e^{u_5 - u_3} = 0,\\
&\gamma_S e^{u_3 - u_4} - (\mu + \sigma) + \mu_0 \delta e^{u_6 - u_4} = 0,\\
&\gamma_D e^{u_6 - u_5} - (\mu + \varepsilon + \varphi) + \mu_0 \alpha e^{u_3 - u_5} = 0,\\
&\sigma e^{u_4 - u_6} - (\mu + \gamma_D + \delta) = 0,
\end{aligned}$$

with $\mu_0 = 1$. From (4.44) we have that $\|(u_1, u_2, u_3, u_4, u_5, u_6)^T\| < R_0$ which is in contradiction to $\|(u_1, u_2, u_3, u_4, u_5, u_6)^T\| = R_T$. It follows that for each $u \in \partial\Omega \cap KerL$, $QNu \neq 0$. This shows that condition 2 of the Theorem 4.2.28 is satisfied.
In order to verify the condition 3 of theorem 4.2.28, we define $\phi : (DomL \cap KerL) \times [0,1] \longrightarrow X$ by

$$\phi(u_1, u_2, u_3, u_4, u_5, u_6, \mu_0) = \begin{bmatrix} \mu e^{-u_1} - \overline{\beta} e^{u_3} \\ \overline{\beta} e^{u_1 - u_2} e^{u_3} - (\mu + \gamma_L) \\ \gamma_L e^{u_2 - u_3} - (\mu + \gamma_S + \alpha) \\ \gamma_S e^{u_3 - u_4} - (\mu + \sigma) \\ \gamma_D e^{u_6 - u_5} - (\mu + \varepsilon + \varphi) \\ \sigma e^{u_4 - u_6} - (\mu + \gamma_D + \delta) \end{bmatrix}$$

$$+ \mu_0 \begin{bmatrix} \varepsilon e^{u_5 - u_1} - \mu - \overline{\beta}(e^{u_2} + e^{u_4} \\ \overline{\beta} e^{u_1 - u_2}(e^{u_2} + e^{u_4}) \\ \varphi e^{u_5 - u_3} \\ \delta e^{u_6 - u_4} \\ \alpha e^{u_3 - u_5} \\ 0 \end{bmatrix},$$

where $\mu_0 \in [0,1]$ is a parameter. When $u = (u_1, u_2, u_3, u_4, u_5, u_6)^T \in \partial\Omega \cap KerL = \partial\Omega \cap \mathbb{R}^6_-$, $u$ is a constant vector in $\mathbb{R}^6_+$ with $\|u\| = R_T$ and $\phi(u_1, u_2, u_3, u_4, u_5, u_6, \mu_0) \neq 0$. Thus, due to homotopy invariance of topology degree Gaines and Mawhin (1977), we have

$$\begin{aligned} &deg\left(JQN((u_1, u_2, u_3, u_4, u_5, u_6)^T), \partial\Omega \cap KerL, (0,0,0,0,0,0)^T\right) \\ &= deg\left(\phi(u_1, u_2, u_3, u_4, u_5, u_6, 1), \Omega \cap KerL, (0,0,0,0,0,0)^T\right) \\ &= deg\left(\phi(u_1, u_2, u_3, u_4, u_5, u_6, 0), \Omega \cap KerL, (0,0,0,0,0,0)^T\right) \\ &= deg\Big\{\Big(\mu e^{-u_1} - \overline{\beta} e^{u_3},\ \overline{\beta} e^{u_1 - u_2} e^{u_3} - (\mu + \gamma_L),\ \gamma_L e^{u_2 - u_3} - (\mu + \gamma_S + \alpha), \\ &\gamma_S e^{u_3 - u_4} - (\mu + \sigma), \gamma_D e^{u_6 - u_5} - (\mu + \varepsilon + \varphi), \sigma e^{u_4 - u_6} - (\mu + \gamma_D + \delta)\Big)^T, \\ &\Omega \cap KerL,\ (0,0,0,0,0,0)^T\Big\}. \end{aligned}$$

If the conditions of Theorem 4.4.1 are satisfied, it follows that the system of algebraic equations

$$\begin{aligned}
\mu e^{-u_1} - \overline{\beta} e^{u_3} &= 0,\\
\overline{\beta} e^{u_1-u_2} e^{u_3} - (\mu + \gamma_L) &= 0,\\
\gamma_L e^{u_2-u_3} - (\mu + \gamma_S + \alpha) &= 0,\\
\gamma_S e^{u_3-u_4} - (\mu + \sigma) &= 0,\\
\gamma_D e^{u_6-u_5} - (\mu + \varepsilon + \varphi) &= 0,\\
\sigma e^{u_4-u_6} - (\mu + \gamma_D + \delta) &= 0,
\end{aligned}$$

has a unique solution $(x_1, x_2, x_3, x_4, x_5, x_6)^T = (e^{u_1^*}, e^{u_2^*}, e^{u_3^*}, e^{u_4^*}, e^{u_5^*}, e^{u_6^*})^T$ which satisfies

$$\begin{aligned}
&e^{u_2^*} = \tfrac{\mu}{\gamma_L+\mu} > 0, \quad e^{u_3^*} = \tfrac{\gamma_L e^{u_2^*}}{\mu+\gamma_S+\alpha} > 0, \quad e^{u_4^*} = \tfrac{\gamma_S e^{u_3^*}}{\mu+\sigma} > 0,\\
&e^{u_6^*} = \tfrac{\sigma e^{u_4^*}}{\mu+\delta+\gamma_D} > 0, \quad e^{u_5^*} = \tfrac{\gamma_D e^{u_6^*}}{\mu+\epsilon+\varphi} > 0, \quad e^{u_1^*} = \tfrac{\mu}{\overline{\beta} e^{u_3^*}} > 0.
\end{aligned}$$

Hence, $deg\left(JQN((u_1, u_2, u_3, u_4, u_5, u_6)^T), \partial\Omega \cap KerL, (0,0,0,0,0,0)^T\right) =$

$$sgn \begin{vmatrix}
-\mu e^{-u_1^*} & 0 & -\overline{\beta} e^{u_3^*} & 0 & 0 & 0\\
\overline{\beta} e^{u_1^*-u_2^*} e^{u_3^*} & -\overline{\beta} e^{u_1^*-u_2^*} e^{u_3^*} & \overline{\beta} e^{u_1^*-u_2^*} e^{u_3^*} & 0 & 0 & 0\\
0 & \gamma_L e^{u_2^*-u_3^*} & -\gamma_L e^{u_2^*-u_3^*} & 0 & 0 & 0\\
0 & 0 & \gamma_S e^{u_3^*-u_4^*} & -\gamma_S e^{u_3^*-u_4^*} & 0 & 0\\
0 & 0 & 0 & 0 & -\gamma_D e^{u_6^*-u_5^*} & \gamma_D e^{u_6^*-u_5^*}\\
0 & 0 & 0 & \sigma e^{u_4^*-u_6^*} & 0 & -\sigma e^{u_4^*-u_6^*}
\end{vmatrix}$$

$$= sgn\{\overline{\beta}^2 \sigma \gamma_L \gamma_S \gamma_D e^{u_1+2u_3^*-u_5^*}\} = 1.$$

Thus, based on the above computations, we have complete the proof for the condition (3) of Theorem 4.2.28. Therefore, the system (4.3) has at least one T-periodic $\left(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t), u_5^*(t), u_6^*(t)\right)^T \in DomL \cap \overline{\Omega}$. Thus the result has been established.

## 4.5 Numerical Simulations

In this section some numerical simulations are shown in order to illustrate the dynamics of the model (4.1). The model includes seasonal effect justified in several works related to overweight and obesity by Plasqui and Westerterp (2004); Westerterp (2001); Van Staveren et al. (1986); Kobayashi (2006); Katzmarzyk and Leonard (1998); Tobe et al. (1994).

The numerical examples are run with the periodic function

$$\beta(t) = b\left(1 + a\cos\left(\frac{\pi t}{26}\right)\right),$$

where $a$ is the relative amplitude varying between 0 and 1 and $b$ is the transmission coefficient or baseline. Notice that the periodic function is chosen with a period of 52 weeks which is approximately one year, since the one year periodic behavior is expected. Furthermore, numerical simulations are performed using the time units in weeks.

Periodic behavior of the solution of the model (4.1) with the particular parameters values used in modeling childhood obesity in the Spanish region of Valencia (1.1), is illustrated in Figure 4.1. As it can be observed the time horizon has been selected after the populations have reached a stable periodic behavior, which plays a similar role as a globally stable equilibrium does in the autonomous model.

