On the Spectra of Some Combinations of Two Generalized Quadratic Matrices

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Abstract

Let A and B be two generalized quadratic matrices with respect to idempotent matrices P and Q, respectively, such that $(A-\alpha P)(A-\beta P)=\mathbf{0}$, $AP=PA=A$, $(B-\gamma Q)(B-\delta Q)=\mathbf{0}$, $BQ=QB=B$, $PQ=QP$, $AB\neq BA$, and $(A+B)(\alpha\beta P-\gamma\delta Q)=(\alpha\beta P-\gamma\delta Q)(A+B)$ with $\alpha,\beta,\gamma,\delta\in\mathbb{C}$. Let $A+B$ be diagonalizable. The relations between the spectrum of the matrix $A+B$ and the spectra of some matrices produced from A and B are considered. Moreover, some results on the spectrum of the matrix $A+B$ are obtained when $A+B$ is not diagonalizable. Finally, some results and examples illustrating the applications of the results in the work are given.

AMS classification: 15A18; 15A21

Keywords: quadratic matrix, generalized quadratic matrix, idempotent matrix, spectrum, linear combination, diagonalization

1 Introduction and Notations

Let $\mathbb{C}$ be the set of all complex numbers and $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$. The symbols $\mathbb{C}_{n,m}$, $\mathbb{C}_n$, I_n, and $\mathbf{0}$ will denote the set of all $n\times m$ complex matrices, the set of all $n\times n$ complex matrices, the identity matrix (of size n), and the zero matrix of suitable size, respectively. The rank of a $A\in\mathbb{C}_{n,m}$ will be denoted by $\mathrm{rk}(A)$. For $A\in\mathbb{C}_n$, the spectrum of A will be symbolized by $\sigma(A)$.

Let $P\in\mathbb{C}_n$ be an idempotent (i.e., $P^2=P$). We say that $A\in\mathbb{C}_n$ is a *generalized quadratic matrix with respect to* P if there exist $\alpha,\beta\in\mathbb{C}$ such that

$$(A-\alpha P)(A-\beta P)=\mathbf{0}, \qquad AP=PA=A. \tag{1.1}$$

The notation $\mathfrak{L}(P;\alpha,\beta)$ will indicate the set of matrices A satisfying (1.1). From (1.1), we get the equality

$$A^2=(\alpha+\beta)A-\alpha\beta P.$$

Taking $P=I_n$ in (1.1), we get that the matrix A is an $\{\alpha,\beta\}$-quadratic matrix. Therefore, the results which will be obtained in this work are more general than the results provided in [6].

The set of $\{\alpha, \beta\}$-quadratic matrices has been extensively studied by many authors. For example, in [10] Wang obtained many results related to sums and products of two quadratic matrices. Also, the author characterized when a complex matrix T is the sum of an idempotent matrix and a square-zero matrix in [10]. In [11], the problem of characterizing matrices which can be expressed as a product of finitely many quadratic matrices were considered by Wang. Wang, considering that every complex $n \times n$ matrix T is a product of four quadratic matrices, showed that if T is invertible, then the number of required quadratic matrices can be reduced to three in [12]. In [13], Wang characterized the products of two and four invertible quadratic operators among normal operators and showed that every invertible operator is the product of six invertible quadratic operators. Aleksiejczyk and Smoktunowicz studied many properties of quadratic matrices in [1]. Later, Farebrother and Trenkler, extending the concept of quadratic matrix to generalized quadratic matrix, examined the Moore-Penrose and group inverse of matrices of that type in [3]. In [2], Deng gave explicit expressions for the Moore-Penrose inverse, the Drazin inverse, and the nonsingularity of the difference of two generalized quadratic operators. Also, Deng obtained spectral characterizations of generalized quadratic operators. In [9], Pazzis considered the problem of determining when a matrix is the sum of an idempotent and a square-zero matrix over an arbitrary field, introducing the concept of an (a, b, c, d)-quadratic sum.

In [5], the authors discussed the spectra of some matrices depending on two idempotent matrices. Later, in [6] it was extended those results to a pair of two quadratic matrices. In this work, we will obtain the generalization of some results given in [6] and give some additional results related to the subject.

These type of matrices should be of interest not only from the algebraic point of view but also from the role they play in applied sciences, for example, in the statistical theory: Let A be a generalized quadratic matrix such that $(A - \alpha P)(A - \beta P) = \mathbf{0}$ with $\alpha \neq \beta$ and $AP = PA = A$. Then, there exist two idempotents $X, Y \in \mathbb{C}_n$ such that $A = \alpha X + \beta Y$, $X + Y = P$, and $XY = YX = \mathbf{0}$ (Theorem 1.1, [7]). If the matrices X and Y are also real symmetric, then the matrix A becomes a linear combination of two disjoint real symmetric idempotent matrices. On the other hand, it is a well known fact that if C is an $n \times n$ real symmetric matrix and $\mathbf{x}$ is an $n \times 1$ real random vector having the multivariate normal distribution $N_n(0, I_n)$, then a necessary and sufficient condition for the quadratic form $\mathbf{x}'C\mathbf{x}$ to be distributed as a chi-square variable is that $C^2 = C$. Now, let $\mathbf{x}$ be an $n \times 1$ real random vector mentioned above. Then, the quadratic form $\mathbf{x}'A\mathbf{x}$ is a random variable distributed as a linear combination of two independent chi-square distributions.

2 Results

In this section, first it is given a theorem which examines the spectrum of a sum of generalized quadratic matrices A and B with $AB = BA$. Later, it is presented a lemma which helps to establish a relation between the spectrum of the sum of these matrices and the spectra of various combinations of these matrices in the case $AB \neq BA$.

As is easy to see, one has $A \in \mathfrak{L}(P; \alpha, \beta)$ if and only if $aA \in \mathfrak{L}(P; a\alpha, a\beta)$ for any $a \in \mathbb{C}^*$. Thus, instead of studying the spectrum of $aA + bB$ when $a, b \in \mathbb{C}^*$ and A and B are generalized quadratic, we will study the spectrum of $A + B$.

We shall use the following notation for the sake of simplicity: If $\Gamma_1, \Gamma_2 \subset \mathbb{C}$, then we denote $\Gamma_1 + \Gamma_2 = \{z_1 + z_2 : z_1 \in \Gamma_1, z_2 \in \Gamma_2\}$. Note that, in general, $\Gamma + \Gamma \neq 2\Gamma$.

Theorem 2.1. *Let A, $B \in \mathbb{C}_n$ be two generalized quadratic matrices and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P, \alpha, \beta)$, $B \in \mathfrak{L}(Q, \gamma, \delta)$. If $AB = BA$, then $\sigma(A + B) \subset \{0, \alpha, \beta\} + \{0, \gamma, \delta\}$.*

Proof. Since P is an idempotent matrix, there exists a nonsingular matrix $S \in \mathbb{C}_n$ such that $P = S(I_r \oplus \mathbf{0})S^{-1}$ with $r = \mathrm{rk}(P)$. From $AP = PA = A$, we get that A can be written as $A = S(X \oplus \mathbf{0})S^{-1}$ where $X \in \mathbb{C}_r$. Also, we have $(X - \alpha I_r)(X - \beta I_r) = \mathbf{0}$ since $(A - \alpha P)(A - \beta P) = \mathbf{0}$. From $X^2 - (\alpha + \beta)X + \alpha\beta I_r = \mathbf{0}$, we have that if $\lambda \in \sigma(X)$, then $\lambda^2 - (\alpha + \beta)\lambda + \alpha\beta = 0$, and therefore, $\lambda \in \{\alpha, \beta\}$. From $A = S(X \oplus \mathbf{0})S^{-1}$, we get that $\sigma(A) \subset \{0\} \cup \sigma(X) \subset \{0, \alpha, \beta\}$. In the same way, we get $\sigma(B) \subset \{0, \gamma, \delta\}$. Thus, applying Theorem 2.4.9 of [4] to the matrices A and B, we get the desired result. □

Lemma 2.1. *Let $A, B, P, Q \in \mathbb{C}_n$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P; \alpha, \beta)$, $B \in \mathfrak{L}(Q; \gamma, \delta)$, $(\alpha\beta P - \gamma\delta Q)(A + B) = (A + B)(\alpha\beta P - \gamma\delta Q)$, and $AB \neq BA$. Let $A + B$ be diagonalizable. Then the following statements are true:*

(i) There exist a nonsingular $S \in \mathbb{C}_n$ and $A_0, \ldots, A_k$, $B_0, \ldots, B_k$, $P_0, \ldots, P_k$, $Q_0, \ldots, Q_k$ such that $A_i, B_i, P_i, Q_i \in \mathbb{C}_{m_i}$, $A_i \in \mathfrak{L}(P_i; \alpha, \beta)$, $B_i \in \mathfrak{L}(Q_i; \gamma, \delta)$, for $i = 0, \ldots, k$,

$$A = S((\oplus_{i=1}^k A_i) \oplus A_0)S^{-1}, \qquad B = S((\oplus_{i=1}^k B_i) \oplus B_0)S^{-1},$$

$$P = S((\oplus_{i=1}^k P_i) \oplus P_0)S^{-1}, \qquad Q = S((\oplus_{i=1}^k Q_i) \oplus Q_0)S^{-1},$$

$A_0B_0 = B_0A_0$, $P_0Q_0 = Q_0P_0$, $A_iB_i \neq B_iA_i$ for $i = 1, \ldots, k$.

(ii) There exist distinct complex numbers $\mu_1, \nu_1, \ldots, \mu_k, \nu_k$ such that

$$\alpha + \beta + \gamma + \delta = \mu_i + \nu_i, \quad \sigma(A_i + B_i) = \{\mu_i, \nu_i\},$$

$$A_iB_i + B_iA_i + \mu_i\nu_i I_{m_i} = (\gamma + \delta)A_i + (\alpha + \beta)B_i + \alpha\beta P_i + \gamma\delta Q_i$$

for $i = 1, \ldots, k$.

