# THE DIRICHLET-BOHR RADIUS

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ABSTRACT. Denote by $\Omega(n)$ the number of prime divisors of $n \in \mathbb{N}$ (counted with multiplicities). For $x \in \mathbb{N}$ define the Dirichlet-Bohr radius $L(x)$ to be the best $r > 0$ such that for every finite Dirichlet polynomial $\sum_{n \leq x} a_n n^{-s}$ we have

$$\sum_{n \leq x} |a_n| r^{\Omega(n)} \leq \sup_{t \in \mathbb{R}} \Big| \sum_{n \leq x} a_n n^{-it} \Big| .$$

We prove that the asymptotically correct order of $L(x)$ is $(\log x)^{1/4} x^{-1/8}$. Following Bohr's vision our proof links the estimation of $L(x)$ with classical Bohr radii for holomorphic functions in several variables. Moreover, we suggest a general setting which allows to translate various results on Bohr radii in a systematic way into results on Dirichlet-Bohr radii, and vice versa.

## 1. INTRODUCTION

The study of problems on absolute convergence of Dirichlet series (of the form $\sum_n a_n n^{-s}$, where $s$ is a complex variable) led H. Bohr to relate properties on absolute convergence with properties of boundedness (on the right half-plane) of the holomorphic function defined by the Dirichlet series. One of his first results in this direction is the following inequality [6, Satz XIII]: for every Dirichlet series of the form $\sum_{p \text{ prime}} a_p p^{-s}$ we have

$$\sum_{p \text{ prime}} |a_p| \leq \sup_{\operatorname{Re} s > 0} \Big| \sum_{p \text{ prime}} a_p p^{-s} \Big| . \tag{1.1}$$

In his research [6, 7] he then established a close relationship between Dirichlet series and power series in infinitely many variables (this relationship was presented in a modern, systematic way much later by Hedenmalm, Lindqvist and Seip [14]). Bohr then looked at holomorphic functions and proved his well known power series theorem [8]: for every holomorphic function $f$ on the open unit disc $\mathbb{D}$ we have

$$\sum_n \Big| \frac{f^{(n)}(0)}{n!} \Big| \frac{1}{3^n} \leq \|f\|_\infty , \tag{1.2}$$

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and that here moreover the number 1/3 is optimal. As a simple consequence of the maximum modulus principle, it can be seen that for each Dirichlet series $\sum_n a_{2^n} 2^{-ns}$ we have

$$\sup_{z \in \mathbb{D}} \Big| \sum_n a_{2^n} z^n \Big| = \sup_{\operatorname{Re} s > 0} \Big| \sum_n a_{2^n} 2^{-ns} \Big| .$$

Hence (1.2) can be reformulated as follows:

$$\sum_n \Big| a_{2^n} \frac{1}{3^n} \Big| \le \sup_{\operatorname{Re} s > 0} \Big| \sum_n a_{2^n} 2^{-ns} \Big| , \tag{1.3}$$

for every Dirichlet series $\sum_n a_{2^n} 2^{-ns}$.

The work of Dineen and Timoney [13] renewed the interest on Bohr's theorem and Boas and Khavinson [5] defined the $n$-dimensional Bohr radius $K_n$ to be the best $0 < r < 1$ such that

$$\sum_{\alpha \in \mathbb{N}_0^n} \Big| \frac{\partial^\alpha f(0)}{\alpha!} \Big| r^{|\alpha|} \le \sup_{z \in \mathbb{D}^n} \Big| \sum_{\alpha \in \mathbb{N}_0^n} \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha \Big| ,$$

for every bounded, holomorphic function $f$ on $\mathbb{D}^n$. That was the starting point of a long search on the optimal asymptotic behaviour of $K_n$ as $n$ grows that was finally closed in [10] and [4] (see Section 3 for more details).