In Figure 4.2 we have a numerical simulation for the obese population from the start of the numerical simulation, i.e. $t = 0$. Notice that despite the fact that the equation related to the obese population in the system (4.1) does not contain the periodic function $\beta(t)$, the solution for the obesity population is periodic. Based on these results we expect that solutions of model (4.1) are periodic. Therefore, the next section is devoted to show the conditions for the existence of periodic solutions in system (4.1).

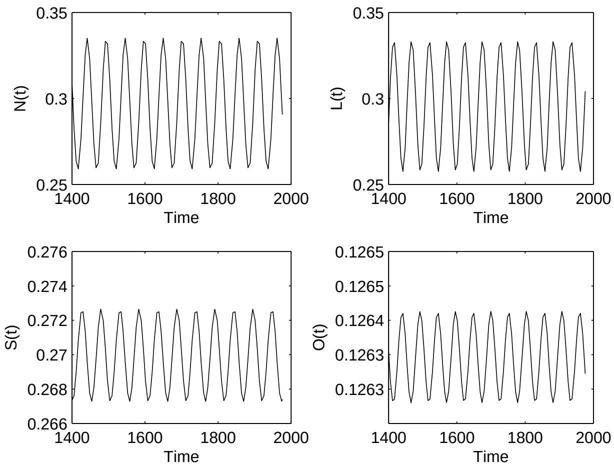



Figure 4.1: Evolution of different populations using the seasonal obesity model with $\beta(t) = 0.02\,(1 + \cos(\frac{\pi t}{26}))$.

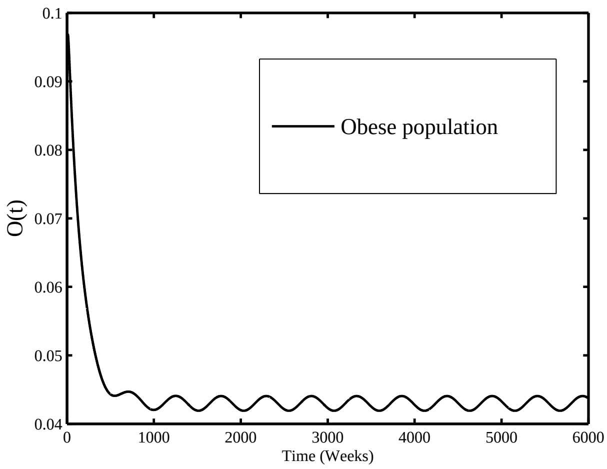



Figure 4.2: Evolution of obese population using the seasonal model (4.1) with $\beta(t) = 0.02\,(1 + 0.9\,\cos(\frac{\pi t}{26}))$.

# Chapter 5

# Piecewise finite series solutions of the seasonal obesity model using multistage Adomian method †

The aim of this chapter is to apply the multistage Adomian Decomposition Method $MADM$ to obtain solutions of the seasonal obesity model that is based on a nonautonomous system of nonlinear differential equations. This obesity seasonal model has periodic behavior due to the periodic transmission parameter. Here the concept of the $MADM$ is introduced and then it is employed to obtain a piecewise finite series solution. The $MADM$ is used here as a hybrid analytical-numerical technique for approximating the solutions of the epidemic models. In order to show the efficiency of the method, the obtained numerical results are compared with the solutions given by the fourth-order Runge-Kutta method. Numerical comparisons

†This chapter is based on González-Parra et al. (2009a)

show that the $MADM$ is accurate, easy to apply and the calculated solutions preserve the periodic behavior of the continuous models. Moreover, the method has the advantage of giving a functional form of the solution for any time interval. Furthermore, it is shown that if the truncation order and the time step size are not properly chosen large computational work is required and inaccurate solutions may be obtained.

## 5.1 Introduction

Ordinary differential initial value problems appear in biological applications and commonly in the modeling of infectious diseases. These models describe the behavior and relationship between the different subpopulations: susceptible, infective and recovered, which together constitute the total population of a certain region or environment. Generally, the exact solutions of these models are unavailable being necessary to obtain accurate numerical approximations to the solutions in order to understand the dynamics of the systems.

Many epidemiological models have been studied using computer simulations to examine the effect of a seasonally varying contact rate on the behavior of the disease. Most of these models performed computer simulations using sinusoidal functions of period 1 year ($\beta(t) = \beta_0 + \beta_1 \cos(\omega t + \phi)$) for the seasonal varying contact rate. Examples of such studies include Dowell (2005); Grossman (2006); Ma and Ma (2006); Moneim and Greenhalgh (2005); Schwartz (1992); Weber et al. (2001); White et al. (2007).

The Adomian Decomposition Method $ADM$ is a method that is useful in various fields of mathematics Adomian (1988, 1994); Achouri and Omrani (2009); Hosseini and Jafari (2009). The $ADM$ decompose the nonlinear terms in the differential equations into a peculiar series of polynomials $\sum_{n=1}^{\infty} A_n$ where the $A_n$ are the so-called Adomian polynomials. In this way $ADM$ compute the solution using a rapidly convergent infinite series depending on the Adomian polynomials. In practice the infinite series is truncated to obtain a practical solution. However, the classical $ADM$

has some drawback: the obtained truncated series solution gives a good approximation to the true solution, only in a small region Ghosh et al. (2007); Ruan and Lu (2007); Chowdhury et al. (2009).

In Repaci (1990) it is proved that it is hopeless to get solutions globally in time by using the classical Adomian decomposition. Nevertheless in Adomian (1994) the author proposes a domain split process with the Adomian method, which later on was applied by pioneers papers Olek (1994); Guellal et al. (1997); Vadasz and Olek (2000).

In the case of ODEs the time $T$ is partitioned in a sequence of time subdomains $[0, t_1), [t_1, t_2), \ldots, [t_{n-1}, T)$ such that the result at $t_p$ is taken as an initial condition in the next subdomain $[t_p, t_{p+1})$. The main advantage of the domain split process is that only a few series terms are required to obtain a good approximation in a small time interval $H_i$. Therefore, the system of differential equations can then be solved in each subdomain Adomian (1994); Ruan and Lu (2007); Chowdhury et al. (2009); Shawagfeh and Kaya (2004). Thus this $MADM$ minimizes the aforementioned drawback.

It is important to remark that the application of $ADM$ is generally less reliable if the exact solutions is oscillating, since the obtained truncated series solution gives a good approximation in smaller regions than in general equations Ghosh et al. (2007). For instance in Venkatarangan and Rajalakshmi (1995) the authors find that $ADM$ solutions of Duffing, Van der Pol and Rayleigh equations were not periodic. They proposed an alternative technique, where Laplace transformation and Padé approximant were applied to obtain a better periodic solution. In the context of integration of equations of motion of nonlinear oscillators, the global accuracy of the $ADM$ needs to be addressed more cautiously Ghosh et al. (2007). However, recently interesting works investigating the accuracy of the Adomian method for ordinary differential equations systems capable of exhibiting chaotic behavior have been developed successfully in Hashim et al. (2006); Noorani et al. (2007); Abdulaziz et al. (2008); Al-Sawalha et al. (2008). Here the main aim is to investigate numerically the application of $MADM$ to seasonal epidemic models represented by systems of nonautonomous nonlinear ordinary differential equations in order to obtain periodic behaviors.

In order to improve the accuracy of $MADM$, there are two alternative approaches, one is computing more terms and the other dividing the time interval $T$ into smaller subintervals. In Ruan and Lu (2007) an algorithm is proposed to choose the time step size $h$ based on a preestablished given threshold value. Despite the fact that numerical solutions are accurate, the effects of the time step size and truncation order $n$ on the global error is not studied. However, in Ghosh et al. (2007) the authors apply the $MADM$ where only collects lower powers of the truncated series on a fixed time step size and an error estimate of the global error were presented. However, an important question remains open, how to chose the number of series terms and the time step size, so that the global error is less than a prefixed admissible error.

In the $MADM$ there is a local truncation error over a particular time interval and this introduces an error in the initial condition of the next subinterval. In Chen et al. (2007) an analysis of this kind of error was performed, when Chebyshev and Frobenius methods are applied to construct approximate solutions of initial value problems for nonlinear ordinary differential equations. The authors developed a procedure to obtain a given admissible global error using the time step size and truncation degree.

For the $MADM$ a similar approach to the one developed in Chen et al. (2007) can be applied. Let $t_i = iT/N$ for $i = 1, 2, ..., N$ be the nodes of the divided subintervals and $\delta_\epsilon$ be an upper bound of the local error at $t_1, t_2, ..., t_N$ due to the truncation of terms. Then, based on the ideas of (Kincaid and Cheney, 2002, p.606), one gets that the global error $\epsilon_G$ at any point of the domain $[0, T]$ is bounded by

$$\epsilon_G \leq \delta_\epsilon \sum_{i=0}^{N-1} e^{Lih} = \delta_\epsilon (e^{LNh} - 1)(e^{Lh} - 1)^{-1}, \tag{5.1}$$

where $h = T/N$ and $L$ is the Lipschitz constant. From (5.1), in order to guarantee that the global error be smaller than a given admissible error $\epsilon_T$, it is sufficient to take $\delta_\epsilon$ satisfying

$$\delta_\epsilon \leq \epsilon_T \frac{(e^{Lh}-1)}{(e^{LNh}-1)}, \tag{5.2}$$

As it can be seen from (5.2), the clever part is to find an expression of $\delta_\epsilon$ in terms of the time step size $h$ and the number of series terms. This dependence will not be discussed here furthermore, because it is outside the aim of this thesis and it is not a straightforward task. But the main idea is already provided here and the work in Chen et al. (2007) can help to solve this issue.