(iii) If $\alpha \neq \beta$ and $PQ = QP$, then there exist nonsingular matrices S_i such that

$$A_i = S_i \begin{bmatrix} \alpha I_{x_i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta I_{y_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} S_i^{-1}, \qquad P_i = S_i \begin{bmatrix} I_{x_i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & I_{y_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} S_i^{-1},$$

$$B_i = S_i \begin{bmatrix} M_{11} & M_{12} & \mathbf{0} \\ M_{21} & M_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_{33} \end{bmatrix} S_i^{-1}, \qquad Q_i = S_i \begin{bmatrix} Y_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & Y_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & Y_{33} \end{bmatrix} S_i^{-1},$$

where $\mathrm{rk}(P_i) = x_i + y_i$ for $i = 1, \ldots, k$, $M_{11}, Y_{11} \in \mathbb{C}_{x_i}$, $M_{22}, Y_{22} \in \mathbb{C}_{y_i}$, $M_{33}, Y_{33} \in \mathbb{C}_{m_i-(x_i+y_i)}$, and

$$(\alpha - \beta)M_{11} - \gamma\delta Y_{11} = (\alpha(\beta + \gamma + \delta) - \mu_i\nu_i)I_{x_i},$$

$$(\beta - \alpha)M_{22} - \gamma\delta Y_{22} = (\beta(\alpha + \gamma + \delta) - \mu_i\nu_i)I_{y_i},$$

and

$$(\alpha + \beta)M_{33} + \gamma\delta Y_{33} = \mu_i\nu_i I_{m_i-(x_i+y_i)}.$$

Proof. First, we prove the parts (i) and (ii) of the lemma:

Since the matrix $X = A + B$ is diagonalizable, there exists a nonsingular matrix $S \in \mathbb{C}_n$ such that

$$X = S(\lambda_1 I_{p_1} \oplus \cdots \oplus \lambda_m I_{p_m})S^{-1}, \tag{2.1}$$

where the scalars $\lambda_1, \ldots, \lambda_m$ are distinct complex numbers and $p_1 + \cdots + p_m = n$. We write the matrices A, P, and Q as follows:

$$A = S \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix} S^{-1}, \qquad P = S \begin{bmatrix} P_{11} & \cdots & P_{1m} \\ \vdots & \ddots & \vdots \\ P_{m1} & \cdots & P_{mm} \end{bmatrix} S^{-1},$$

and

$$Q = S \begin{bmatrix} Q_{11} & \cdots & Q_{1m} \\ \vdots & \ddots & \vdots \\ Q_{m1} & \cdots & Q_{mm} \end{bmatrix} S^{-1}$$

with $P_{ii}, Q_{ii}, A_{ii} \in \mathbb{C}_{p_i}$ for $i = 1, \ldots, m$. From this, we have

$$AX = S \begin{bmatrix} \lambda_1 A_{11} & \cdots & \lambda_m A_{1m} \\ \vdots & \ddots & \vdots \\ \lambda_1 A_{m1} & \cdots & \lambda_m A_{mm} \end{bmatrix} S^{-1}, \qquad XA = S \begin{bmatrix} \lambda_1 A_{11} & \cdots & \lambda_1 A_{1m} \\ \vdots & \ddots & \vdots \\ \lambda_m A_{m1} & \cdots & \lambda_m A_{mm} \end{bmatrix} S^{-1}.$$

Since $AB \neq BA$, we have $AX \neq XA$. So, there exist $i_0, j_0 \in \{1, \ldots, m\}$ such that $i_0 \neq j_0$ and $\lambda_{i_0} A_{i_0 j_0} \neq \lambda_{j_0} A_{i_0 j_0}$. In particular, we have $A_{i_0 j_0} \neq \mathbf{0}$.

In view of $X = A + B$, $A^2 = (\alpha + \beta)A - \alpha\beta P$, and $B^2 = (\gamma + \delta)B - \gamma\delta Q$, we get

$$X^2 + (\alpha + \beta + \gamma + \delta)A - \alpha\beta P = (\gamma + \delta)X + AX + XA - \gamma\delta Q.$$

Hence, if $i, j \in \{1, \ldots, m\}$ and $i \neq j$, then

$$(\alpha + \beta + \gamma + \delta)A_{ij} - \alpha\beta P_{ij} = (\lambda_i + \lambda_j)A_{ij} - \gamma\delta Q_{ij}. \tag{2.2}$$

Let us denote $W = \alpha\beta P - \gamma\delta Q$ and partition the matrix W as $W = S[W_{ij}]_{i,j=1}^m S^{-1}$, where $W_{ii} \in \mathbb{C}_{p_i}$ for $i = 1, \cdots, m$. From (2.1), we get

$$WX = S \begin{bmatrix} \lambda_1 W_{11} & \cdots & \lambda_m W_{1m} \\ \vdots & \ddots & \vdots \\ \lambda_1 W_{m1} & \cdots & \lambda_m W_{mm} \end{bmatrix} S^{-1}, \quad XW = S \begin{bmatrix} \lambda_1 W_{11} & \cdots & \lambda_1 W_{1m} \\ \vdots & \ddots & \vdots \\ \lambda_m W_{m1} & \cdots & \lambda_m W_{mm} \end{bmatrix} S^{-1}.$$

If $i \neq j$, then $WX = XW$ and $\lambda_i \neq \lambda_j$ imply $W_{ij} = \mathbf{0}$. Thus,

$$\alpha\beta P_{ij} = \gamma\delta Q_{ij} \quad \text{for } i \neq j \text{ and } i, j \in \{1, \ldots, m\}. \tag{2.3}$$

Considering the equality (2.3) together with (2.2), we get

$$(\alpha + \beta + \gamma + \delta)A_{ij} = (\lambda_i + \lambda_j)A_{ij} \quad \text{for } i \neq j \text{ and } i, j \in \{1, \ldots, m\}. \tag{2.4}$$

Since $A_{i_0 j_0} \neq \mathbf{0}$, we have $\lambda_{i_0} + \lambda_{j_0} = \alpha + \beta + \gamma + \delta$.

By a rearrangement of the indices, we can assume $i_0 = 1$ and $j_0 = 2$. From this, we have $\lambda_1 + \lambda_2 = \alpha + \beta + \gamma + \delta$. Suppose that there exists $t \in \{3, \ldots, m\}$ such that $\alpha + \beta + \gamma + \delta = \lambda_1 + \lambda_t$. So, we get $\lambda_1 + \lambda_2 = \lambda_1 + \lambda_t$, that is $\lambda_2 = \lambda_t$. But, this is a contradiction since $\lambda_i \neq \lambda_j$ for $i \neq j$. Hence, $\lambda_1 + \lambda_t \neq \alpha + \beta + \gamma + \delta$ for any $t \in \{3, \ldots, m\}$. From (2.4), we get $A_{1t} = \mathbf{0}$ for all $t \in \{3, \ldots, m\}$. And from a symmetric reasoning, we obtain $A_{2t} = \mathbf{0}$, $A_{t1} = \mathbf{0}$, and $A_{t2} = \mathbf{0}$ for all $t \in \{3, \ldots, m\}$. Thus, A can be written as

$$A = S(A_1 \oplus \tilde{A}_1)S^{-1}, \qquad A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad A_{11} \in \mathbb{C}_{p_1},\ A_{22} \in \mathbb{C}_{p_2}, \tag{2.5}$$

where $\tilde{A}_1$ is some square matrix of suitable size. Since $A^2 = (\alpha+\beta)A - \alpha\beta P$, if $\alpha\beta \neq 0$, then we get

$$P = S(P_1 \oplus \tilde{P}_1)S^{-1}, \qquad P_1 = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \qquad P_{11} \in \mathbb{C}_{p_1},\ P_{22} \in \mathbb{C}_{p_2},$$

where $A_1^2 = (\alpha+\beta)A_1 - \alpha\beta P_1$, $\tilde{A_1}^2 = (\alpha+\beta)\tilde{A}_1 - \alpha\beta\tilde{P}_1$, and $\tilde{P}_1$ is some square matrix of suitable size. From the idempotency of P, the matrices P_1 and $\tilde{P}_1$ are idempotents. So, $A_1 \in \mathfrak{L}(P_1;\alpha,\beta)$ and $\tilde{A}_1 \in \mathfrak{L}(\tilde{P}_1;\alpha,\beta)$. If $\alpha\beta = 0$, then we take $P = I_n$. In both cases, the blocks (1,2) and (2,1) of P are null.

From now on, we denote $\mu_1 = \lambda_1$, $\nu_1 = \lambda_2$, $r_1 = p_1$, and $s_1 = p_2$. From (2.1), we have

$$X = S((\mu_1 I_{r_1} \oplus \nu_1 I_{s_1}) \oplus \Lambda_2)S^{-1}, \tag{2.6}$$

where Λ_2 is a diagonal matrix of suitable size. In view of $A + B = X$, from (2.5) and (2.6), we get

$$B = X - A = S(((\mu_1 I_{r_1} \oplus \nu_1 I_{s_1}) - A_1) \oplus (\Lambda_2 - \tilde{A}_1))S^{-1}.$$

If we define $B_1 = (\mu_1 I_{r_1} \oplus \nu_1 I_{s_1}) - A_1$ and $\tilde{B}_1 = \Lambda_2 - \tilde{A}_1$, then we have $B = S(B_1 \oplus \tilde{B}_1)S^{-1}$. Since $B^2 = (\gamma+\delta)B - \gamma\delta Q$, if $\gamma\delta \neq 0$, then we obtain

$$Q = S(Q_1 \oplus \tilde{Q}_1)S^{-1}, \qquad Q_1 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \qquad Q_{11} \in \mathbb{C}_{p_1},\ Q_{22} \in \mathbb{C}_{p_2},$$

where $B_1^2 = (\gamma+\delta)B_1 - \gamma\delta Q_1$ and $\tilde{B_1}^2 = (\gamma+\delta)\tilde{B}_1 - \gamma\delta\tilde{Q}_1$. The matrices Q_1 and $\tilde{Q}_1$ are idempotents because Q is idempotent. So, $B_1 \in \mathfrak{L}(Q_1;\gamma,\delta)$ and $\tilde{B}_1 \in \mathfrak{L}(\tilde{Q}_1;\gamma,\delta)$. If $\gamma\delta = 0$, then we take $Q = I_n$. In both cases, the blocks (1,2) and (2,1) of Q are null. If $\tilde{A}_1\tilde{B}_1 = \tilde{B}_1\tilde{A}_1$, then it is enough to take $A_0 = \tilde{A}_1$ and $B_0 = \tilde{B}_1$ for the proof of the part (i) of lemma. In addition, in case $A_0 = \tilde{A}_1$ and $B_0 = \tilde{B}_1$, we have $\alpha+\beta+\gamma+\delta = \mu_1 + \nu_1$ and $A_1 + B_1 = \mu_1 I_{r_1} \oplus \nu_1 I_{s_1}$, and hence $\sigma(A_1 + B_1) = \{\mu_1, \nu_1\}$.

Assume that $\tilde{A}_1\tilde{B}_1 \neq \tilde{B}_1\tilde{A}_1$. Since $\tilde{A}_1$, $\tilde{B}_1$, $\tilde{P}_1$, and $\tilde{Q}_1$ satisfy the hypothesis of the theorem, we can apply the first step of the proof to get $\tilde{A}_1 = A_2 \oplus \tilde{A}_2$, $\tilde{B}_1 = B_2 \oplus \tilde{B}_2$, $\tilde{P}_1 = P_2 \oplus \tilde{P}_2$, and $\tilde{Q}_1 = Q_2 \oplus \tilde{Q}_2$, where $A_2 \in \mathfrak{L}(P_2;\alpha,\beta)$, $\tilde{A}_2 \in \mathfrak{L}(\tilde{P}_2;\alpha,\beta)$, $B_2 \in \mathfrak{L}(Q_2;\gamma,\delta)$, and $\tilde{B}_2 \in \mathfrak{L}(\tilde{Q}_2;\gamma,\delta)$. Now, by an exhaustion process we prove (i), the first and second relations of (ii).