Because of the link between Dirichlet series and power series, each result in either framework has an immediate translation into the other. This is of course the case with the behaviour of $K_n$ (a fact which is stated in more detail in Example 3.6). But, as it happens, what is natural in one side may not be as natural in the other; and while taking $n$ variables (or, equivalently, $n$-dimensional spaces) is natural in the side of holomorphic functions, in the side of Dirichlet series we would rather take finite sums of (the first) $n$ terms. So, inspired by the Bohr radius for holomorphic functions, our main aim in this note is to determine, for each $x \ge 2$, the best $r = r(x) \ge 0$ such that for every finite Dirichlet polynomial $\sum_{n \le x} a_n n^{-s}$ of length $x$

$$\sum_{n \le x} |a_n| r^{\Omega(n)} \le \sup_{\operatorname{Re} s > 0} \Big| \sum_{n \le x} a_n n^{-s} \Big| ,$$

where $\Omega(n)$ denotes the number of prime divisors of $n \in \mathbb{N}$ (counted with multiplicities). We do this in our main result Theorem 2.1, that gives the asymptotically correct order of this best radius.

We then take a general point of view and, for a given subset $J$ of $\mathbb{N}$, we define the *Dirichlet-Bohr radius* $L(J)$ of $J$ to be the best $r = r(J) \ge 0$ such that for every Dirichlet series $\sum_{n \in J} a_n n^{-s}$ convergent on the open half-plane

[$\operatorname{Re} s > 0$], we have

$$\sum_{n\in J} |a_n| r^{\Omega(n)} \le \sup_{\operatorname{Re} s>0} \Big| \sum_{n\in J} a_n n^{-s} \Big| . \tag{1.4}$$

With this, denoting by $P$ the set of prime numbers, (1.1) and (1.3) can be rephrased as

$$L(P) = 1 \qquad \text{and} \qquad L\big(\{2^k \mid k \in \mathbb{N}\}\big) = \frac{1}{3} . \tag{1.5}$$

Then, Theorem 2.1 gives the correct asymptotic order of $L(\{n \in \mathbb{N} \,|1 \le n \le x\})$. We will see that, following an idea of H. Bohr based on Diophantine approximation, this study can be extended to other sets $J$ of indices.

Finally, we mention another estimate which seems of relevance when motivating our results: For every $\varepsilon > 0$ there is $C = C(\varepsilon) \ge 1$ such that for every $x$ and finite Dirichlet polynomial $\sum_{n\le x} a_n n^{-s}$

$$\sum_{n\le x} |a_n| \frac{e^{\left(\frac{1}{\sqrt{2}}-\varepsilon\right)\sqrt{\log n \log\log n}}}{n^{1/2}} \le C \sup_{\operatorname{Re} s>0} \Big| \sum_{n\le x} a_n n^{-s} \Big| . \tag{1.6}$$

This result is under several different aspects optimal, and it is the final outcome of a long series of results due to [2, 9, 10, 15, 17, 18]. Our main result, Theorem 2.1, can be considered to be a relative of (1.6).

1.1. **Notations.** As we have already mentioned, $\Omega(n)$ denotes, for $n \in \mathbb{N}$, the number of prime divisors of $n$, counted with their multiplicity. We denote by $(p_n)_n$ the sequence of prime numbers. The set of multiindices $\alpha$ that eventually become 0 is denoted by $\mathbb{N}_0^{(\mathbb{N})}$. For $\alpha = (\alpha_1, \dots, \alpha_k, 0, \dots)$ we write $p^\alpha = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_k$.
Along this note $\pi$ denotes the prime counting function, i.e., $\pi(x)$ is the number of prime numbers less than or equal to $x$.
Given two real functions $f$ and $g$ we write $f(x) \ll g(x)$ if there exists a constant $C > 0$ such that $f(x) \le Cg(x)$ for every $x$. If $f(x) \ll g(x)$ and $g(x) \ll f(x)$ we write $f(x) \approx g(x)$.
For each $N$ we denote by $H_\infty(\mathbb{D}^N)$ the space of bounded, holomorphic functions on $\mathbb{D}^N$. If $f \in H_\infty(\mathbb{D}^N)$ and $\alpha \in \mathbb{N}_0^N$ we write $c_\alpha(f) = \frac{\partial^\alpha f(0)}{\alpha!}$, the $\alpha$-th coefficient of the monomial expansion.