The $MADM$ is applied here to obtain numeric-analytic solutions of the seasonal obesity model (4.1) that is based on a nonautonomous system of nonlinear differential equations. The seasonality of this model is given by a periodic forcing term in the transmission rate $\beta(t)$.

The organization of this chapter is as follows. In Section 2, basic principles of the Adomian Decomposition Method are presented. Section 3 is devoted to present the numerical results of the application of the method to the obesity seasonal model. Comparisons between the $MADM$ and the fourth-order Runge-Kutta (RK4) solutions are included In order to illustrate the accuracy of the $MADM$. The $MADM$ is used here in some cases with arbitrarily large time step sizes, saving computational cost when integrating over long time periods. This fact is important since Euler's method and other well-known methods produce bad approximations in the simulation of the numerical solutions for the models using large time step sizes. Finally in Section 4 conclusions are presented.

## 5.2 Basic principles of $ADM$ and $MADM$

We recall the basic principles of the Adomian Decomposition Method for solving differential equations. Let us consider the equation $Fu = g$, where $F$ is a general nonlinear differential operator involving both linear and nonlinear terms, the linear term is decomposed into $L+R$, where $L$ is always invertible and $R$ is the remainder of the linear operator. For convenience,

$L$ may be taken as the highest order derivative term. Therefore $Fu = g$ may be written as

$$Lu + Ru + Nu = g, \tag{5.3}$$

where $Nu$ are the nonlinear terms. Solving $Lu$ from (5.3), one gets

$$Lu = g - Ru - Nu.$$

Since $L$ is invertible, then

$$L^{-1}Lu = L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{5.4}$$

If $L$ is a first-order operator, then $L^{-1}$ is a integration operator and $L^{-1}Lu = u - u(0)$. Thus, the equation (5.4) for $u$ can be written as

$$u = a + L^{-1}g - L^{-1}Ru - L^{-1}Nu. \tag{5.5}$$

Therefore, $u$ can be presented as a series

$$u = \sum_{n=0}^{\infty} u_n, \tag{5.6}$$

with $u_0 = a + L^{-1}g$, and $u_n$ $(n > 0)$ are to be determined. The nonlinear term $Nu$ can be decomposed by the infinite series of Adomian polynomials

$$Nu = \sum_{n=0}^{\infty} A_n, \tag{5.7}$$

where $A_n^{'}s$ are obtained by writing

$$\nu(\lambda) = \sum_{n=0}^{\infty} \lambda^n u_n, \tag{5.8}$$

$$N(\nu(\lambda)) = \sum_{n=0}^{\infty} \lambda^n A_n. \tag{5.9}$$

Here $\lambda$ is a parameter introduced for convenience. From (5.8) and (5.9), one gets

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N(\nu(\lambda))\right]_{\lambda=0}, \quad n = 0, 1, ... \tag{5.10}$$

Now, substituting (5.6) and (5.7) into (5.5), we have

$$\sum_{n=0}^{\infty} u_n = u_0 - L^{-1}R\sum_{n=0}^{\infty} u_n - L^{-1}\sum_{n=0}^{\infty} A_n,$$

and component wise, one gets that

$$\begin{aligned} u_0 &= a + L^{-1}g, \\ u_1 &= -L^{-1}Ru_0 - L^{-1}A_0, \\ &\vdots \\ u_{n+1} &= -L^{-1}Ru_n - L^{-1}A_n. \end{aligned} \tag{5.11}$$

If $n$ is the truncation order and $u_0$ is known, then it is possible to compute $u_1$,$u_2$,..., $u_n$, using (5.11). Then the $n$-term partial sum $u \cong \sum_{k=0}^{n} u_k$ can be used as an approximate solution, since the series converges Abbaoui and Cherruault (1994).

**Multistage Adomian Decomposition Method**

For practical problems of numerical modeling, the computation interval $[0, H]$ is not always small, and to accelerate the rate of convergence and improve the accuracy of the calculations, it is necessary to divide the entire domain $H$ into $n$ subdomains, as it is shown in Fig. 6.1 in the previous chapter. The main advantage of domain split process is that only a few series terms are required to construct a good approximation in a small time interval $H_i$, where $H = \sum_{i=1}^{n} H_i$. It is important to remark that, $H_i$ can be chosen arbitrarily small if necessary. Thus, the system of differential equations can then be solved in each subdomain (Adomian, 1994; Ruan and Lu, 2007; Chowdhury et al., 2009). Therefore, in order to apply the

$MADM$ to solve nonlinear ODEs over a finite time $H$ it is necessary to choose previously the value of two parameters, the truncation order $n$ and the time step size $h$.

Let us consider $[0, H]$ the interval of simulation with $H$ the time horizon of interest. We take a partition of the interval $[0, H]$ as $\{0 = t_0, t_1, \dots, t_n = H\}$ such that $x_i < x_{i+1}$ and $H_i = t_{i+1} - t_i$ for $i = 0, \dots, n-1$. Therefore, $MADM$ can solve a system of differential equation with initial value of the form:

$$\dot{x}(t) = f\big(x(t), t\big) \quad t \in [a, b], \text{ with the initial condition } x(a) = x_a,$$

where $x(t) = (x^1(t), x^2(t), ..., x^j(t), ..., x^n(t))^T$ ($T$ transposed) and that are well-posed. Thus, applying the $MADM$ to a system of differential equations in the domain of interest, it can be obtained a finite-term series for each subdomain $(i - th)$.

$$\phi^i = \sum_{n=0}^{\infty} \phi_n^i, \tag{5.12}$$

The $ADM$ is used to obtain an approximate analytic solution $\phi^{(1)}(t)$ for the first subdomain $[0, t_1)$. Then, the solution $\phi^{(1)}(t)$, valid only in this first subinterval, can be used to obtain a new initial condition for the next subinterval $[t_1, t_2)$. In the second subinterval $[t_1, t_2)$ another approximated analytic solution $\phi^{(2)}(t)$ is obtained and as before it is used to compute a new initial condition for the subinterval $[t_2, t_3)$. This procedure is applied over $n$ subintervals, thus obtaining an analytic approximated solution $\phi^{(i)}(t)$ over the i-th subinterval $[t_{i-1}, t_i)$. Using this domain split process, the functions $\phi^i(t)$ can be obtained throughout the entire domain, for all $i$.

It is important to remark that the number of subdomains depends on the time step size $h$ and the number of terms of each $\phi^{(i)}(t)$ depends on the truncation order value $n$ which eventually could be different for each subinterval. However in this work the same truncation order $n$ is used for all subdomains.

## 5.3 Application to the seasonal obesity model

In this section, $MADM$ is applied to obtain a numeric-analytic solution of the seasonal obesity mathematical model (4.1).This model considers seasonal forcing given by a continuous function, positive, nonconstant and periodic for the transmission rate $\beta(t)$. Thus, we assume,

$$\beta(t) = b_0 \left(1 + b_1 \, cos\left(\frac{2\pi}{T}(t+\phi)\right)\right), \tag{5.13}$$

where $b_0 \geq 0$ is the baseline transmission parameter, $0 \leq b_1 \leq 1$ measures the amplitude of the seasonal variation in the transmission and $0 \leq \varphi \leq 1$ is the phase angle normalized.