Now, we shall prove the last equality of (ii). For any $i \in \{1,\dots,k\}$, we have $A_i + B_i = \mu_i I_{r_i} \oplus \nu_i I_{s_i}$. Hence, denoting $m_i = r_i + s_i$, we have

$$(A_i + B_i - \mu_i I_{m_i})(A_i + B_i - \nu_i I_{m_i}) = \mathbf{0}.$$

By doing a little algebra and using the proved equalities of (i) and (ii), we get the last equality of (ii).

Next, we prove the part (iii) of the lemma. Let us fix $i \in \{1,\dots,k\}$. Since P_i is an idempotent, there exists a nonsingular matrix $R_i \in \mathbb{C}_{m_i}$ such that $P_i = R_i(I_{r_i} \oplus \mathbf{0})R_i^{-1}$, where $r_i = \mathrm{rk}(P_i)$. Let us write

$$A_i = R_i \begin{bmatrix} K & L \\ M & N \end{bmatrix} R_i^{-1}, \qquad K \in \mathbb{C}_{r_i}.$$

From $A_iP_i = P_iA_i = A_i$, we obtain $L = \mathbf{0}$, $M = \mathbf{0}$, and $N = \mathbf{0}$. From $(A_i - \alpha P_i)(A_i - \beta P_i) = \mathbf{0}$, we get $(K - \alpha I_{r_i})(K - \beta I_{r_i}) = \mathbf{0}$, and using $\alpha \neq \beta$, we get that the matrix K is diagonalizable and $\sigma(K) \subset \{\alpha, \beta\}$. So, there exist $T_i \in \mathbb{C}_{r_i}$ and $x_i, y_i \in \{0, \dots, r_i\}$ such that $K = T_i(\alpha I_{x_i} \oplus \beta I_{y_i})T_i^{-1}$.

Let S_i be the nonsingular matrix defined by $S_i = R_i(T_i \oplus I_{m_i - r_i})$ and $D = \alpha I_{x_i} \oplus \beta I_{y_i}$. It is simple to see that

$$A_i = S_i(D \oplus \mathbf{0})S_i^{-1} \qquad \text{and} \qquad P_i = S_i(I_{r_i} \oplus \mathbf{0})S_i^{-1}. \tag{2.7}$$

We write the matrices B_i and Q_i as

$$B_i = S_i \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} S_i^{-1} \qquad \text{and} \qquad Q_i = S_i \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix} S_i^{-1}, \tag{2.8}$$

where $M_1, Y_1 \in \mathbb{C}_{r_i}$. From $P_iQ_i = Q_iP_i$, we get $Y_2 = \mathbf{0}$ and $Y_3 = \mathbf{0}$. Thus, we have

$$Q_i = S_i(Y_1 \oplus Y_4)S_i^{-1}.$$

From $(A_i + B_i)(\alpha\beta P_i - \gamma\delta Q_i) = (\alpha\beta P_i - \gamma\delta Q_i)(A_i + B_i)$, $A_iP_i = P_iA_i = A_i$, and $B_iQ_i = Q_iB_i = B_i$, we arrive at the equality $\alpha\beta(P_iB_i - B_iP_i) = \gamma\delta(Q_iA_i - A_iQ_i)$.

From (2.7) and (2.8), we get

$$P_iB_i - B_iP_i = S_i \begin{bmatrix} \mathbf{0} & M_2 \\ -M_3 & \mathbf{0} \end{bmatrix} S_i^{-1} \quad \text{and} \quad Q_iA_i - A_iQ_i = S_i \begin{bmatrix} Y_1D - DY_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} S_i^{-1}. \tag{2.9}$$

Now we have four possibilities:

(a) $\alpha, \beta, \gamma, \delta \neq 0$.

From $\alpha\beta(P_iB_i - B_iP_i) = \gamma\delta(Q_iA_i - A_iQ_i)$ and $\alpha, \beta, \gamma, \delta \neq 0$, we get

$$M_2 = \mathbf{0}, \qquad M_3 = \mathbf{0}, \qquad Y_1D = DY_1. \tag{2.10}$$

In particular, we have

$$B_i = S_i(M_1 \oplus M_4)S_i^{-1}.$$

Now, let us write

$$M_1 = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \qquad Y_1 = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}, \qquad M_{11}, Y_{11} \in \mathbb{C}_{x_i}. \tag{2.11}$$

In view of the last equality of (2.10) and $\alpha \neq \beta$, we get $Y_{12} = \mathbf{0}$ and $Y_{21} = \mathbf{0}$. From now on, we denote $Y_4 = Y_{33}$ and $M_4 = M_{33}$.

(b) $\alpha\beta \neq 0$ and $\gamma\delta = 0$.

From $\alpha\beta(P_iB_i - B_iP_i) = \gamma\delta(Q_iA_i - A_iQ_i)$, we get $P_iB_i = B_iP_i$ because $\alpha\beta \neq 0$ and $\gamma\delta = 0$. From this, considering (2.9), we arrive at $M_2 = \mathbf{0}$ and $M_3 = \mathbf{0}$. So, we have $B_i = S_i(M_1 \oplus M_4)S_i^{-1}$. Since $\gamma\delta = 0$, we can take $Q = I_n$, and hence $Q_i = I_{m_i} = S_i(I_{x_i} \oplus I_{y_i} \oplus I_{m_i - (x_i + y_i)})S_i^{-1}$. In this case, the idempotent matrices Y_{11}, Y_{22}, and Y_{33} are particularly identity matrices of suitable size. So, we get a particular case of the case in (a).

(c) $\alpha\beta = 0$ and $\gamma\delta \neq 0$.

From $\alpha\beta(P_iB_i - B_iP_i) = \gamma\delta(Q_iA_i - A_iQ_i)$, we get $Q_iA_i = A_iQ_i$ because $\alpha\beta = 0$ and $\gamma\delta \neq 0$. Thus, $Y_1D = DY_1$ in view of (2.9). Let the matrix Y_1 be as in (2.11). In view of $\alpha \neq \beta$ and $Y_1D = DY_1$, we arrive at $Y_{12} = \mathbf{0}$, $Y_{21} = \mathbf{0}$. Furthermore, since $\alpha\beta = 0$, we can take $P = I_n$, and hence $P_i = I_{m_i}$, which means that the last summand in the direct sums occurring in (2.7) are not present. Also, by considering the first equality of (2.9) together with $P_i = I_{m_i}$, we obtain $M_2 = \mathbf{0}$ and $M_3 = \mathbf{0}$. So, we get a particular case of (a).

(d) $\alpha\beta = 0$ and $\gamma\delta = 0$.

In this case, we can take $P = Q = I_n$, and hence $P_i = Q_i = I_{m_i}$. Since $\alpha\beta = 0$, the last blocks of A_i and P_i are absent. In view of $Q_i = I_{m_i}$, we can write $Y_{11} = I_{x_i}$ and $Y_{22} = I_{y_i}$. Also, the blocks Y_{33} (of Q_i) and M_{33} (of B_i) are absent. Thus, again we get a particular case of (a). So, without loss of the generality, from now on, we will consider the case in (a). Hence, the matrices A_i, B_i, P_i, and Q_i can be written as in (iii).

We know from Lemma 2.1 that

$$A_iB_i = S_i \begin{bmatrix} \alpha M_{11} & \alpha M_{12} & \mathbf{0} \\ \beta M_{21} & \beta M_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} S_i^{-1} \quad \text{and} \quad B_iA_i = S_i \begin{bmatrix} \alpha M_{11} & \beta M_{12} & \mathbf{0} \\ \alpha M_{21} & \beta M_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} S_i^{-1}. \tag{2.12}$$

Also, by the last relation of (ii) of Lemma 2.1, we have

$$A_iB_i + B_iA_i + \mu_i\nu_i I_{m_i} = (\gamma + \delta)A_i + (\alpha + \beta)B_i + \alpha\beta P_i + \gamma\delta Q_i \quad \text{for } i = 1, \dots, k.$$

Considering the last equality, (2.12), and the statements of the matrices A_i, P_i, B_i, and Q_i, we get

$$(\alpha - \beta)M_{11} - \gamma\delta Y_{11} = (\alpha\beta + (\gamma + \delta)\alpha - \mu_i\nu_i)I_{x_i},$$
$$(\beta - \alpha)M_{22} - \gamma\delta Y_{22} = (\alpha\beta + (\gamma + \delta)\beta - \mu_i\nu_i)I_{y_i},$$

and

$$(\alpha + \beta)M_{33} + \gamma\delta Y_{33} = \mu_i\nu_i I_{m_i-(x_i+y_i)}.$$

So, the proof of (iii) is completed. □

Let us observe that the blocks A_0 and B_0 in (i) of Lemma 2.1, and therefore, P_0 and Q_0 may be absent.

Now, let us give the following remark in view of Lemma 2.1:

Remark 2.1. At first, by Lemma 2.1 (iii), under the condition $\alpha \neq \beta$, we have already $P_iB_i = B_iP_i$ and $Q_iA_i = A_iQ_i$ with $i \in \{1, \dots, k\}$ for all $\alpha, \beta, \gamma, \delta \in \mathbb{C}$. On the other hand;

(i) If $\alpha\beta\gamma\delta \neq 0$ and $\alpha \neq \beta$, the condition $(\alpha\beta P - \gamma\delta Q)(A + B) = (A + B)(\alpha\beta P - \gamma\delta Q)$ in Lemma 2.1 is equivalent to the conditions $PB = BP$ and $QA = AQ$ since in this case, we get $P_0B_0 = B_0P_0$ and $Q_0A_0 = A_0Q_0$.

(ii) If $\alpha\beta \neq 0 = \gamma\delta$ and $\alpha\beta = 0 \neq \gamma\delta$, then the same condition is equivalent to the condition $PB = BP$ and $QA = AQ$, respectively.

(iii) In the case $\alpha\beta = \gamma\delta = 0$, the condition $(\alpha\beta P - \gamma\delta Q)(A + B) = (A + B)(\alpha\beta P - \gamma\delta Q)$ already vanishes.

The following corollary is a simple consequence of Theorem 2.1 and Lemma 2.1.