## 2. Main result

For any $x \ge 2$, we write

$$L(x) = L\big(\{n \in \mathbb{N} \,\big|\, 1 \le n \le x\}\big) ,$$

where $L$ is defined in (1.4), and call this number the $x$-th Dirichlet-Bohr radius. The main result of this note then reads as follows.

**Theorem 2.1.** *We have*

$$L(x) \approx \frac{\sqrt[4]{\log x}}{x^{1/8}}\,.$$

*In particular, there is a universal constant $C > 0$ such that*

$$\sum_{n \leq x} |a_n| \left( \frac{C\sqrt[4]{\log n}}{n^{1/8}} \right)^{\Omega(n)} \leq \sup_{\operatorname{Re} s > 0} \Big| \sum_{n \leq x} a_n n^{-s} \Big|$$

*for every $x \geq 2$ and every finite Dirichlet polynomial $\sum_{n \leq x} a_n n^{-s}$.*

The rest of this section is devoted to the proof of this result.

2.1. **Reduction I.** We start with a device which reduces the estimation of Dirichlet-Bohr radii $L(x)$ to the estimation of their *homogeneous parts* $L_m(x)$ which we are going to define now. For $x \geq 2$ define the finite dimensional Banach space

$$\mathcal{H}_\infty^{(x)} := \Big\{ D = \sum_{n=1}^{\infty} a_n n^{-s} \;\Big|\; a_n \neq 0 \quad \text{only if} \quad n \leq x \Big\}$$

$$\|D\|_\infty = \sup_{t \in \mathbb{R}} \Big| \sum_{n \leq x} a_n \frac{1}{n^{it}} \Big| = \sup_{\operatorname{Re} s > 0} \Big| \sum_{n \leq x} a_n \frac{1}{n^s} \Big|$$

together with its closed subspace

$$\mathcal{H}_\infty^{(x,m)} := \Big\{ \sum_{n=1}^{\infty} a_n n^{-s} \;\Big|\; a_n \neq 0 \quad \text{only if} \quad n \leq x \text{ and } \Omega(n) = m \Big\}\,.$$

Then

$$L(x) = \sup \Big\{ 0 \leq r \leq 1 \,\Big|\, \forall D \in \mathcal{H}_\infty^{(x)} : \sum_{n \leq x} |a_n| r^{\Omega(n)} \leq \|D\|_\infty \Big\}\,,$$

and therefore for $m \in \mathbb{N}$ we define the $m$-homogeneous $x$-th Dirichlet-Bohr radius by

$$L_m(x) := \sup \Big\{ 0 \leq r \leq 1 \;\Big|\; \forall D \in \mathcal{H}_\infty^{(x,m)} : \sum_{n \leq x} |a_n| \leq r^{-m} \|D\|_\infty \Big\}\,. \tag{2.1}$$

The following result is the announced *reduction theorem.*

**Proposition 2.2.** *With the previous notation, we have*

$$\frac{1}{3} \inf_m L_m(x) \;\leq\; L(x) \leq \;\inf_m L_m(x) \quad \textit{for all} \quad x \geq 2\,.$$