Applying the $MADM$ the functions $\tilde{N}^j(t)$, $\tilde{L}^j(t)$, $\tilde{S}^j(t)$, $\tilde{O}^j(t)$, $\tilde{D_S}^j(t)$ and $\tilde{D_O}^j(t)$ can be obtained throughout the entire domain, for all $j$, provided that the solutions holds with:

$$\tilde{N}^j(t) = \sum_{i=0}^{n} N_i^j(t),\ \tilde{L}^j(t) = \sum_{i=0}^{n} L_i^j(t),\ \tilde{S}^j(t) = \sum_{i=0}^{n} S_i^j(t),$$
$$\tilde{O}^j(t) = \sum_{i=0}^{n} O_i^j(t),\ \tilde{D_S}^j(t) = \sum_{i=0}^{n} Ds_i^j(t),\ \tilde{D_O}^j(t) = \sum_{i=0}^{n} Do_i^j(t). \tag{5.14}$$

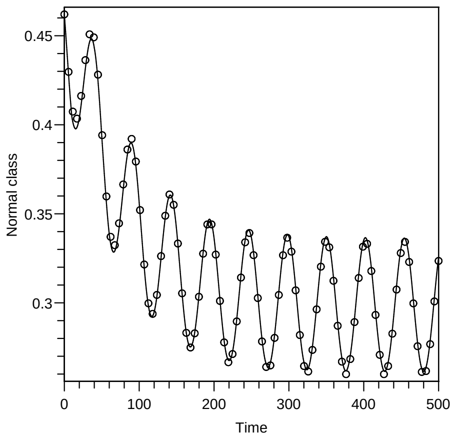
## 5.4 Numerical solution with $MADM$ of the seasonal obesity model

This section is devoted to present the numerical results of the application of the method to the the obesity seasonal social epidemic model (4.1). Comparisons between the $MADM$ and the fourth-order Runge-Kutta (RK4) solutions are included. The $MADM$ algorithm is coded using mathematical software package Maple in a computer with 1.6 GHz and 1 GB of RAM. Moreover the calculations are based upon different values of the truncation order $n$ and time step size $H_i$. The time step size of the Runge-Kutta method is selected in order to obtain an accurate solution for all the time domain.

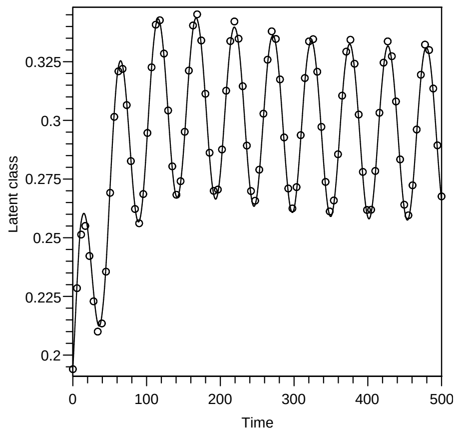
Numerical results shown here illustrate that if the truncation order $n$ and the time step size $H_i$ are chosen in an appropriate way the numerical solutions given by $MADM$ reproduce the correct periodic behavior, positivity and boundedness of the different subpopulations for the obesity seasonal social epidemic model, which are in line with the periodic behavior of the continuous model shown in Arenas et al. (2009b). Additionally, it is briefly shown the effect of the time step size $H_i$ and the truncation order $n$.

Here we want to show that $MADM$ produces accurate solutions with a right choice of the time step size $H_i$ and the truncation order $n$. It is expected that a decrease in the time step size improves the accuracy of the solutions as well as increasing the value of the truncation order. The numerical simulations were made using the parameter values shown in Table 6.1 and using a period $T = 1/52$ for the seasonal obesity social epidemic model (4.1).

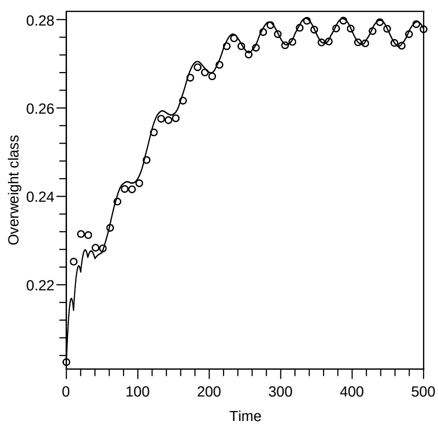
In Fig. 5.1 it can be seen that the solution given by $MADM$ with a truncation order value $n = 2$ and time step size $H_i = 10$ does not reproduce the correct solution. In this case the computation time required to obtain the $MADM$ solution in the time domain $[0, 500]$ was 19 seconds. On the other hand, in Fig. 5.2 it can be observed that with a time step size $H_i = 5$ the numerical results agree with the solution given by the 4-th order Runge-Kutta method. As it was expected computation time increases to 34 seconds. It is important to remark that a large time step size $H_i = 5$ has been sufficient in this model to achieve good agreement with the 4-th order Runge-Kutta method. In regard to computation time, as we stated above the Runge-Kutta method is faster than the $MADM$, but a closed analytical form is not obtained. In addition the accuracy of the $MADM$ solution can be improved by two different ways that is by decreasing the time step size and taking more terms in the $MADM$.



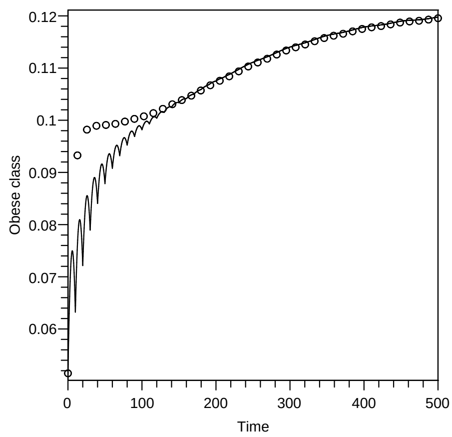

(a) $N(0) = 0.33410$




(b) $L(0) = 0.254659$

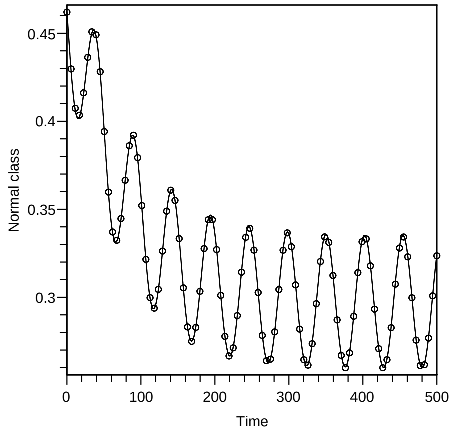



(c) $S(0) = 0.26968$

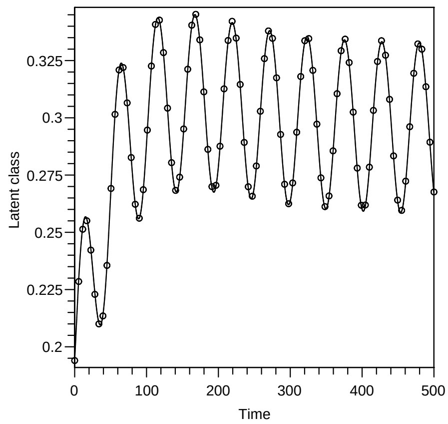



(d) $O(0) = 0.12642$

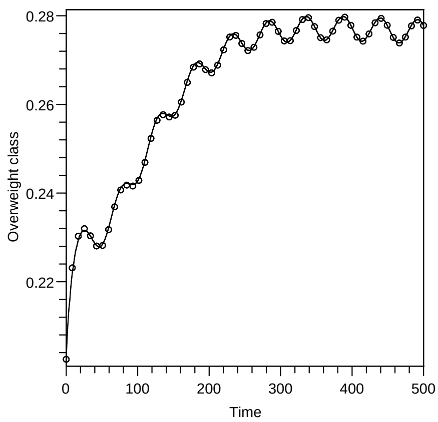
Figure 5.1: Comparison of the numerical approximations of solutions between the $MADM$ ($H_i = 10$, $n = 2$, line) and Runge-Kutta ($h = 0.01$, circles) results for the system (4.1).



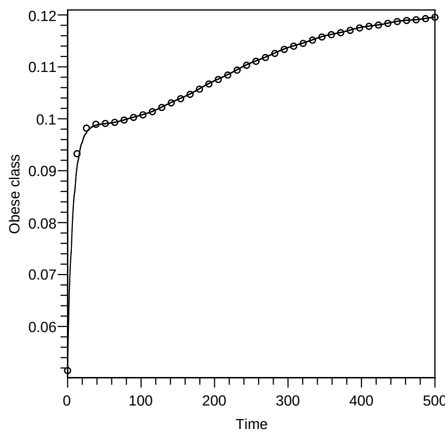

(a) $N(0) = 0.33410$




(b) $L(0) = 0.254659$




(c) $S(0) = 0.26968$




(d) $O(0) = 0.12642$

Figure 5.2: Comparison of the numerical approximations of solutions between the $MADM$ ($H_i = 5$, $n = 2$, line) and Runge-Kutta ($h = 0.01$, circles) results for the system (4.1).

## 5.5 Conclusions

In this chapter the obesity seasonal social epidemic model is solved numerically using the $MADM$ for approximating the solutions. The $ADM$ decompose the nonlinear terms in the differential equations into a peculiar series of polynomials. In this way $ADM$ compute the solution using a rapidly convergent infinite series depending on the Adomian polynomials. In practice the infinite series is truncated to obtain an approximated solution. In order to obtain accurate solutions, the domain region is splitted into subintervals and the solutions are obtained in a sequence of time intervals with the $MADM$. The main advantage of the domain split process is that only a few series terms are required to obtain the solution in a small time interval. Therefore, an approximated solution of the system of differential equation is obtained in each subdomain.