Corollary 2.1. *Let $P, Q \in \mathbb{C}_n$ be two idempotents, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, and $A \in \mathfrak{L}(P; \alpha, \beta)$, $B \in \mathfrak{L}(Q; \gamma, \delta)$. Assume that $(A + B)(\alpha\beta P - \gamma\delta Q) = (\alpha\beta P - \gamma\delta Q)(A + B)$, $AB \neq BA$, and $A + B$ is diagonalizable. If $\lambda \in \sigma(A + B) \setminus [\{0, \alpha, \beta\} + \{0, \gamma, \delta\}]$, then there exists $\mu \in \sigma(A + B)$ such that $\lambda \neq \mu$ and $\lambda + \mu = \alpha + \beta + \gamma + \delta$.*

The following theorem provides a tool for proofs of the next theorems.

Theorem 2.2. *Let $A, B, P, Q \in \mathbb{C}_n$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P; \alpha, \beta)$ and $B \in \mathfrak{L}(Q; \gamma, \delta)$. If $AB = BA$, then*

(i) $\sigma((\gamma + \delta)A + (\alpha + \beta)B + \alpha\beta P + \gamma\delta Q - AB - BA) \subset \Gamma_1 + \Gamma_2$ with $\Gamma_1 = \{0, \alpha(\gamma + \delta), \beta(\gamma + \delta), (\alpha + \beta)\gamma, (\alpha + \beta)\delta, \alpha\delta + \beta\gamma, \alpha\gamma + \beta\delta\}$ and $\Gamma_2 = \{0, \alpha\beta, \gamma\delta, \alpha\beta + \gamma\delta\}$ if $PQ = QP$,

(ii) $\sigma(AB - \beta B - \gamma\delta PQ) \subset \Psi_1 + \Psi_2$ with $\Psi_1 = \{0, -\beta\gamma, -\beta\delta, -\gamma\delta, -\gamma\delta - \beta\gamma, -\gamma\delta - \beta\delta\}$ and $\Psi_2 = \{0, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta\}$ if $BP = PB$,

(iii) $\sigma((\alpha - \beta)B + BA - AB - \gamma\delta Q) \subset \Gamma$ with $\Gamma = \{0, -\gamma\delta, (\alpha - \beta)\gamma, -(\beta - \alpha + \delta)\gamma, (\alpha - \beta)\delta, -(\beta - \alpha + \gamma)\delta\}$.

Proof. (i) Since P and Q are two commuting idempotents, (and therefore, $\sigma(\alpha\beta P) \subset \{0, \alpha\beta\}$ and $\sigma(\gamma\delta P) \subset \{0, \gamma\delta\}$), we have

$$\sigma(\alpha\beta P + \gamma\delta Q) \subset \Gamma_2 \tag{2.13}$$

by Theorem 2.4.9 of [4]. Also, we get

$$\sigma((\gamma + \delta)A + (\alpha + \beta)B - AB - BA) \subset \Gamma_1 \tag{2.14}$$

by Theorem 2.4.9 of [4] because A and B commute, $\sigma(A) \subset \{0, \alpha, \beta\}$, and $\sigma(B) \subset \{0, \gamma, \delta\}$. On the other hand, the matrices $\alpha\beta P + \gamma\delta Q$ and $(\gamma + \delta)A + (\alpha + \beta)B - AB - BA$ are commuting matrices. So, it is obtained the desired result in view of (2.13), (2.14), and Theorem 2.4.9 of [4].

(ii) Since $BP = PB$ and $BQ = QB = B$, we get $(\beta B)(\gamma\delta PQ) = (\gamma\delta PQ)(\beta B)$. Also, P and Q are commuting idempotents, and therefore PQ is idempotent. So, we have $\sigma(-\gamma\delta PQ) \subset \{0, -\gamma\delta\}$. Again, we have $\sigma(-\beta B) \subset \{0, -\beta\gamma, -\beta\delta\}$ since B is a generalized quadratic matrix such that $(B - \gamma Q)(B - \delta Q) = \mathbf{0}$. So, by Theorem 2.4.9 of [4], we obtain

$$\sigma(-\beta B - \gamma\delta PQ) \subset \Psi_1 \tag{2.15}$$

since $(\beta B)(\gamma\delta PQ) = (\gamma\delta PQ)(\beta B)$. In view of $AB = BA$, $PB = BP$, $AP = PA = A$, and $BQ = QB = B$, we get that AB commutes with $-\beta B - \gamma\delta PQ$. Since $\sigma(A) \subset \{0, \alpha, \beta\}$, $\sigma(B) \subset \{0, \gamma, \delta\}$, and $AB = BA$, we get

$$\sigma(AB) \subset \Psi_2. \tag{2.16}$$

So, we have from (2.15), (2.16), and Theorem 2.4.9 of [4],

$$\sigma(AB - \beta B - \gamma\delta PQ) \subset \Psi_1 + \Psi_2 \tag{2.17}$$

since AB and $-\beta B - \gamma\delta PQ$ commute.

(iii) Since $AB = BA$, we get $(\alpha-\beta)B - AB + BA - \gamma\delta Q = (\alpha-\beta)B - \gamma\delta Q$. In view of $\sigma(B) \subset \{0,\gamma,\delta\}$, $\sigma(Q) \subset \{0,1\}$, and $BQ = QB$, we obtain

$$\sigma((\alpha-\beta)B - \gamma\delta Q) \subset \{0, -\gamma\delta, (\alpha-\beta)\gamma, \gamma(\alpha-\beta-\delta), (\alpha-\beta)\delta, \delta(\alpha-\beta-\gamma)\} = \Gamma$$

by Theorem 2.4.9 of [4].

□

From the part (ii) of Lemma 2.1, we get the following result.

Theorem 2.3. *Let $A, B, P, Q \in \mathbb{C}_n$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P;\alpha,\beta)$, $B \in \mathfrak{L}(Q;\gamma,\delta)$, $(\alpha\beta P - \gamma\delta Q)(A+B) = (A+B)(\alpha\beta P - \gamma\delta Q)$, $AB \neq BA$ and let $A+B$ be diagonalizable.*

(i) If $\mu \in \sigma(A+B) \setminus [\{0,\alpha,\beta\} + \{0,\gamma,\delta\}]$, then

$$\mu(\alpha+\beta+\gamma+\delta-\mu) \in \sigma[(\gamma+\delta)A + (\alpha+\beta)B + \alpha\beta P + \gamma\delta Q - AB - BA].$$

(ii) If $\lambda \in \sigma[(\gamma+\delta)A + (\alpha+\beta)B + \alpha\beta P + \gamma\delta Q - AB - BA] \setminus [\Gamma_1 + \Gamma_2]$, where $\Gamma_1 = \{0, \alpha(\gamma+\delta), \beta(\gamma+\delta), (\alpha+\beta)\gamma, (\alpha+\beta)\delta, \alpha\delta+\beta\gamma, \alpha\gamma+\beta\delta\}$, $\Gamma_2 = \{0, \alpha\beta, \gamma\delta, \alpha\beta+\gamma\delta\}$, then the roots of the polynomial $x^2 - (\alpha+\beta+\gamma+\delta)x + \lambda$ are eigenvalues of the matrix $A+B$.

Proof. Let the matrices A, B, P, and Q be as in Lemma 2.1.

Now, take any $\mu \in \sigma(A+B) \setminus [\{0,\alpha,\beta\} + \{0,\gamma,\delta\}]$. By Theorem 2.1 and the part (ii) of Lemma 2.1, there exists $i \in \{1,\ldots,k\}$ such that $\sigma(A_i + B_i) = \{\mu, \alpha+\beta+\gamma+\delta-\mu\}$. In view of the last relation of (ii) of Lemma 2.1, we have

$$\begin{aligned} \mu(\alpha+\beta+\gamma+\delta-\mu) &\in \sigma[(\gamma+\delta)A_i + (\alpha+\beta)B_i + \alpha\beta P_i + \gamma\delta Q_i - A_iB_i - B_iA_i] \\ &\subset \sigma[(\gamma+\delta)A + (\alpha+\beta)B + \alpha\beta P + \gamma\delta Q - AB - BA]. \end{aligned}$$

Hence, the proof of (i) is completed.

Next, take any $\lambda \in \sigma[(\gamma+\delta)A + (\alpha+\beta)B + \alpha\beta P + \gamma\delta Q - AB - BA] \setminus [\Gamma_1 + \Gamma_2]$. So, there exists $i \in \{1,\ldots,k\}$ such that $\lambda \in \sigma[(\gamma+\delta)A_i + (\alpha+\beta)B_i + \alpha\beta P_i + \gamma\delta Q_i - A_iB_i - B_iA_i]$ by Theorem 2.2 (i). Thus, by the last relation of (ii) of Lemma 2.1, there exist $\mu, \nu \in \sigma(A_i + B_i)$ such that $\alpha+\beta+\gamma+\delta = \mu+\nu$ and $\lambda = \mu\nu$, and therefore, μ and ν are the roots of the polynomial $x^2 - (\alpha+\beta+\gamma+\delta)x + \lambda$. In view of $\sigma(A_i+B_i) \subset \sigma(A+B)$, we have the desired result in (ii). □

Theorem 2.4. *Let $A, B, P, Q \in \mathbb{C}_n$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P;\alpha,\beta)$ with $\alpha \neq \beta$, $B \in \mathfrak{L}(Q;\gamma,\delta)$, $(\alpha\beta P - \gamma\delta Q)(A+B) = (A+B)(\alpha\beta P - \gamma\delta Q)$, $AB \neq BA$ and let $A+B$ be diagonalizable.*

(i) If $\lambda \in \sigma(AB - \beta B - \gamma\delta PQ) \setminus [(\Psi_1 + \Psi_2) \cup \Psi_3]$ with $\Psi_1 = \{0, -\beta\gamma, -\beta\delta, -\gamma\delta, -\gamma\delta - \beta\gamma, -\gamma\delta - \beta\delta\}$, $\Psi_2 = \{0, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta\}$, and $\Psi_3 = \{0, -\beta\gamma, -\beta\delta, -\gamma\delta\}$, then the roots of the polynomial $x^2 - (\alpha+\beta+\gamma+\delta)x + \alpha(\beta+\gamma+\delta) - \lambda$ are eigenvalues of the matrix $A+B$.

(ii) If $\mu \in \sigma(A+B) \setminus [\{0,\alpha,\beta\} + \{0,\gamma,\delta\}]$, then $\mu^2 - \mu(\alpha+\beta+\gamma+\delta) + \alpha(\beta+\gamma+\delta) \in \sigma(AB - \beta B - \gamma\delta PQ)$.