We start with a reformulation in terms of holomorphic functions. Note that if $n = p^\alpha$ and $1 \leq n \leq x$ then clearly $\alpha$ has at most the first $\pi(x)$ coordinates different from zero; in other words $\alpha \in \mathbb{N}_0^{\pi(x)}$. Then, by Bohr's fundamental lemma (see [18]) we know that for every finite Dirichlet polynomial $\sum_{n \leq x} a_n n^{-s}$ we have

$$\sup_{t \in \mathbb{R}} \Big| \sum_{n \leq x} a_n n^{-it} \Big| = \sup_{z \in \mathbb{D}^{\pi(x)}} \Big| \sum_{\substack{\alpha \in \mathbb{N}_0^{\pi(x)} \\ 1 \leq p^\alpha \leq x}} a_{p^\alpha} z^\alpha \Big| . \tag{2.2}$$

With this identity in mind we define the Banach space

$$H_\infty^{(x)} := \Big\{ f \in H_\infty(\mathbb{D}^{\pi(x)}) \;\Big|\; c_\alpha(f) \neq 0 \quad \text{only if} \quad p^\alpha \leq x \Big\} ,$$

(the norm clearly given by the right side of (2.2)) and its closed subspace

$$H_\infty^{(x,m)} := \Big\{ f \in H_\infty(\mathbb{D}^{\pi(x)}) \;\Big|\; c_\alpha(f) \neq 0 \quad \text{only if} \quad p^\alpha \leq x \text{ and } |\alpha| = m \Big\} .$$

Identifying Dirichlet series $\sum_{n \leq x} a_n n^{-s}$ with functions $\sum_{\substack{\alpha \in \mathbb{N}_0^{\pi(x)} \\ 1 \leq p^\alpha \leq x}} a_{p^\alpha} z^\alpha$ we then obtain the following isometric equalities

$$\mathcal{H}_\infty^{(x)} = H_\infty^{(x)} \quad \text{and} \quad \mathcal{H}_\infty^{(x,m)} = H_\infty^{(x,m)} ,$$

and this in turn shows that

$$L(x) = \sup \Big\{ 0 \leq r \leq 1 \;\Big|\; \forall f \in H_\infty^{(x)} : \sum_{\substack{\alpha \in \mathbb{N}_0^{\pi(x)} \\ 1 \leq p^\alpha \leq x}} \big| c_\alpha(f) \big| r^{|\alpha|} \leq \|f\|_\infty \Big\} , \tag{2.3}$$

and

$$L_m(x) = \sup \Big\{ 0 \leq r \leq 1 \;\Big|\; \forall f \in H_\infty^{(x,m)} : \sum_{\substack{1 \leq p^\alpha \leq x \\ |\alpha| = m}} \big| c_\alpha(f) \big| \leq r^{-m} \|f\|_\infty \Big\} . \tag{2.4}$$

*Proof of Proposition 2.2.* The proof of the upper estimate is obvious, and for the proof of the lower estimate we follow [11, Section 2]. Fix $f \in H_\infty^{(x)}$ with $\|f\|_\infty \leq 1$, and write for its $m$-homogeneous part

$$f_m(\omega) = \sum_{\substack{1 \leq p^\alpha \leq x \\ |\alpha| = m}} c_\alpha(f) \omega^\alpha , \; \omega \in \mathbb{D}^{\pi(x)} ;$$

obviously, $f_m \in H_\infty^{(x,m)}$ and using Cauchy inequalities we see that $\|f_m\|_\infty \leq 1$ for all $m$. We fix now some $z_0 \in \mathbb{D}^{\pi(x)}$ and $\theta \in \mathbb{T}$ such that $|c_0(f)| = \theta c_0(f)$, and define

$$g : \mathbb{D} \to \mathbb{C} , \; g(\omega) := f(\omega z_0) = \sum_{m=1}^{\infty} f_m(z_0) \omega^m ,$$

$$h : \mathbb{D} \to \mathbb{C} , \; h := 1 - \theta g .$$

Since $\|g\|_\infty \le 1$, we have that $\operatorname{Re} h \ge 0$ on $\mathbb{D}$, and by Caratheodory's theorem (for an elementary proof, see [1, Lemma 1.1]) we have for all $m$

$$\big|f_m(z_0)\big| = \frac{h^{(m)}(0)}{m!} \le 2\operatorname{Re} h(0) = 2(1-|c_0(f)|)\,. \tag{2.5}$$