In order to illustrate the accuracy of the $MADM$, the obtained results were compared with the fourth-order Runge-Kutta method. For the obesity seasonal social epidemic model studied we found that the $2-term$ $MADM$ solutions achieve comparable results with the Runge-Kutta solutions. Here, it is shown that the $MADM$ is easy to apply and their numerical solutions preserves the properties of the continuous models, such as periodic behavior, positivity and boundedness. Furthermore, the calculated results demonstrate the reliability and efficiency of the method when is applied to seasonal epidemiological models. Also the method has the advantage of giving a functional form of the solution within each time interval. In addition, the analytical form makes it easier to study the effect that some epidemic parameters have on the dynamics of the diseases. This is not possible in purely numerical techniques like the Runge–Kutta method, which provides solution only at discrete times, provided that the interval is chosen small enough for convergence.

Based on the numerical results it can be inferred that the $MADM$ is a mathematical tool which enables to find approximate accurate solutions for seasonal epidemiological models represented by systems of nonautonomous nonlinear ordinary differential equations. In general, by splitting the time domain, the numerical solutions can be approximated quite well using a small number of terms and small time interval. $MADM$ is based theoretically on an infinite series, but the numerical results show that in practice a small number of terms of the series are sufficient to provided an accurate solution.

# Chapter 6

# Numerical solutions for the mathematical seasonal obesity model using the differential transformation method †

The aim of this chapter is to apply the differential transformation method ($DTM$) to solve a system of nonautonomous nonlinear differential equations that describe the seasonal obesity in the population. The solution of this model exhibits periodic behavior due to the seasonal transmission rate. The dynamics of this model describe the evolution of the different classes of the population. Here the concept of $DTM$ is introduced and then it is employed to derive a set of difference equations for the seasonal obesity social epidemic model. The $DTM$ is used here as an algorithm for approximating the solutions of the seasonal obesity model in a sequence of time intervals. In order to show the efficiency of the method, the ob-

†This chapter is based on (Arenas et al., 2009a)

tained numerical results are compared with the fourth-order Runge-Kutta method solutions. The numerical comparisons show that the $DTM$ is accurate, easy to apply and the calculated solutions preserve the properties of the continuous models, such as the periodic behavior. Furthermore, it is showed that the $DTM$ avoids large computational work and symbolic computation.

## 6.1 Introduction

Ordinary differential initial value problems appear in biological applications and commonly in the modeling of infectious diseases. These models describe the behavior and relationship between the different subpopulations: susceptible, infective and recovered, which together constitute the total population of a certain region o environment. Generally, the exact solutions of these models are unavailable and usually are very complex. Therefore, it is necessary to obtain accurate numerical approximations to the solutions to be able to understand the dynamics of the systems.

Many epidemiological models have been studied using computer simulations to examine the effect of a seasonally varying contact rate on the behavior of the disease. Most of these models performed computer simulations using sinusoidal functions of period 1 year ($\beta(t) = \beta_0 + \beta_1 \cos(\omega t + \phi)$) for the seasonal varying contact rate. Examples of such studies include (Ma and Ma, 2006; Grossman, 2006; Moneim and Greenhalgh, 2005; Dowell, 2005; Schwartz, 1992; Weber et al., 2001; White et al., 2007).

The numerical solution of seasonal epidemic models has been obtained in several papers in order to investigate numerically the reliability and efficiency of the different methods. For instance in (Piyawong et al., 2003), a nonstandard numerical method was tested numerically using a seasonally forced epidemic model. Additionally, in (Roberts and Grenfell, 1992) a Fourier transform method was studied and applied to analyze the population dynamics of nematode infections of ruminants with the effect of seasonality in the free-living stages. Also, in (Arenas et al., 2008b) a nonstandard numerical method for the solution of a mathematical model for the $RSV$ epidemiological transmission is used to investigate the numerical efficiency of the method.

In this chapter the mathematical seasonal obesity model is solved using the $DTM$ for approximating the solutions in a sequence of time intervals. In order to illustrate the accuracy of the $DTM$, the obtained results are compared with the fourth-order Runge-Kutta method. It is showed that the $DTM$ is easy to apply and their numerical solutions preserve the properties of the continuous models, such as periodic behavior, positivity and boundedness.

Furthermore, the proposed numerical method is used in some cases with arbitrarily large time step sizes, saving computational cost when integrating over long time periods. In fact Euler's method and other well-known methods produce bad approximations in the simulation of the numerical solutions for the models when using large time step sizes. It is important to remark that this method is applied directly to system of nonlinear ordinary differential equations without requiring linearization, discretization or perturbation.

The $DTM$ is a semi-analytical numerical technique depending on Taylor series that promises to be useful in various fields of mathematics. The $DTM$ derives from the differential equation system with initial conditions a system of recurrence equations that finally leads to a system of algebraic equations whose solutions are the coefficients of a power series solution. However, the classical $DTM$ has some drawbacks: the obtained truncated series solution does not exhibit the periodic behavior which is characteristic of seasonal disease models and gives a good approximation to the true solution, only in a small region. Therefore, in order to accelerate the rate of convergence and improve the accuracy of the calculations, it is necessary to divide the entire domain $H$ into $n$ subdomains. The main advantage of domain split process is that only a few series terms are required to get the solution in a small time interval $H_i$. Therefore, the system of differential equations can then be solved in each subdomain (Chen et al., 1996). After the system of recurrence equations has been solved, each solution $x^j(t)$ can be obtained by a finite-term Taylor series. Thus this proposed $DTM$ does not have the above drawbacks.

The differential transformation technique is applied here to solve the mathematical seasonal obesity model. This model is at the population level and the forcing seasonality of this model is given by a transmission rate $\beta(t)$.

## 6.2 Basic definitions of differential transformation method

Pukhov (Pukhov, 1980) proposed the concept of differential transformation, where the image of a transformed function is computed by differential operations, which is different from the traditional integral transforms as are Laplace and Fourier. Thus, this method becomes a numerical-analytic technique that formalizes the Taylor series in a totally different manner. The differential transformation is a computational method can be used to solve linear (or non-linear) ordinary (or partial) differential equations with their corresponding boundary conditions. A pioneer using this method to solve initial value problems (Zhou, 1986), who introduced it in a study of electrical circuits. Additionally, differential transformation has been applied to solve a variety of problems that are modeled with differential equations (Chen et al., 1996; Yeh et al., 2007; Hassan, 2008; Jang and Chen, 1997).

The method consists of, given a system of differential equations and related initial conditions, these are transformed into a system of recurrence equation that finally leads to a system of algebraic equations whose solutions are the coefficients of a power series solution. The numerical solution of the system of differential equation in the time domain can be obtained in the form of a finite-term series in terms of a chosen basis system. For this case, we take $\{t^k\}_{k=0}^{+\infty}$ as a basis system, therefore the solution is obtained in the form of the Taylor series. Other bases may be chosen, see (Hwang et al., 2008). The advantage of this method is that it is not necessary to do linearization or perturbations. Furthermore, large computational work and round-off errors are avoided. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations. However, to the best of our knowledge, the differential transformation has not been applied yet in seasonal epidemic models.

For the sake of clarity in the presentation of the $DTM$ and in order to help to the reader we summarize the main issues of the method that may be found in (Zhou, 1986).

**Definition 6.2.1** *Let $x(t)$ be analytic in the time domain $D$, then it has derivatives of all orders with respect to time $t$. Let*

$$\varphi(t,k) = \frac{d^k x(t)}{dt^k}, \qquad \forall t \in D. \tag{6.1}$$

*For $t = t_i$, then $\varphi(t,k) = \varphi(t_i,k)$, where $k$ belongs to a set of non-negative integers, denoted as the $K$ domain. Therefore, (6.1) can be rewritten as*

$$X(k) = \varphi(t_i,k) = \left[\frac{d^k x(t)}{dt^k}\right]_{t=t_i} \tag{6.2}$$

*where $X(k)$ is called the spectrum of $x(t)$ at $t = t_i$.*

**Definition 6.2.2** *Suppose that $x(t)$ is analytic in the time domain $D$, then it can be represented as*

$$x(t) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} X(k). \tag{6.3}$$

Thus, the equation (6.3) represents the inverse transformation of $X(k)$.