Proof. Let us write A, B, P, and Q as in Lemma 2.1. By Lemma 2.1 (i), we have

$$AB-\beta B-\gamma\delta PQ = S[(\oplus_{i=1}^{k}(A_iB_i-\beta B_i-\gamma\delta P_iQ_i))\oplus(A_0B_0-\beta B_0-\gamma\delta P_0Q_0)]S^{-1}. \quad (2.18)$$

From Lemma 2.1 (iii), we get

$$A_iB_i-\beta B_i-\gamma\delta P_iQ_i = S_i\begin{bmatrix}(\alpha-\beta)M_{11}-\gamma\delta Y_{11} & (\alpha-\beta)M_{12} & \mathbf{0}\\ \mathbf{0} & -\gamma\delta Y_{22} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & -\beta M_{33}\end{bmatrix}S_i^{-1} \quad (2.19)$$

for all $i \in \{1,\ldots,k\}$ and also, we have

$$(\alpha-\beta)M_{11}-\gamma\delta Y_{11} = (\alpha(\beta+\gamma+\delta)-\mu_i\nu_i)I_{x_i}. \quad (2.20)$$

On the other hand, the matrix Y_{22} is an idempotent matrix and M_{33} satisfies the equality

$$(M_{33}-\gamma Y_{33})(M_{33}-\delta Y_{33}) = \mathbf{0}.$$

So, we have $\sigma(-\gamma\delta Y_{22}) \subset \{0,-\gamma\delta\}$ and $\sigma(-\beta M_{33}) \subset \{0,-\beta\gamma,-\beta\delta\}$, and thus,

$$\sigma(-\gamma\delta Y_{22}\oplus -\beta M_{33}) \subset \Psi_3. \quad (2.21)$$

Observe that we have $B_0P_0 = P_0B_0$ because it does not invalidate the generality to take $\alpha\beta\gamma\delta \neq 0$. Since $A_0B_0 = B_0A_0$, $B_0P_0 = P_0B_0$, and $P_0Q_0 = Q_0P_0$, we get

$$\sigma(A_0B_0-\beta B_0-\gamma\delta P_0Q_0) \subset \Psi_1+\Psi_2 \quad (2.22)$$

by Theorem 2.2 (ii).

Now, take any $\lambda \in \sigma(AB-\beta B-\gamma\delta PQ)\backslash[(\Psi_1+\Psi_2)\cup\Psi_3]$. So, by (2.18), (2.19), (2.20), (2.21), and (2.22), there exists $i \in \{1,\ldots,k\}$ such that $\lambda = \alpha(\beta+\gamma+\delta)-\mu_i\nu_i$. Moreover, from Lemma 2.1 (ii), we get $\mu_i,\nu_i \in \sigma(A+B)$ and $\mu_i+\nu_i = \alpha+\beta+\gamma+\delta$. So, μ_i and ν_i are the roots of the polynomial $x^2-(\alpha+\beta+\gamma+\delta)x+\alpha(\beta+\gamma+\delta)-\lambda$.

Next, take any $\mu \in \sigma(A+B)\backslash[\{0,\alpha,\beta\}+\{0,\gamma,\delta\}]$. Lemma 2.1 and Theorem 2.1 ensure that there exists $i \in \{1,\ldots,k\}$ such that μ and $\alpha+\beta+\gamma+\delta-\mu$ are eigenvalues of A_i+B_i. Let us denote $\nu = \alpha+\beta+\gamma+\delta-\mu$. From (2.19) and (2.20), we get

$$\alpha(\beta+\gamma+\delta)-\mu\nu \in \sigma(A_iB_i-\beta B_i-\gamma\delta P_iQ_i) \subset \sigma(AB-\beta B-\gamma\delta PQ).$$

Hence, the proof is completed. □

Theorem 2.5. *Let $A,B,P,Q \in \mathbb{C}_n$ and $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P;\alpha,\beta)$ with $\alpha \neq \beta$, $B \in \mathfrak{L}(Q;\gamma,\delta)$, $(\alpha\beta P-\gamma\delta Q)(A+B) = (A+B)(\alpha\beta P-\gamma\delta Q)$, $AB \neq BA$ and let $A+B$ be diagonalizable.*

(i) If $\mu \in \sigma(A+B)\backslash[\{0,\alpha,\beta\}+\{0,\gamma,\delta\}]$, then

(i.a) $\alpha(\beta+\gamma+\delta)-\mu(\alpha+\beta+\gamma+\delta-\mu) \in \sigma[(\alpha-\beta)B+BA-AB-\gamma\delta Q]$.

(i.b) $-\beta(\alpha+\gamma+\delta)+\mu(\alpha+\beta+\gamma+\delta-\mu) \in \sigma[(\alpha-\beta)B+BA-AB-\gamma\delta Q]$ or $-2\gamma\delta-\beta(\alpha+\gamma+\delta)+\mu(\alpha+\beta+\gamma+\delta-\mu) \in \sigma[(\alpha-\beta)B+BA-AB-\gamma\delta Q]$.

(ii) If $\lambda \in \sigma[(\alpha-\beta)B + BA - AB - \gamma\delta Q]\backslash\Gamma$, where $\Gamma = \{0, -\gamma\delta, (\alpha-\beta)\gamma, -(\beta-\alpha+\delta)\gamma, (\alpha-\beta)\delta, -(\beta-\alpha+\gamma)\delta\}$, then the roots of one of the following polynomials are eigenvalues of the matrix $A+B$.

(a) $x^2 - (\alpha+\beta+\gamma+\delta)x + \alpha(\beta+\gamma+\delta) - \lambda$,

(b) $x^2 - (\alpha+\beta+\gamma+\delta)x + \lambda + \beta(\alpha+\gamma+\delta)$,

(c) $x^2 - (\alpha+\beta+\gamma+\delta)x + \lambda + \beta(\alpha+\gamma+\delta) + 2\gamma\delta$.

Proof. Let us write A, B, P, and Q as in Lemma 2.1. By Lemma 2.1 (i), we have that

$$(\alpha-\beta)B - AB + BA - \gamma\delta Q$$

is similar to

$$[(\oplus_{i=1}^{k}(\alpha-\beta)B_i - A_iB_i + B_iA_i - \gamma\delta Q_i) \oplus ((\alpha-\beta)B_0 - A_0B_0 + B_0A_0 - \gamma\delta Q_0)].$$

From Lemma 2.1 (iii), we arrive at

$$\begin{aligned}&(\alpha-\beta)B_i - A_iB_i + B_iA_i - \gamma\delta Q_i\\ &\quad = S_i \begin{bmatrix} (\alpha-\beta)M_{11} - \gamma\delta Y_{11} & \mathbf{0} & \mathbf{0} \\ 2(\alpha-\beta)M_{21} & (\alpha-\beta)M_{22} - \gamma\delta Y_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\alpha-\beta)M_{33} - \gamma\delta Y_{33} \end{bmatrix} S_i^{-1} \end{aligned} \tag{2.23}$$

Also, we know from Lemma 2.1 (iii) that $(\alpha-\beta)M_{11} - \gamma\delta Y_{11} = (\alpha(\beta+\gamma+\delta) - \mu_i\nu_i)I_{x_i}$, and therefore,

$$\sigma((\alpha-\beta)M_{11} - \gamma\delta Y_{11}) = \{\alpha(\beta+\gamma+\delta) - \mu_i\nu_i\}. \tag{2.24}$$

Again, from Lemma 2.1(iii), we have $(\alpha-\beta)M_{22} + \gamma\delta Y_{22} = (-\beta(\alpha+\gamma+\delta) + \mu_i\nu_i)I_{y_i}$, and therefore

$$(\alpha-\beta)M_{22} - \gamma\delta Y_{22} = -2\gamma\delta Y_{22} + (-\beta(\alpha+\gamma+\delta) + \mu_i\nu_i)I_{y_i}. \tag{2.25}$$

Since Y_{22} is an idempotent, we get

$$\sigma((\alpha-\beta)M_{22} - \gamma\delta Y_{22}) \subset \{-\beta(\alpha+\gamma+\delta) + \mu_i\nu_i, -2\gamma\delta - \beta(\alpha+\gamma+\delta) + \mu_i\nu_i\} \tag{2.26}$$

by Theorem 2.4.9 of [4] and the equality (2.25). On the other hand, since $(M_{33} - \gamma Y_{33})(M_{33} - \delta Y_{33}) = \mathbf{0}$, $M_{33}Y_{33} = Y_{33}M_{33} = M_{33}$, and $Y_{33}^2 = Y_{33}$, we get that

$$\sigma((\alpha-\beta)M_{33} - \gamma\delta Y_{33}) \subset \{0, -\gamma\delta, (\alpha-\beta)\gamma, (\alpha-\beta)\gamma - \gamma\delta, (\alpha-\beta)\delta, (\alpha-\beta)\delta - \gamma\delta\} = \Gamma. \tag{2.27}$$

Observe that in Lemma 2.1 (iii) the blocks M_{11} and M_{22} of B_i must be present, since otherwise, $A_iB_i = B_iA_i$, which is not possible.

Now, take any $\mu \in \sigma(A+B)\backslash[\{0,\alpha,\beta\} + \{0,\gamma,\delta\}]$. So, by Lemma 2.1 and Theorem 2.1, there exists $i \in \{1,\ldots,k\}$ such that μ and $\alpha+\beta+\gamma+\delta-\mu$ are eigenvalues of $A_i + B_i$. Let us denote $\nu = \alpha+\beta+\gamma+\delta-\mu$. By (2.23) and (2.24), we can write

$$\alpha(\beta+\gamma+\delta) - \mu\nu \in \sigma((\alpha-\beta)M_{11} - \gamma\delta Y_{11}) \subset \sigma((\alpha-\beta)B + BA - AB - \gamma\delta Q).$$

From (2.26), we have

$$-\beta(\alpha+\gamma+\delta) + \mu\nu \in \sigma((\alpha-\beta)M_{22} - \gamma\delta Y_{22})$$

or

$$-2\gamma\delta - \beta(\alpha+\gamma+\delta) + \mu\nu \in \sigma((\alpha-\beta)M_{22} - \gamma\delta Y_{22}).$$

Since $\sigma((\alpha-\beta)M_{22} - \gamma\delta Y_{22}) \subset \sigma((\alpha-\beta)B + BA - AB - \gamma\delta Q)$, it is completed the proof of (i).

Next, take any $\lambda \in \sigma((\alpha-\beta)B - AB + BA - \gamma\delta Q)\setminus\Gamma$. So, considering Theorem 2.2 (iii), from (2.23), (2.24), (2.26), and (2.27), it is seen that there exists $i \in \{1,\dots,k\}$ such that $\lambda = \alpha(\beta+\gamma+\delta) - \mu_i\nu_i$ or $\lambda = -\beta(\alpha+\gamma+\delta) + \mu_i\nu_i$ or $\lambda = -2\gamma\delta - \beta(\alpha+\gamma+\delta) + \mu_i\nu_i$, where $\mu_i, \nu_i \in \sigma(A_i + B_i)$ satisfy $\alpha+\beta+\gamma+\delta = \mu_i + \nu_i$. Observe that we have obtained three possibilities where the sum and the product of μ_i and ν_i are known. Therefore, μ_i, ν_i are the roots of one of the polynomials written in (ii) of the theorem. □

Next results deal with similar results, but deleting the hypothesis of the diagonalizability of $A+B$. We begin with establishing several lemmas.