We take now some $r < \inf_m L_m(x)$. Then for all $z \in \mathbb{D}^{\pi(x)}$ and all $m$ we have by (2.4) and (2.5)

$$\sum_{\substack{1\le p^\alpha \le x\\ |\alpha|=m}} \big|c_\alpha(f)(\tfrac{r}{3}z)^\alpha\big| \le \frac{1}{3^m}\|f_m\|_\infty \le \frac{1}{3^m}2(1-|c_0(f)|)\,,$$

and hence for all $z \in \frac{r}{3}\mathbb{D}^{\pi(x)}$

$$\sum_{1\le p^\alpha \le x} \big|c_\alpha(f) z^\alpha\big| \le |c_0(f)| + \sum_{m=1}^{\infty} \frac{1}{3^m}2(1-|c_0(f)|) = 1\,.$$

The conclusion now follows from (2.3). □

2.2. **The tool.** The following proposition is our main tool – a reelaboration of a result due to Balasubramanian, Calado, and Queffélec [2, Theorem 1.4] (see also [12, Theorem 4.2]).

**Proposition 2.3.** *Let $m \ge 2$ and $\kappa > 1$. There exists $C(\kappa) > 0$ such that for every $m$-homogeneous Dirichlet polynomial $D = \sum_{n\le x} a_n n^{-s}$ in $\mathcal{H}_\infty^{(x,m)}$ we have*

$$\sum_{n\le x} |a_n| \frac{(\log n)^{\frac{m-1}{2}}}{n^{\frac{m-1}{2m}}} \le C(\kappa) m^{m-1} (2\kappa)^m \|D\|_\infty\,.$$

Our proof follows from a careful analysis of the original proof of [2], that allows us to obtain the constant $C(\kappa)m^{m-1}(2\kappa)^m$, smaller than the original one. Since this fact is essential for our purpose, we for the sake of completeness prefer to add the proof. Every $m$-homogeneous polynomial in $n$ variables admits two possible representations:

$$P(z) = \sum_{\substack{\alpha\in\mathbb{N}^n\\ |\alpha|=m}} c_\alpha z^\alpha = \sum_{1\le j_1\le\cdots\le j_m\le n} c_{j_1,\ldots,j_m} z_{j_1}\cdot\ldots\cdot z_{j_m}\,, \quad \text{for } z\in\mathbb{C}^n.$$

We need the following lemma [10, page 492] (see also [12, Lemma 4.3] or [3, Lemma 2.6]).

**Lemma 2.4.** *Let $n \ge 1$, $m \ge 1$ and $\kappa > 1$. Then there exists $C(\kappa) > 0$ such that, for every $m$-homogeneous polynomial $P$ on $\mathbb{C}^n$ we have*

$$\sum_{j_m=1}^{n} \Big( \sum_{1\le j_1\le\cdots\le j_m} |c_{j_1,\ldots,j_m}|^2 \Big)^{\frac{1}{2}} \le C(\kappa)\big(2\kappa\big)^m \sup\{|P(z)| : z\in\mathbb{D}^n\}\,.$$

*Proof of Proposition 2.3 .* We begin by fixing some finite Dirichlet polynomial

$$D = \sum_{n \leq x} a_n n^{-s} \in \mathcal{H}_\infty^{(x,m)}.$$

Now we define the following $m$-homogeneous polynomial in $\pi(x)$ variables

$$P(z) = \sum_{1 \leq j_1 \leq \cdots \leq j_m \leq \pi(x)} c_{\mathbf{i}} z_{j_1} \cdot \ldots \cdot z_{j_m}\,,\ z \in \mathbb{C}^{\pi(x)}\,,$$