**Definition 6.2.3** *If $X(k)$ is defined as*

$$X(k) = M(k)\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_i} \tag{6.4}$$

*where $k \in \mathbb{Z}^+ \cup \{0\}$, then the function $x(t)$ can be described as*

$$x(t) = \frac{1}{q(t)} \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \frac{X(k)}{M(k)}, \tag{6.5}$$

*where $M(k) \neq 0$ and $q(t) \neq 0$. $M(k)$is the weighting factor and $q(t)$ is regarded as a kernel corresponding to $x(t)$.*

Note, that if $M(k) = 1$ and $q(t) = 1$, then Eqs. (6.2) and (6.3) and (6.4) and (6.5) are equivalent.

**Definition 6.2.4** *Let $[0, H]$ the interval of simulation with $H$ the time horizon of interest. We take a partition of the interval $[0, H]$ as $\{0 = t_0, t_1, \dots, t_n = H\}$ such that $t_i < t_{i+1}$ and $H_i = t_{i+1} - t_i$ for $i = 0, \dots, n$. Let $M(k) = \frac{H_i^k}{k!}$, $q(t) = 1$ and $x(t)$ be a analytic function in $[0, H]$. It then defines the differential transformation as*

$$X(k) = \frac{H_i^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=t_i} \quad \text{where } k \in \mathbb{Z}^+ \cup \{0\}, \tag{6.6}$$

*and its differential inverse transformation of $X(k)$ is defined as follow*

$$x(t) = \sum_{k=0}^{\infty}\left(\frac{t}{H_i}\right)^k X(k), \quad \text{for } t \in [t_i, t_{i+1}]. \tag{6.7}$$

From the definitions above, we can see that the concept of differential transformation is based upon the Taylor series expansion. Note that, the original functions are denoted by lowercase and their transformed functions are indicated by uppercase letter. The $DTM$ can solve a system of differential equation with initial-value of the form:

$$\dot{x}(t) = f\big(x(t), t\big) \quad t \in [a, b], \text{ with the initial condition } x(a) = x_a,$$

where $x(t) = (x^1(t), x^2(t), ..., x^j(t), ..., x^n(t))^T$ ($T$ transposed) and that are well-posed. Thus, applying the $DTM$ a system of differential equations in the domain of interest can be transformed to a algebraic equation system in the $K$ domain and each $x^j(t)$ can be obtained by the finite-term Taylor series plus a remainder, i.e.,

$$x^j(t) = \frac{1}{q(t)}\sum_{k=0}^{n}\frac{(t-t_i)^k}{k!}\frac{X^j(k)}{M(k)} + R_{n+1} = \sum_{k=0}^{n}\left(\frac{t}{H}\right)^k X^j(k) + R_{n+1}, \tag{6.8}$$

where

$$R_{n+1} = \sum_{k=n+1}^{\infty} \left(\frac{t}{H}\right)^k X^j(k), \text{ and } R_{n+1} \to 0 \text{ as } n \to \infty.$$

For practical problems of simulation, the computation interval $[0, H]$ is not always small, and to accelerate the rate of convergence and improve the accuracy of the calculations, it is necessary to divide the entire domain $H$ into $n$ subdomains, as shown in Fig. 6.1. The main advantage of domain split process is that only a few Taylor series terms are required to construct the solution in a small time interval $H_i$, where $H = \sum_{i=1}^{n} H_i$. It is important to remark that, $H_i$ can be chosen arbitrarily small if necessary. Thus, the system of differential equations can then be solved in each subdomain (Chen et al., 1996). The approach described above is known as the $D$ spectra method. Considering the function $x^j(t)$ in the first subdomain ($0 \leq t \leq t_1$, $t_0 = 0$), the one dimensional differential transformation is given by

$$x^j(t) = \sum_{k=0}^{n} \left(\frac{t}{H_0}\right)^k X_0^j(k)\,, \text{ where } X_0^j(0) = x_0^j(0). \tag{6.9}$$

Therefore, the differential transformation and system dynamic equations can be solved for the first subdomain and $X_0^j$ can be solved entirely in the first subdomain. The end point of function $x^j(t)$ in the first subdomain is $x_1^j(H_0)$. Thus, $x_1^j(t)$ is obtained by the differential transformation method as

$$x_1^j(H_0) = x^j(H_0) = \sum_{k=0}^{n} X_0^j(k). \tag{6.10}$$

Since that $x_1^j(H_0)$ represents the initial condition in the second subdomain, then $X_1^j(0) = x_1^j(H_0)$. And so the function $x^j(t)$ can be expressed in the second subdomain as

$$x_2^j(H_1) = x^j(H_1) = \sum_{k=0}^{n} X_1^j(k). \tag{6.11}$$

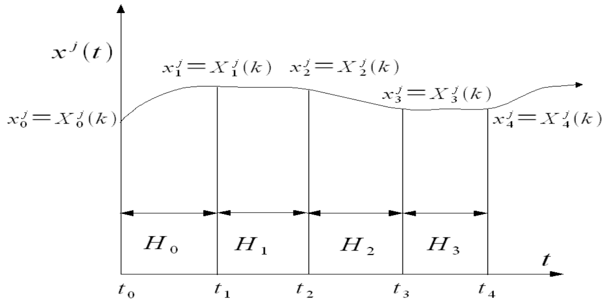



Figure 6.1: Time step diagram.

In general, the function $x^j(t)$ can be expressed in the $i-1$ subdomain as

$$x_i^j(H_i) = x_{i-1}^j(H_{i-1}) + \sum_{k=1}^{n} X_{i-1}^j(k) = X_{i-1}^j(0) + \sum_{k=1}^{n} X_{i-1}^j(k), \quad i = 1, 2, ..., n. \tag{6.12}$$

Using the $D$ spectra method described above, the functions $x^j(t)$ can be obtained throughout the entire domain, for all $j$.

## 6.3 The operation properties of the differential transformation

We consider $q(t) = 1$, $M(k) = \frac{H_i^k}{k!}$ and $x^1(t)$, $x^2(t)$, $x^3(t)$ three uncorrelated functions of time $t$ and $X^1(k)$, $X^2(k)$, $X^2(k)$ the transformed functions corresponding to $x^1(t)$, $x^2(t)$, $x^2(t)$. With $\mathcal{D}$ we denote the Differential Transformation Operator. Thus, the following basic properties hold:

1. **Linearity**. If $X^1(k) = \mathcal{D}[x^1(t)]$, $X^2(k) = \mathcal{D}[x^2(t)]$ and $c_1$ and $c_2$ are independent of $t$ and $k$ then

$$\mathcal{D}[c_1 x^1(t) \pm c_2 x^2(t)] = c_1 X^1(k) \pm c_2 X^2(k). \tag{6.13}$$

Thus, if $c$ is a constant, then $\mathcal{D}[c] = c\delta(k)$, where $\delta(k)$ is the Dirac delta function.

2. **Convolution**. If $X^1(k) = \mathcal{D}[x^1(t)]$, $X^2(k) = \mathcal{D}[x^2(t)]$, then

$$\mathcal{D}[x^1(t)x^2(t)] = X^1(k) * X^2(k) = \sum_{l=0}^{k} X^1(l)X^2(k-l). \text{ Therefore,}$$

$$\begin{aligned}\mathcal{D}[x^1(t)x^2(t)x^3(t)] &= X^1(k) * \big(X^2(k) * X^3(k)\big)\\ &= \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} X^3(k_1)X^2(k_2-k_1)X^3(k-k_2).\end{aligned} \tag{6.14}$$

3. **Derivative**. If $x^1(t) \in C^n[0, H]$, then

$$\mathcal{D}\left[\frac{d^n x^1(t)}{dt^n}\right] = \frac{(k+1)(k+2)\cdots(k+n)}{H_i^n} X^1(k+n). \tag{6.15}$$

4. If $x^1(t) = \cos(\omega t + \alpha)$, then

$$\mathcal{D}[x^1(t)] = \frac{(H_i\omega)^k}{k!}\cos\left(\frac{\pi k}{2} + \alpha + 2\pi i H_i\right), \tag{6.16}$$

   where $i$ denotes the $i$-th split domain.

The proof of the above properties is deduced from the definition of the differential transformation.

## 6.4 Application to the seasonal obesity mathematical model

In this section, the differential transformation technique is applied to solve the seasonal obesity mathematical model (4.1), which is given by a non-linear differential equations system. The seasonality of the model is given by the transmission rate $\beta(t)$ and biological considerations mean that must be a continuous function, positive, nonconstant and periodic of period $T$. Thus, $0 < \beta^l := \min_{t\in R}\beta(t) \leq \beta(t) \leq \beta^u := \max_{t\in R}\beta(t)$. The transmission rate $\beta(t)$ is taken as it is shown in (5.13).