Notice that $xP = yQ$ (when $x, y \in \mathbb{C}$ and P, Q are idempotents) implies one of the following situations:

a) $x = 0$. Thus, $yQ = \mathbf{0}$. Hence $y = 0$ or $Q = \mathbf{0}$.

b) $x \neq 0$. Thus, $P = zQ$. Therefore, $zQ = P = P^2 = z^2Q^2 = z^2Q$.

Now, we have three possibilities:

b.1) $z = 0$; b.2) $z = 1$; b.3) $Q = \mathbf{0}$.

On the other hand, observe that the matrix Q can not be zero, since otherwise, in view of $BQ = QB = B$, we get $B = \mathbf{0}$. From this, we obtain $AB = BA$, which is a contradiction. So, the results in this section include the following situations:

(i) A and B are scalar-potent matrices.

(ii) $A \in \mathfrak{L}(P;\alpha,\beta)$ and $B \in \mathfrak{L}(P;\gamma,\delta)$ with $\alpha\beta = \gamma\delta$.

Lemma 2.2. *Let $A, B, P, Q \in \mathbb{C}_2$ and $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P;\alpha,\beta)$, $B \in \mathfrak{L}(Q;\gamma,\delta)$, $\alpha\beta P = \gamma\delta Q$, and $AB \neq BA$. If $A+B$ is a Jordan block corresponding to $\lambda \in \mathbb{C}$, then $\alpha+\beta+\gamma+\delta = 2\lambda$.*

Proof. Let us define $X = A + B$. From $B^2 = (\gamma+\delta)B - \gamma\delta Q$, we have $(X-A)^2 = (\gamma+\delta)(X-A) - \gamma\delta Q$. By expanding this equality and using $A^2 = (\alpha+\beta)A - \alpha\beta P$ and $\alpha\beta P = \gamma\delta Q$, we get

$$X^2 + (\alpha+\beta+\gamma+\delta)A = AX + XA + (\gamma+\delta)X. \tag{2.28}$$

By hypothesis, we have $X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. Let us write $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$. We get from (2.28)

$$\begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix} + (\alpha+\beta+\gamma+\delta)\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \tag{2.29}$$
$$= \begin{bmatrix} \lambda a_1 & a_1 + \lambda a_2 \\ \lambda a_3 & a_3 + \lambda a_4 \end{bmatrix} + \begin{bmatrix} \lambda a_1 + a_3 & \lambda a_2 + a_4 \\ \lambda a_3 & \lambda a_4 \end{bmatrix} + (\gamma+\delta)\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

Since $AX \neq XA$, we have two possibilities: (i) $a_3 \neq 0$ or (ii) $a_1 \neq a_4$.

(i) By looking at the entry (2,1) of (2.29), one gets $(\alpha+\beta+\gamma+\delta)a_3 = 2\lambda a_3$. Now $a_3 \neq 0$ leads to $\alpha+\beta+\gamma+\delta = 2\lambda$.

(ii) By looking at the entries (1,1) and (2,2) of (2.29), one gets $\lambda^2 + (\alpha+\beta+\gamma+\delta)a_1 = 2\lambda a_1 + a_3 + (\gamma+\delta)\lambda$ and $\lambda^2 + (\alpha+\beta+\gamma+\delta)a_4 = 2\lambda a_4 + a_3 + (\gamma+\delta)\lambda$. By subtracting these last two equalities, we have $(\alpha+\beta+\gamma+\delta)(a_1-a_4) = 2\lambda(a_1-a_4)$. From $a_1 \neq a_4$, we get $\alpha+\beta+\gamma+\delta = 2\lambda$.

This finishes the proof. □

Lemma 2.3. *Let $A, B, P, Q \in \mathbb{C}_n$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P;\alpha,\beta)$, $B \in \mathfrak{L}(Q;\gamma,\delta)$, $\alpha\beta P = \gamma\delta Q$, $AB \neq BA$, and $\alpha \neq \beta$, $\gamma \neq \delta$. If $A+B$ is a Jordan block corresponding to $\lambda \in \mathbb{C}$, then $\alpha+\beta+\gamma+\delta = 2\lambda$.*

Proof. We shall prove this lemma by induction on $n \geq 2$. First of all, we must note that $n = 1$ is not possible since $n = 1$ implies that the matrices A and B would be scalars, which would contradict $AB \neq BA$. The case $n = 2$ was proved in Lemma 2.2.

Assume that $n > 2$ and the lemma holds for complex $(n-1)\times(n-1)$ matrices. Let $X = A+B$. Since X is a Jordan block whose size is n corresponding to $\lambda \in \mathbb{C}$, we can write

$$X = \left[\begin{array}{c|cccccc} \lambda & 1 & 0 & \cdots & 0 & 0 \\ \hline 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{array}\right] = \begin{bmatrix} \lambda & \mathbf{u} \\ \mathbf{0} & J \end{bmatrix}, \qquad J \in \mathbb{C}_{n-1}. \tag{2.30}$$

Let us remark that J is a Jordan block corresponding to λ. Let us write A as follows:

$$A = \begin{bmatrix} a_1 & \mathbf{a}_2 \\ \mathbf{a}_3 & A_0 \end{bmatrix}, \qquad a_1 \in \mathbb{C},\ \mathbf{a}_2 \in \mathbb{C}_{1,n-1},\ \mathbf{a}_3 \in \mathbb{C}_{n-1,1},\ A_0 \in \mathbb{C}_{n-1}. \tag{2.31}$$

The equality (2.28), which can be used, leads to

$$\begin{bmatrix} \lambda^2 & \lambda\mathbf{u} + \mathbf{u}J \\ \mathbf{0} & J^2 \end{bmatrix} + (\alpha+\beta+\gamma+\delta)\begin{bmatrix} a_1 & \mathbf{a}_2 \\ \mathbf{a}_3 & A_0 \end{bmatrix}$$
$$= (\gamma+\delta)\begin{bmatrix} \lambda & \mathbf{u} \\ \mathbf{0} & J \end{bmatrix} + \begin{bmatrix} \lambda a_1 & a_1\mathbf{u} + \mathbf{a}_2 J \\ \lambda\mathbf{a}_3 & \mathbf{a}_3\mathbf{u} + A_0 J \end{bmatrix} + \begin{bmatrix} \lambda a_1 + \mathbf{u}\mathbf{a}_3 & \lambda\mathbf{a}_2 + \mathbf{u}A_0 \\ J\mathbf{a}_3 & JA_0 \end{bmatrix} \tag{2.32}$$

We obtain from the "south-west block" of (2.32) the equality $(\alpha+\beta+\gamma+\delta)\mathbf{a}_3 = \lambda\mathbf{a}_3 + J\mathbf{a}_3$. If $\mathbf{a}_3 \neq \mathbf{0}$, then $\alpha+\beta+\gamma+\delta-\lambda \in \sigma(J) = \{\lambda\}$. Therefore, $\alpha+\beta+\gamma+\delta = 2\lambda$.

Now, assume $\mathbf{a}_3 = \mathbf{0}$. If $\alpha\beta \neq 0$, then from $A^2 = (\alpha+\beta)A - \alpha\beta P$ we can write

$$P = \begin{bmatrix} p_1 & \mathbf{p}_2 \\ \mathbf{0} & P_0 \end{bmatrix}, \qquad p_1 \in \mathbb{C},\ \mathbf{p}_2 \in \mathbb{C}_{1,n-1},\ P_0 \in \mathbb{C}_{n-1}. \tag{2.33}$$

If $\alpha\beta = 0$, we can take $P = I_n$, and we can also write P as in (2.33). On account of $X = A+B$, (2.30), (2.31), $\mathbf{a}_3 = \mathbf{0}$, and $B^2 = (\gamma+\delta)B - \gamma\delta Q$, we can write

$$B = \begin{bmatrix} b_1 & \mathbf{b}_2 \\ \mathbf{0} & B_0 \end{bmatrix}, \qquad Q = \begin{bmatrix} q_1 & \mathbf{q}_2 \\ \mathbf{0} & Q_0 \end{bmatrix},$$

where $b_1, q_1 \in \mathbb{C}$, $\mathbf{b}_2, \mathbf{q}_2 \in \mathbb{C}_{1,n-1}$, and $B_0, Q_0 \in \mathbb{C}_{n-1}$. Hence A_0, B_0 satisfy all conditions of the lemma, except maybe $A_0B_0 \neq B_0A_0$.

If $A_0B_0 = B_0A_0$, since A_0 and B_0 are diagonalizable (because $\alpha \neq \beta$, $\gamma \neq \delta$), [4, Theorem 1.13.19] allows us to get that $A_0 + B_0$ is diagonalizable; but this is impossible because $J = A_0 + B_0$ is a Jordan block whose size is greater than 1. Therefore $A_0B_0 \neq B_0A_0$. Applying the induction hypothesis to A_0 and B_0, we get $\alpha + \beta + \gamma + \delta = 2\lambda$. □

Lemma 2.4. *Let $A, B, P, Q \in \mathbb{C}_n$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P; \alpha, \beta)$, $B \in \mathfrak{L}(Q; \gamma, \delta)$, $\alpha\beta P = \gamma\delta Q$, $AB \neq BA$, and $\alpha \neq \beta$, $\gamma \neq \delta$. If $A + B$ is a direct sum of Jordan blocks corresponding to $\lambda \in \mathbb{C}$, then $\alpha + \beta + \gamma + \delta = 2\lambda$.*

Proof. Write $X = A + B$ and $X = J_1 \oplus \cdots \oplus J_m$, where each $J_i \in \mathbb{C}_{k_i}$ is a Jordan block corresponding to λ. Write $A = [A_{ij}]_{i,j=1}^m$, where $A_{ii} \in \mathbb{C}_{k_i}$. The condition $AB \neq BA$ is equivalent to $AX \neq XA$. Two possibilities can happen:

(i) $A_{ij} = \mathbf{0}$ for any $i, j \in \{1, \dots, m\}$ with $i \neq j$. Since $A^2 = (\alpha + \beta)A - \alpha\beta P$, we get that P can be written as $P_1 \oplus \cdots \oplus P_m$, where $P_i \in \mathbb{C}_{k_i}$ (if $\alpha\beta = 0$, we can take $P = I_n$). By denoting $B_i = J_i - A_{ii}$ we get $B = B_1 \oplus \cdots \oplus B_m$. As we did with P, matrix Q can be written as $Q_1 \oplus \cdots \oplus Q_m$, where $Q_i \in \mathbb{C}_{k_i}$. Applying Lemma 2.3 for A_i and B_i, we get the conclusion of the lemma.

(ii) There exist $i, j \in \{1, \dots, m\}$ such that $i \neq j$ and $A_{ij} \neq \mathbf{0}$.