where $c_{j_1 \cdots j_m} = a_n$ for $1 \leq n = p_{j_1} \cdots p_{j_m} \leq x$ and 0 otherwise. Then

$$\begin{aligned}
\sum_{n \leq x} |a_n| \frac{(\log n)^{\frac{m-1}{2}}}{n^{\frac{m-1}{2m}}} &= \sum_{1 \leq j_1 \leq \cdots \leq j_m \leq \pi(x)} |c_{j_1,\ldots,j_m}| \frac{\big(\log(p_{j_1} \cdots p_{j_m})\big)^{\frac{m-1}{2}}}{(p_{j_1} \cdots p_{j_m})^{\frac{m-1}{2m}}} \\
&\leq \sum_{j_m=1}^{\pi(x)} \frac{(m \log p_{j_m})^{\frac{m-1}{2}}}{p_{j_m}^{\frac{m-1}{2m}}} \sum_{1 \leq j_1 \leq \ldots \leq j_{m-1} \leq j_m} \frac{|c_{j_1,\ldots,j_m}|}{(p_{j_1} \cdots p_{j_{m-1}})^{\frac{m-1}{2m}}} \\
&\leq \sum_{j_m=1}^{\pi(x)} \frac{(m \log p_{j_m})^{\frac{m-1}{2}}}{p_{j_m}^{\frac{m-1}{2m}}} \Bigg( \sum_{1 \leq j_1 \leq \ldots \leq j_{m-1} \leq j_m} |c_{j_1,\ldots,j_m}|^2 \Bigg)^{\frac{1}{2}} \times \\
&\qquad\qquad \times \Bigg( \sum_{1 \leq j_1 \leq \ldots \leq j_{m-1} \leq j_m} \frac{1}{(p_{j_1} \cdots p_{j_{m-1}})^{\frac{m-1}{m}}} \Bigg)^{\frac{1}{2}},
\end{aligned}$$

where the last step follows from the Cauchy-Schwarz inequality. We use now the fact that for $0 < \alpha < 1$ (see [16, Satz 4.2, p. 22])

$$\sum_{p \leq x} p^{-\alpha} \ll \frac{1}{1-\alpha} \frac{x^{1-\alpha}}{\log x}$$

to bound the last factor. By taking $\alpha = \frac{m-1}{m}$,

$$\begin{aligned}
\Bigg( \sum_{1 \leq j_1 \leq \ldots \leq j_{m-1} \leq j_m} \frac{1}{(p_{j_1} \cdots p_{j_{m-1}})^{\frac{m-1}{m}}} \Bigg)^{\frac{1}{2}} &\leq \Bigg( \sum_{j \leq j_m} \Big( \frac{1}{p_j} \Big)^{\frac{m-1}{m}} \Bigg)^{\frac{m-1}{2}} \\
&\ll \Bigg( m \frac{p_{j_m}^{\frac{1}{m}}}{\log p_{j_m}} \Bigg)^{\frac{m-1}{2}}.
\end{aligned}$$

With this we have

$$\begin{aligned}
&\sum_{n \leq x} |a_n| \frac{(\log n)^{\frac{m-1}{2}}}{n^{\frac{m-1}{2m}}} \\
&\qquad \ll m^{m-1} \sum_{j_m=1}^{\pi(x)} \frac{(\log p_{j_m})^{\frac{m-1}{2}}}{p_{j_m}^{\frac{m-1}{2m}}} \Bigg( \frac{p_{j_m}^{\frac{1}{m}}}{\log p_{j_m}} \Bigg)^{\frac{m-1}{2}} \Bigg( \sum_{1 \leq j_1 \leq \ldots \leq j_{m-1} \leq j_m} |c_{\mathbf{j}}|^2 \Bigg)^{\frac{1}{2}}.
\end{aligned}$$