### 6.4.1 Computation of the differential transformation method to the seasonal obesity model

From the properties given in Section 6.3, the corresponding spectrum can be determined for the system (4.1) by

$$\begin{aligned}
&\mathbf{N}(k+1) = \frac{H_i}{k+1}\Big\{\mu\,\delta(k) + \varepsilon\,\mathbf{D}_S(k) - \mu\,\mathbf{N}(k) \\
&- \sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} \mathbf{B}(k_1)\mathbf{N}(k_2-k_1)\Big(\mathbf{L}(k-k_2) + \mathbf{S}(k-k_2) + \mathbf{O}(k-k_2)\Big)\Big\}, \\
&\mathbf{L}(k+1) = \frac{H_i}{k+1}\Big\{\sum_{k_2=0}^{k}\sum_{k_1=0}^{k_2} \mathbf{B}(k_1)\mathbf{N}(k_2-k_1)\Big(\mathbf{L}(k-k_2) + \mathbf{S}(k-k_2) \\
&+ \mathbf{O}(k-k_2)\Big) - (\mu+\gamma_L)\,\mathbf{L}(k)\Big\}, \\
&\mathbf{S}(k+1) = \frac{H_i}{k+1}\Big\{\gamma_L\,\mathbf{L}(k) + \varphi\,\mathbf{D}_S(k) - (\mu+\gamma_S+\alpha)\,\mathbf{S}(k)\Big\},
\end{aligned}$$

$$\begin{aligned}
\mathbf{O}(k+1) &= \frac{H_i}{k+1}\Big\{\gamma_S\,\mathbf{S}(k) + \delta\,\mathbf{D}_O(k) - (\mu+\sigma)\,\mathbf{O}(k)\Big\}, \\
\mathbf{D}_S(k+1) &= \frac{H_i}{k+1}\Big\{\gamma_D\,\mathbf{D}_O(k) + \alpha\,\mathbf{S}(k) - (\mu+\varepsilon+\varphi)\,\mathbf{D}_S(k)\Big\}, \\
\mathbf{D}_O(k+1) &= \frac{H_i}{k+1}\Big\{\sigma\,\mathbf{O}(k) - (\mu+\gamma_D+\delta)\,\mathbf{D}_O(k)\Big\},
\end{aligned} \tag{6.17}$$

with $\mathbf{N}(0) = N(0)$, $\mathbf{S}(0) = S(0)$, $\mathbf{L}(0) = L(0)$, $\mathbf{O}(0) = O(0)$, $\mathbf{D_S}(0) = D_S(0)$, $\mathbf{D_O}(0) = D_O(0)$, where $\mathbf{B}(k_1)$ given as follows:

$$\mathbf{B}(k_1) = b_0\delta(k) + b_0 b_1 \frac{(H_i\omega)^k}{k!}\cos\Big(\frac{\pi k}{2} + \phi + 2\pi i H_i\Big). \tag{6.18}$$

Thus, from a process of inverse differential transformation, the solutions of each sub-domain can be obtained taking $n+1$ terms for the power series like Eq. (6.9), i.e.,

$$\begin{aligned}
N_i(t) &= \sum_{k=0}^{n} \left(\frac{t}{H_i}\right)^k \mathbf{N}_i(k),\ 0 \le t \le H_i,\\
L_i(t) &= \sum_{k=0}^{n} \left(\frac{t}{H_i}\right)^k \mathbf{L}_i(k),\ 0 \le t \le H_i,\\
S_i(t) &= \sum_{k=0}^{n} \left(\frac{t}{H_i}\right)^k \mathbf{S}_i(k),\ 0 \le t \le H_i,\\
O_i(t) &= \sum_{k=0}^{n} \left(\frac{t}{H_i}\right)^k \mathbf{O}_i(k),\ 0 \le t \le H_i,\\
D_{S_i}(t) &= \sum_{k=0}^{n} \left(\frac{t}{H_i}\right)^k \mathbf{D_{S_i}}(k),\ 0 \le t \le H_i,\\
D_{O_i}(t) &= \sum_{k=0}^{n} \left(\frac{t}{H_i}\right)^k \mathbf{D_{S_i}}(k),\ 0 \le t \le H_i,
\end{aligned} \tag{6.19}$$

provided that the solutions holds with:

$$\begin{aligned}
&N(t) = \sum_{i=0}^{n} N_i(t),\ L(t) = \sum_{i=0}^{n} L_i(t),\ S(t) = \sum_{i=0}^{n} S_i(t), O(t) = \sum_{i=0}^{n} O_i(t),\\
&D_S(t) = \sum_{i=0}^{n} D_{S_i}(t), \quad \text{and} \quad D_O(t) = \sum_{i=0}^{n} D_{O_i}(t).
\end{aligned} \tag{6.20}$$

### 6.4.2 Numerical results

The numerical simulations were made using the parameter values shown in Table 6.1 and using a period $T = 1/52\, weeks^{-1}$. The $DTM$ algorithm is coded in the computer using Fortran and the variables are in double precision in all the calculations done in this chapter. Moreover the calculations are based upon a value of $n = 5$ in the Taylor series. From Fig. 6.2 it can be seen that the numerical solutions given by $DTM$ reproduce the correct periodic behavior, positivity and boundedness of the different subpopulations for the obesity seasonal model, which are in line with the periodic behavior of the continuous model proved in Arenas et al. (2009b).

It is clear from Fig. 6.2 that excellent agreement exists between the two sets of results, i.e., that the numerical solutions obtained are as accurate as the 4-th order Runge-Kutta method solution, which shows that the results obtained from the $DTM$ are highly consistent with those obtained from the Runge-Kutta method.

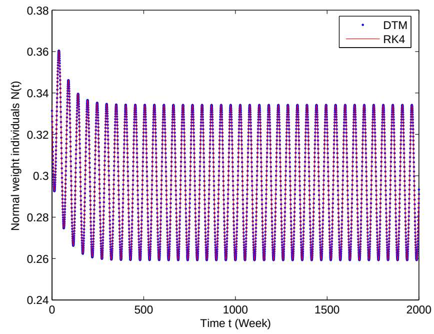
In Table 6.2 we can see that both methods are consistent between 2 and 4 decimal places for a step size $h = 0.5$ for $DTM$ and step size $h = 0.01$ for the Runge-Kutta method, while with a step size $h = 0.01$, both methods are consistent between 4 and 7 decimal places as its can be seen in Table 6.3. In Fig. 6.2 a noteworthy observation is that after a certain period the subpopulations achieve a correct periodic behavior.

Table 6.1: Parameter values in the seasonal obesity model (4.1) for the region of Valencia.

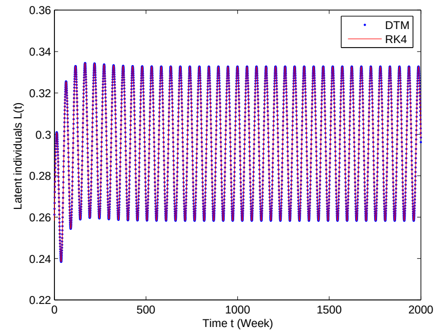
| $\mu$ | $b_0$ | $b_1$ | $\phi$ | $\epsilon$ | $\sigma$ |
|---|---|---|---|---|---|
| 0.0064 | 0.0222949 | 1 | 0 | 0.0028 | 0.0044 |
| $\delta$ | $\gamma_L$ | $\gamma_S$ | $\gamma_D$ | $\varphi$ | $\alpha$ |
| 0.1597 | 0.0089 | 0.0031 | $4.2269e-004$ | 0.1273 | 0.0041 |

Table 6.2: Differences between the 5-term $DTM$ and $RK4$ solutions.

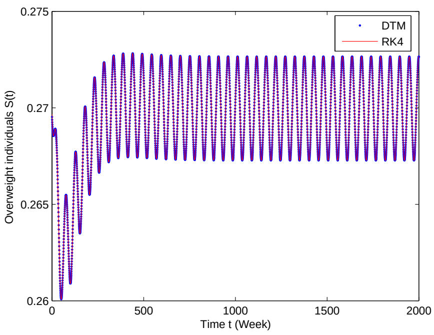
| $\Delta = \lvert DTM_{0.5} - RK4_{0.01}\rvert$ | | | | | | |
|---|---|---|---|---|---|---|
| *Time* | $\Delta N$ | $\Delta L$ | $\Delta S$ | $\Delta O$ | $\Delta D_S$ | $\Delta D_O$ |
| 000.00 | .0000E+00 | .0000E+00 | .0000E+00 | .0000E+00 | .0000E+00 | .0000E+00 |
| 200.00 | .1582E-02 | .1553E-02 | .2676E-04 | .1580E-05 | .1894E-06 | .4253E-07 |
| 400.00 | .8343E-03 | .1033E-02 | .1832E-03 | .1012E-04 | .5112E-05 | .1896E-06 |
| 600.00 | .1078E-02 | .9264E-03 | .1589E-03 | .8747E-05 | .1692E-05 | .2812E-06 |
| 800.00 | .2020E-02 | .1957E-02 | .6069E-04 | .2739E-05 | .1145E-05 | .8977E-07 |
| 1000.00 | .2271E-02 | .2275E-02 | .5972E-05 | .7829E-05 | .1976E-05 | .1848E-06 |
| 1200.00 | .9799E-03 | .1141E-02 | .1551E-03 | .1145E-05 | .4481E-05 | .4879E-07 |
| 1400.00 | .4568E-03 | .3229E-03 | .1326E-03 | .3199E-06 | .1655E-05 | .4538E-07 |
| 1600.00 | .1748E-02 | .1605E-02 | .1422E-03 | .6270E-06 | .1144E-05 | .4405E-07 |
| 1800.00 | .2166E-02 | .2188E-02 | .1216E-04 | .6990E-05 | .2853E-05 | .1485E-06 |
| 2000.00 | .1626E-02 | .1707E-02 | .7852E-04 | .4129E-06 | .2950E-05 | .6469E-07 |