We can use (2.28). The block (i, j) of this latter equality gives

$$(\alpha + \beta + \gamma + \delta)A_{ij} = J_iA_{ij} + A_{ij}J_j. \tag{2.34}$$

Since $A_{ij} = [\mathbf{v}_1 \cdots \mathbf{v}_{k_j}] \neq \mathbf{0}$, let k be the least index such that $\mathbf{v}_k \neq \mathbf{0}$. The kth column of (2.34) allows us to get $(\alpha + \beta + \gamma + \delta)\mathbf{v}_k = J_i\mathbf{v}_k + \lambda\mathbf{v}_k$. Hence $\alpha + \beta + \gamma + \delta - \lambda \in \sigma(J_i) = \{\lambda\}$. Therefore, the conclusion of the lemma is obtained.

Both cases permit to prove the lemma. □

Theorem 2.6. *Let $A, B, P, Q \in \mathbb{C}_n$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P; \alpha, \beta)$, $B \in \mathfrak{L}(Q; \gamma, \delta)$, $\alpha\beta P = \gamma\delta Q$, $AB \neq BA$, and $\alpha \neq \beta$, $\gamma \neq \delta$. If $A + B$ is not diagonalizable, then there exist $\lambda, \mu \in \sigma(A + B)$ such that $\alpha + \beta + \gamma + \delta = \lambda + \mu$.*

Proof. Let us define $X = A + B$ and let $X = SJS^{-1}$ be the Jordan canonical form of X. Here, the matrix S is nonsingular and $J = J_1 \oplus \cdots \oplus J_m$, where each $J_i \in \mathbb{C}_{k_i}$ is a direct sum of Jordan blocks corresponding to $\lambda_i \in \mathbb{C}$ for $i = 1, \dots, m$. We assume that $\lambda_1, \dots, \lambda_m$ are pairwise distinct. Also, we define $K = S^{-1}AS$, $\widetilde{P} = S^{-1}PS$, $\widetilde{Q} = S^{-1}QS$, and let us decompose $K = (K_{ij})_{i,j=1}^m$, $\widetilde{P} = (P_{ij})_{i,j=1}^m$, and $\widetilde{Q} = (Q_{ij})_{i,j=1}^m$, where $K_{ii}, P_{ii}, Q_{ii} \in \mathbb{C}_{k_i}$ for $i = 1, \dots, m$. Observe that $AX - XA = A(A+B) - (A+B)A = AB - BA \neq \mathbf{0}$. Therefore, $KJ \neq JK$.

Now, two possibilities can occur: (i) There exist $i, j \in \{1, \dots, m\}$ such that $i \neq j$ and $K_{ij} \neq \mathbf{0}$. (ii) For any $i, j \in \{1, \dots, m\}$ with $i \neq j$ one has $K_{ij} = \mathbf{0}$.

(i) Obviously, the equality (2.28) can be used. By doing so, we get

$$J^2 + (\alpha + \beta + \gamma + \delta)K = JK + KJ + (\gamma + \delta)J. \tag{2.35}$$

By looking at the block (i,j) of (2.35) one obtains

$$(\alpha+\beta+\gamma+\delta)K_{ij} = J_i K_{ij} + K_{ij} J_j. \tag{2.36}$$

Let us write

$$J_j = \begin{bmatrix} \lambda_j & 1 & \cdots & 0 & 0 \\ 0 & \lambda_j & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_j & 1 \\ 0 & 0 & \cdots & 0 & \lambda_j \end{bmatrix}.$$

If $\mathbf{v}_k$ is the least non-zero column of K_{ij} (exists because $K_{ij} \neq \mathbf{0}$), then the kth column of (2.36) leads to $(\alpha+\beta+\gamma+\delta)\mathbf{v}_k = J_i\mathbf{v}_k + \lambda_j \mathbf{v}_k$. Thus $\alpha+\beta+\gamma+\delta-\lambda_j \in \sigma(J_i) = \{\lambda_i\}$. Therefore $\alpha+\beta+\gamma+\delta = \lambda_j + \lambda_i$.

(ii) In this situation, one has $K = K_{11} \oplus \cdots \oplus K_{mm}$. Since $KJ \neq JK$ there exists $i \in \{1,\dots,m\}$ such that $K_{ii}J_i \neq J_i K_{ii}$. Observe that $B = X - A = S(J-K)S^{-1}$, and thus, $P_{ij} = Q_{ij} = \mathbf{0}$ for any $i \neq j$. By applying Lemma 2.4 to matrices K_{ii} and $J_i - K_{ii}$, one finishes the proof of this case.

Both cases end the proof of the theorem. □

3 Applications and Examples

The purpose of this section is twofold: (i) To provide several examples that serves to show the wide applicability of the former results, (ii) To show how some scattered results in the known literature can be obtained from the established results of this paper.

Example 1: Corollary 2.1 can help someone to solve complex problems. For example, let us permit to point the next one: Let

$$P = \frac{1}{4}\begin{bmatrix} 1 & -9 & 6 \\ -1 & 1 & 2 \\ -1 & -3 & 6 \end{bmatrix}, \quad A = \frac{1}{2}\begin{bmatrix} 2 & -16 & 10 \\ 0 & -6 & 6 \\ 1 & -17 & 14 \end{bmatrix}, \quad B = \frac{1}{2}\begin{bmatrix} 1 & -8 & 5 \\ 1 & -10 & 7 \\ 2 & -19 & 13 \end{bmatrix}.$$

We can check $P^2 = P$, $(A-2P)(A-3P) = \mathbf{0}$, $AP = PA = A$, $(B-P)^2 = \mathbf{0}$, and $BP = PB = B$. The problem to solve is "find all complex numbers z such that $A + zB$ is idempotent".

Since $B \in \mathfrak{L}(P;1,1)$, then $zB \in \mathfrak{L}(P;z,z)$. Since $A+zB$ is idempotent, then $A+zB$ is diagonalizable and $\sigma(A+zB) \subset \{0,1\}$. Observe that all conditions of Corollary 2.1 hold. By Corollary 2.1, if $1 \notin \{0,2,3\} + \{0,z\}$, then $1+0 = 2+3+z+z$. But $\{0,2,3\}+\{0,z\} = \{0,2,3,z,2+z,3+z\}$. Hence we have only four possible cases to consider: $1 = z$, $1 = 2+z$, $1 = 3+z$, and $1 = 5+2z$. With these values of z, it is enough to check whether $A+zB$ is idempotent by a simple numerical computation.

Observe that if we substitute, for example, the condition $(A+zB)^2 = A+zB$ by $(A+zB)^6 = a+zB$, the exposed procedure, although a bit longer, permits finding such values of z, whereas the "brute force" procedure is unfeasible to perform.

Example 2: Let $A \in \mathfrak{L}(P;a,b)$ and $B \in \mathfrak{L}(P;c,d)$ with $a,b,c,d \in \mathbb{C}^*$. Also, the complex numbers x,y,z are given. We want to study the spectrum of $M = xA+yB+zP-AB-BA$.

First, we will find $\alpha,\beta,\gamma,\delta$ such that

$$x = \gamma+\delta, \quad c/d = \gamma/\delta, \quad y = \alpha+\beta, \quad a/b = \alpha/\beta \text{ with } a+b \neq 0 \neq c+d. \tag{3.1}$$

Hence, we have

$$M = (\gamma + \delta)A + (\alpha + \beta)B + zP - AB - BA.$$

On the other hand, the solution of (3.1) is

$$\alpha = \frac{ay}{a+b},\ \beta = \frac{by}{a+b},\ \gamma = \frac{cx}{c+d},\ \delta = \frac{dx}{c+d}.$$

Let

$$N = M - zP + (\alpha\beta + \gamma\delta)P = (\gamma + \delta)A + (\alpha + \beta)B + (\alpha\beta + \gamma\delta)P - AB - BA.$$

We have $MP = PM = M$ (because $AP = PA = A$ and $BP = PB = B$). We can write $P = S(I_r \oplus \mathbf{0})\mathbf{S^{-1}}$. From $MP = PM = M$, we can deduce that $M = S(X \oplus \mathbf{0})\mathbf{S^{-1}}$ and $N = S(X + (\alpha\beta + \gamma\delta - z)I_r \oplus \mathbf{0})\mathbf{S^{-1}}$. Hence we can make an easy relation between $\sigma(M)$ and $\sigma(N)$. But we can study $\sigma(N)$ via Theorem 2.3 because $A \in \mathfrak{L}(P; a, b)$ implies $\frac{y}{a+b}A \in \mathfrak{L}(P; a\frac{y}{a+b}, b\frac{y}{a+b}) = \mathfrak{L}(P; \alpha, \beta)$. Similarly, $\frac{x}{c+d}B \in \mathfrak{L}(P; \gamma, \delta)$. Hence, by finding the eigenvalues of $\frac{y}{a+b}A + \frac{x}{c+d}B$, we can find the eigenvalues of M for arbitrary $z \in \mathbb{C}$.

Now, let us give two more examples which illustrate the situations (i) and (ii) in Remark 2.1 and satisfy the conditions of Corollary 2.1 and Theorems 2.3, 2.4, 2.5.

Example 3:

$$\alpha = 1, \quad \beta = -1, \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

and

$$\gamma = -3, \quad \delta = -2, \quad B = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & -1 & -2 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Example 4:

$$\alpha = 0, \quad \beta = -1, \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 5 & -6 & -2 & 1 \\ 5 & -6 & -2 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & 4 & 2 & -1 \\ -6 & 7 & 2 & -1 \end{bmatrix},$$

$$\gamma = -3, \quad \delta = -2, \quad B = \begin{bmatrix} 9 & \frac{28}{3} & \frac{7}{3} & -2 \\ -6 & 6 & 2 & -2 \\ 6 & -8 & -4 & 2 \\ 9 & -12 & -3 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 4 & -4 & -1 & 1 \\ 3 & -3 & -1 & 1 \\ -3 & 4 & 2 & -1 \\ -3 & 4 & 1 & 0 \end{bmatrix}.$$

Notice that we have $\alpha\beta \neq 0$, $\gamma\delta \neq 0$, $PB = BP$, and $QA = AQ$ in Example 3. On the other hand, observe that $\alpha\beta = 0 \neq \gamma\delta$, $PB \neq BP$, and $QA = AQ$ in Example 4. So, Example 3 and Example 4 explains the situation in the part (i) and (ii) of Remark 2.1, respectively.

Lemma 2.1 permits to deal with many situations when A and B satisfy the hypotheses of this lemma. Some of these situations are the following.