Finally, by Lemma 2.4 and (2.2), there exists $C(\kappa) > 0$ such that

$$\sum_{n \le x} |a_n| \frac{(\log n)^{\frac{m-1}{2}}}{n^{\frac{m-1}{2m}}} \le C(\kappa) m^{m-1} (2\kappa)^m \|P\| = C(\kappa) m^{m-1} (2\kappa)^m \|D\|_\infty .$$

□

### 2.3. **Proofs.**

*Proof of the lower estimate in Theorem 2.1.* We fix some $x \ge 2$. By Proposition 2.2 we only have to control each $m$-homogeneous part, $L_m(x)$. Note first that if $1 \le n \le x$ is such that $\Omega(n) = m$ we have that $2^m \le n \le x$, which gives $m \le \frac{\log x}{\log 2}$. Then $\mathcal{H}_\infty^{x,m} = \{0\}$, and hence $L_m(x) = 1$, for every $m > \frac{\log x}{\log 2}$. Thus

$$\frac{1}{3} \min_{1 \le m \le \frac{\log x}{\log 2}} L_m(x) \ \le\ L_x. \tag{2.6}$$

By (1.5) we have $L_1(x) = 1$ for every $x$. We fix then $m \ge 2$ and observe that, for every $D = \sum_{n \le x} a_n n^{-s} \in \mathcal{H}_\infty^{(x,m)}$ we have $a_1 = a_2 = a_3 = 0$. By Proposition 2.3, for each $\kappa > 1$ there exists $C(\kappa) > 0$ such that

$$\sum_{n \le x} |a_n| = \sum_{4 \le n \le x} |a_n| \le \sum_{4 \le n \le x} |a_n| (\log n)^{\frac{m-1}{2}} \le C(\kappa) m^{m-1} (2\kappa)^m x^{\frac{m-1}{2m}} \|D\|_\infty .$$

This, using (2.1), gives

$$m^{-1} x^{-\frac{m-1}{2m^2}} \ll \Big( C(\kappa) m^{m-1} (2\kappa)^m x^{\frac{m-1}{2m}} \Big)^{-1/m} \le L_m(x) .$$

But the sequence $\big( x^{-\frac{m-1}{2m^2}} \big)_{m=2}^\infty$ is increasing to 1 (recall that $x \ge 2$). This implies that for all $m \ge 3$

$$m^{-1} x^{-\frac{1}{9}} \ll L_m(x) ,$$

and hence for all $3 \le m \le \frac{\log x}{\log 2}$

$$\frac{\sqrt[4]{\log x}}{x^{\frac{1}{8}}} \ll \frac{\log 2}{\log x} \frac{1}{x^{\frac{1}{9}}} \ll L_m(x) . \tag{2.7}$$

We finish our argument by handling the case $m = 2$. We observe first that $f(t) = \frac{\sqrt{\log t}}{t^{\frac{1}{4}}} = e^{g(t)}$ with $g(t) = \frac{1}{2} \log \log t - \frac{1}{4} \log t$, $t \ge 2$. Since $g'(t) = \frac{1}{2t} \frac{2 - \log t}{2 \log t}$, we have that $f$ is strictly decreasing for $t > e^2$. Then the sequence $\big( \frac{\sqrt{\log n}}{n^{\frac{1}{4}}} \big)$ is strictly decreasing for $n \ge 8$. Thus there exists $A > 0$ such that for every $2 \le n \le x$ we have $\frac{\sqrt{\log x}}{x^{\frac{1}{4}}} \le A \frac{\sqrt{\log n}}{n^{\frac{1}{4}}}$. Applying again Proposition 2.3 we see that for every $D \in \mathcal{H}_\infty^{(x,2)}$

$$\frac{\sqrt{\log x}}{x^{\frac{1}{4}}} \sum_{n \le x} |a_n| \le A\, C(\kappa) 8 \kappa^2 \|D\|_\infty ,$$