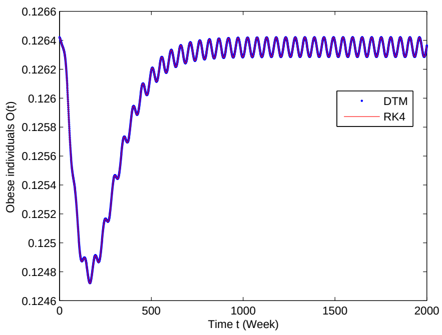

(a) $h = 0.5,\ N_0 = 0.33410,\ H_i = 0.5$



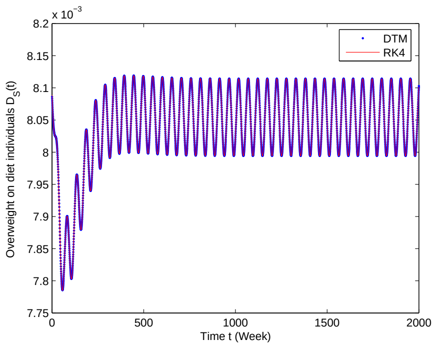

(b) $h = 0.5,\ L_0 = 0.254659,\ H_i = 0.5$



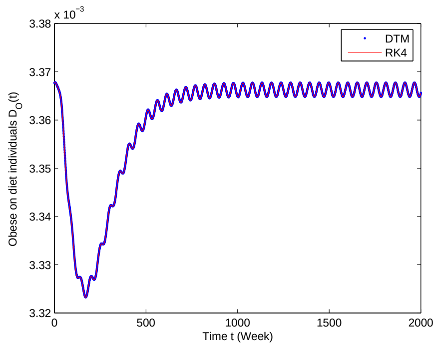

(c) $h = 0.5,\ O_0 = 0.26968,\ H_i = 0.5$




(d) $h = 0.5,\ S_0 = 0.12642,\ H_i = 0.5$




(e) $h = 0.5,\ D_{S0} = 0.008089,\ H_i = 0.5$




(f) $h = 0.5,\ D_{O0} = 0.003681,\ H_i = 0.5$

Figure 6.2: Comparison of the numerical approximations of solutions between the differential transformation and Runge-Kutta results with $n = 5$ to the seasonal obesity model (4.1).

Table 6.3: Differences between the 10-term $DTM$ and $RK4$ solutions.

| $\Delta = \|DTM_{0.01} - RK4_{0.01}\|$ | | | | | | |
|---|---|---|---|---|---|---|
| $Time$ | $\Delta N$ | $\Delta L$ | $\Delta S$ | $\Delta O$ | $\Delta D_S$ | $\Delta D_O$ |
| 000.00 | .0000E+00 | .0000E+00 | .0000E+00 | .0000E+00 | .0000E+00 | .0000E+00 |
| 200.00 | .7192E-04 | .6690E-04 | .4861E-05 | .8351E-07 | .6939E-07 | .9970E-09 |
| 400.00 | .1768E-04 | .4607E-05 | .1213E-04 | .6243E-06 | .3133E-06 | .1178E-07 |
| 600.00 | .1097E-05 | .8816E-05 | .6382E-05 | .1069E-05 | .2431E-06 | .2569E-07 |
| 800.00 | .5951E-04 | .4743E-04 | .1042E-04 | .1370E-05 | .2545E-06 | .3359E-07 |
| 1000.00 | .3263E-04 | .4323E-04 | .8437E-05 | .1818E-05 | .3035E-06 | .4533E-07 |
| 1200.00 | .2777E-04 | .2181E-04 | .3894E-05 | .1876E-05 | .1358E-06 | .4988E-07 |
| 1400.00 | .1824E-04 | .7425E-05 | .8516E-05 | .2022E-05 | .2194E-06 | .5223E-07 |
| 1600.00 | .3161E-04 | .3566E-04 | .1768E-05 | .2094E-05 | .1295E-06 | .5605E-07 |
| 1800.00 | .4776E-04 | .4261E-04 | .3134E-05 | .1912E-05 | .5578E-07 | .5203E-07 |
| 2000.00 | .2929E-04 | .3573E-04 | .4281E-05 | .1967E-05 | .1476E-06 | .5198E-07 |

## 6.5 Conclusions

In this chapter, seasonal epidemiological models are solved numerically using the $DTM$ for approximating the solutions in a sequence of time intervals. In order to obtain very accurate solutions, the domain region has been splitted into subintervals and the approximating solutions are obtained in a sequence of time intervals. The $DTM$ produces from the system of differential equations with initial conditions a system of recurrence equations that finally leads to a system of algebraic equations whose solutions are the coefficients of a power series solution, and applying a process of inverse transformations it obtain the solutions. Moreover, the $DTM$ does not evaluate the derivatives symbolically and this give advantages over other methods such Taylor, power series or Adomian method.

In order to illustrate the accuracy of the $DTM$, the obtained results were compared with the fourth-order Runge-Kutta method. For the seasonal epidemiological models studied we found that the 5-term $DTM$ solutions on a larger time step achieved comparable results with the $RK4$ solutions on a much smaller time step. Here, it is showed that the $DTM$ is easy to apply and their numerical solutions preserves the properties of the continuous models, such as periodic behavior, positivity and boundedness, which when using Runge-Kutta and other numerical methods, we cannot guarantee these properties especially with step size $h$ relatively large. Furthermore, the calculated results demonstrate the reliability and efficiency of the method when is applied to seasonal epidemiological models. It is important to remark that this method is applied directly to the system of nonlinear ordinary differential equations without requiring linearization, discretization or perturbation.

Based on the numerical results it can be concluded that the $DTM$ is a mathematical tool which enables to find approximate accurate analytical solutions for seasonal epidemiological models represented by systems of nonautonomous nonlinear ordinary differential equations. In general, by splitting the time domain, the numerical solutions can be approximated quite well using a small number of terms and small time interval $H_i$. Furthermore, high accuracy can be obtained without using large computer power and the $DTM$ has the advantage of giving an analytical form of the solution within each time interval which is not possible in purely numerical techniques like $RK4$.

Since the Taylor series is an infinite series, the differential transformation should theoretically consist of an infinite series, but the numerical results shown that a small number of terms of the series are sufficient to provided an accurate solution in practice.

# Chapter 7

# Conclusions

This thesis dissertation helps one to understand obesity dynamics in a population. Three mathematical models based on first order systems of nonlinear ordinary differential equations were constructed. The first deals with the mathematical modeling of childhood obesity from a social epidemic point of view for the Spanish region of Valencia for $3-5$ years old population. The second model is an age structured model developed in order to study the influence of age stages in the obesity population dynamics. The proposed model considers the proportion of overweight and obese children populations in the groups $6-8$ and $9-12$ years old. Based on the numerical simulations of different scenarios it is shown that the prevention of children obesity in early years is of paramount importance. Therefore public health strategies should be designed as soon as possible to reduce the worldwide social obesity epidemic.

The third model is a seasonal social epidemic model for obesity which is based on a nonautonomuos systems of nonlinear of ordinary differential equations, where it is proved the existence of periodic solutions using Jean Mawhin's continuation theorem. This seasonal model is simulated numerically using multistage Adomian method and differential transformation method. Numerical results show the reliability of these methods when are applied to this nonautonomous model. In addition numerical simulations of different scenarios are performed using a developed nonstandard finite difference scheme based on Ronald Micken's techniques, that allows to compute numerical solutions with larger step sizes than those normally used by traditional schemes.

As future work, we will develop new mathematical models for other age groups. In addition, we are interested in the developing of an innovative model capable of integrating dynamics of the individual's body and the population. This type of models are a new way and challenge to mathematical modeling of epidemics that need to be addressed. Another interesting possibility for future research is to consider epidemic mathematical models under different uncertainty forms in parameter's model and initial conditions. These considerations are realistic since real world process present different types of uncertainty.

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