Theorem 3.1. *Let $A, B, P, Q \in \mathbb{C}_n$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that $A \in \mathfrak{L}(P; \alpha, \beta)$, $B \in \mathfrak{L}(Q; \gamma, \delta)$, $(\alpha\beta P - \gamma\delta Q)(A + B) = (A + B)(\alpha\beta P - \gamma\delta Q)$, and $AB \neq BA$.*

(i) *If $A+B$ is an idempotent matrix, then $\alpha+\beta+\gamma+\delta=1$.*

(ii) *If $A+B$ is an involutive matrix (i.e., $(A+B)^2=I_n$), then $\alpha+\beta+\gamma+\delta=0$.*

(iii) *If $A+B$ is a tripotent matrix (i.e., $(A+B)^3=A+B$), then $\alpha+\beta+\gamma+\delta\in\{0,-1,1\}$.*

Proof. **(i)** Since $A+B$ is idempotent, this matrix is diagonalizable and $\sigma(A+B)\subset\{0,1\}$. Let us write the matrices A and B as in Lemma 2.1. Since $AB\neq BA$, we get $k\geq 1$. Thus, there exist $\mu,\nu\in\sigma(A_1+B_1)$ such that $\mu\neq\nu$ and $\mu+\nu=\alpha+\beta+\gamma+\delta$. From $\sigma(A_1+B_1)\subset\sigma(A+B)$, we get $\{\mu,\nu\}=\{0,1\}$. Hence $\alpha+\beta+\gamma+\delta=1$.

(ii) Since $A+B$ is involutive, this matrix is diagonalizable and $\sigma(A+B)\subset\{-1,1\}$. The proof follows as in the previous item.

(iii) Since $A+B$ is tripotent, this matrix is diagonalizable and $\sigma(A+B)\subset\{0,-1,1\}$. Let us write matrices A and B as in Lemma 2.1. Since $AB\neq BA$, we get $k\geq 1$. Thus, there exist $\mu,\nu\in\sigma(A_1+B_1)$ such that $\mu\neq\nu$ and $\mu+\nu=\alpha+\beta+\gamma+\delta$. From $\sigma(A_1+B_1)\subset\sigma(A+B)$, we have three possibilities $\{\mu,\nu\}=\{0,1\}$ or $\{\mu,\nu\}=\{0,-1\}$, or $\{\mu,\nu\}=\{-1,1\}$. Hence $\alpha+\beta+\gamma+\delta\in\{1,-1,0\}$. □

Theorem 3.2. *Let $A,B,P,Q\in\mathbb{C}_n$ and $\alpha,\beta,\gamma,\delta\in\mathbb{C}$ such that $A\in\mathfrak{L}(P;\alpha,\beta)$, $B\in\mathfrak{L}(Q;\gamma,\delta)$, $(\alpha\beta P-\gamma\delta Q)(A+B)=(A+B)(\alpha\beta P-\gamma\delta Q)$, and $AB\neq BA$.*

(i) $(A+B)^2=A+B\iff\alpha+\beta+\gamma+\delta=1,\quad AB+BA-\alpha\beta P-\gamma\delta Q=(\alpha+\beta)B+(\gamma+\delta)A.$

(ii) $(A+B)^2=I_n\iff\alpha+\beta+\gamma+\delta=0,\quad AB+BA-\alpha\beta P-\gamma\delta Q=I_n-(\alpha+\beta)(A-B).$

(iii) $(A+B)^3=A+B\iff$ *The matrices A, B, P, and Q satisfy one of the following conditions:*

(a) $\alpha+\beta+\gamma+\delta=0$ *and* $(1-(\alpha+\beta)^2)(A+B)=$
$-2\gamma\delta QA-2\alpha\beta BP-\alpha\beta A-\gamma\delta B-(\alpha+\beta)\alpha\beta P-(\gamma+\delta)\gamma\delta Q+ABA+BAB.$

(b) $\alpha+\beta+\gamma+\delta=1$ *and* $(1-(\alpha+\beta)^2+\alpha\beta)A+(1-(\gamma+\delta)^2+\gamma\delta)B=$
$AB+BA+-2\gamma\delta QA-2\alpha\beta BP-(\alpha+\beta)\alpha\beta P-(\gamma+\delta)\gamma\delta Q+ABA+BAB.$

(c) $\alpha+\beta+\gamma+\delta=-1$ *and* $(1-(\alpha+\beta)^2+\alpha\beta)A+(1-(\gamma+\delta)^2+\gamma\delta)B=$
$-2\gamma\delta QA-2\alpha\beta BP-AB-BA-(\alpha+\beta)\alpha\beta P-(\gamma+\delta)\gamma\delta Q+ABA+BAB.$

Proof. **(i)** By expanding $(A+B)^2=A+B$, we have

$$\begin{aligned}(A+B)^2=A+B\quad&\Leftrightarrow\quad A^2+B^2+AB+BA=A+B\\&\Leftrightarrow(\alpha+\beta)A-\alpha\beta P+(\gamma+\delta)B-\gamma\delta Q+AB+BA=A+B\end{aligned}$$

If we employ the condition $\alpha+\beta+\gamma+\delta=1$, then we get

$$\begin{aligned}&(\alpha+\beta)A-\alpha\beta P+(\gamma+\delta)B-\gamma\delta Q+AB+BA=A+B\\&\iff AB+BA-\alpha\beta P-\gamma\delta Q=(\alpha+\beta)B+(\gamma+\delta)A.\end{aligned}$$

So, the proof of (i) is complete.

(ii) Since $(A+B)^2 = I_n$, we can write

$$\begin{aligned}(A+B)^2 = I_n \Leftrightarrow \quad & A^2 + B^2 + AB + BA = I_n \\ \Leftrightarrow & (\alpha+\beta)A - \alpha\beta P + (\gamma+\delta)B - \gamma\delta Q + AB + BA = I_n\end{aligned}$$

From this, if we insert the condition $\alpha+\beta+\gamma+\delta = 0$, then we get that the equality

$$(\alpha+\beta)A - \alpha\beta P + (\gamma+\delta)B - \gamma\delta Q + AB + BA = I_n$$

holds if and only if the equality

$$AB + BA - \alpha\beta P - \gamma\delta Q = I_n - (\alpha+\beta)(A-B)$$

holds. So, it is obtained the desired result in (ii).

(iii) The equality $(A+B)^3 = A+B$ can be written as

$$A^3 + B^3 + ABA + BAB + AB^2 + BA^2 + B^2A + A^2B = A+B.$$

If we insert $A^2 = (\alpha+\beta)A - \alpha\beta P$, $B^2 = (\gamma+\delta)B - \gamma\delta Q$, then this last equality turns to

$$\begin{aligned}&(\alpha+\beta+\gamma+\delta)(AB+BA) - \alpha\beta(BP+PB+A) - \gamma\delta(QA+AQ+B) \\ &-(\alpha+\beta)\alpha\beta P - (\gamma+\delta)\gamma\delta Q + ABA + BAB = (1-(\alpha+\beta)^2)A + (1-(\gamma+\delta)^2)B.\end{aligned} \tag{3.2}$$

If $\alpha+\beta+\gamma+\delta = 0$, $\alpha+\beta+\gamma+\delta = 1$, and $\alpha+\beta+\gamma+\delta = -1$, respectively, the equality (3.2) is equivalent to the equalities

$$\begin{aligned}&(1-(\alpha+\beta)^2)(A+B) = \\ &-\alpha\beta(BP+PB+A) - \gamma\delta(QA+AQ+B) - (\alpha+\beta)\alpha\beta P - (\gamma+\delta)\gamma\delta Q + ABA + BAB,\end{aligned}$$

$$\begin{aligned}&AB + BA - \alpha\beta(PB+BP) - \gamma\delta(QA+AQ) - (\alpha+\beta)\alpha\beta P \\ &-(\gamma+\delta)\gamma\delta Q + ABA + BAB = (1-(\alpha+\beta)^2+\alpha\beta)A + (1-(\gamma+\delta)^2+\gamma\delta)B,\end{aligned}$$

and

$$\begin{aligned}&-\alpha\beta(PB+BP) - \gamma\delta(QA+AQ) - AB - BA - (\alpha+\beta)\alpha\beta P \\ &-(\gamma+\delta)\gamma\delta Q + ABA + BAB = (1-(\alpha+\beta)^2+\alpha\beta)A + (1-(\gamma+\delta)^2+\gamma\delta)B.\end{aligned}$$

Also, from the hypothesis of theorem, we have $\alpha\beta(PB-BP) = \gamma\delta(QA-AQ)$, and therefore $-\alpha\beta(PB+BP) - \gamma\delta(QA+AQ) = -2\gamma\delta QA - 2\alpha\beta BP$. So, the proof is completed. □

In [8], Sarduvan and Özdemir proved that for $c_1, c_2 \in \mathbb{C}^*$ and idempotent matrices $T_1, T_2 \in \mathbb{C}_n$, under the assumption $T_1T_2 \neq T_2T_1$, the matrix $c_1T_1 + c_2T_2$ is involutive if $c_1 + c_2 = 0$ and $\frac{1}{c_1^2}I_n + T_1T_2 + T_2T_1 = T_1 + T_2$.

Note that this result can be obtained from Theorem 3.2 (ii) by taking $A = c_1T_1$, $B = c_2T_2$, $\alpha = c_1$, $\beta = 0$, $\gamma = c_2$, $\delta = 0$, and $P = Q = I_n$. Observe that it is also true the converse of this result.

Again in [8], Sarduvan and Özdemir established that for $c_1, c_2 \in \mathbb{C}^*$ and involutive matrices $R_1, R_2 \in \mathbb{C}_n$, under the condition $R_1R_2 \neq R_2R_1$, the matrix $c_1R_1 + c_2R_2$ is involutive if $R_1R_2 + R_2R_1 = \frac{1-(c_1^2+c_2^2)}{c_1c_2}I_n$.

Observe that this result is trivially obtained from Theorem 3.2 (ii) by taking $A = c_1R_1$, $B = c_2R_2$, $\alpha = c_1$, $\beta = -c_1$, $\gamma = c_2$, $\delta = -c_2$, and $P = Q = I_n$. Also, it is seen that it is also true the converse of this result.

Example 5: In Theorem 2.6, the eigenvalues λ and μ may be equal. As an example of the applicability of this theorem, we shall study the following situation. Let $X, Y \in \mathbb{C}_n$ be two noncommuting idempotents. We want to find all nonzero complex numbers a, b such that $aX + bY$ is nilpotent (i.e., there exists $k \in \mathbb{N}$ such that $(aX + bY)^k = \mathbf{0}$). If we define $A = aX$, $B = bY$, $P = Q = I_n$, $\alpha = a$, $\beta = 0$, $\gamma = b$, and $\delta = 0$, then all conditions of Theorem 2.6 are satisfied. Furthermore, since $\sigma(A+B) = \{0\}$ (because $A+B$ is nilpotent), we have that $a + b = 0$. Hence $aX + bY$ is nilpotent if and only if $a + b = 0$ and $X - Y$ is nilpotent.

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