and hence

$$\frac{\sqrt[4]{\log x}}{x^{\frac{1}{8}}} \ll L_2(x)\,.$$

This equation combined with (2.7) and (2.6) proves the lower estimate. □

*Proof of the upper estimate in Theorem 2.1.* By Proposition 2.2 it suffices to show that there is a constant $C > 0$ such that for all $x$

$$L_2(x) \leq C\frac{\sqrt[4]{\log x}}{x^{\frac{1}{8}}}\,. \tag{2.8}$$

According to (2.1), fix some $x$ and assume that $r > 0$ satisfies

$$\sum_{n\leq x} |a_n| \leq r^{-2} \sup_{t\in\mathbb{R}} \Big|\sum_{n\leq x} a_n n^{it}\Big| \tag{2.9}$$

for every Dirichlet polynomial $\sum_{n\leq x} a_n n^{-s} \in \mathcal{H}_\infty^{(x,2)}$. We choose $q$ to be the biggest natural number $\leq \frac{\pi(\sqrt{x})}{2}$. Consider the $q\times q$ matrix $(a_{nk})_{n,k}$ defined by $a_{nk} = e^{2\pi i\frac{nk}{q}}$ (sometimes called Fourier matrix). Then it is well known (and a straightforward calculation) that for all $n, k$ we have $|a_{nk}| = 1$ and $\sum_l a_{ln}\overline{a}_{lk} = q\delta_{nk}$.

We define the Dirichlet series

$$\sum_{n,k=1}^{q} a_{nk}\frac{1}{(p_n p_{q+k})^s} \in \mathcal{H}_\infty^{(x,2)}\,.$$

Note that for every $1 \leq n, k \leq q$ we have $p_n p_{q+k} \leq p_{2q}^2 \leq p_{\pi(\sqrt{x})}^2 \leq x$ and the Dirichlet series indeed belongs to $\mathcal{H}_\infty^{(x,2)}$. Obviously, we have

$$\sum_{n,k=1}^{q} |a_{nk}| = q^2\,.$$

On the other hand,

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} \Big|\sum_{n,k=1}^{q} a_{nk}p_n^{it}p_{q+k}^{it}\Big| \leq q^{1/2}\Big(\sum_k \Big|\sum_n a_{nk}p_n^{it}\Big|^2\Big)^{1/2}\\
&= q^{1/2}\Big(\sum_k \sum_{n_1,n_2} a_{kn_1}\overline{a_{kn_2}}p_{n_1}^{it}p_{n_2}^{-it}\Big)^{1/2} = q^{1/2}\Big(\sum_{n_1,n_2} p_{n_1}^{it}p_{n_2}^{-it}\sum_k a_{kn_1}\overline{a_{kn_2}}\Big)^{1/2}\\
&\qquad\qquad = q^{1/2}\Big(\sum_{n_1,n_2} p_{n_1}^{it}p_{n_2}^{-it}q\delta_{n_1,n_2}\Big)^{1/2} = q\Big(\sum_n |p_n^{it}|^2\Big)^{1/2} \leq q^{3/2}\,.
\end{aligned}$$

Then by (2.9) we conclude $q^2 \leq r^{-2}q^{\frac{3}{2}}$. But from the prime number theorem we deduce that there is a (universal) constant $C > 0$ such that $\frac{\sqrt{x}}{\log x} \leq Cq$, and therefore

$$r \leq C\frac{\sqrt[4]{\log x}}{x^{\frac{1}{8}}}\,.$$

Clearly, this gives the desired estimate (2.8). □

## 3. Dirichlet-Bohr radii

The main goal of the previous section was to find the correct asymptotic order of the Dirichlet-Bohr radius $L\big(\{n \in \mathbb{N} \,|\, 1 \leq n \leq x\}\big)$.

Analysing the ideas of our proof, we in the coming subsection show how to reduce the study of Dirichlet-Bohr radii $L(J)$ for index sets to the study of Bohr radii for holomorphic functions in infinitely many variables with lacunary monomial coefficients. Finally, we treat a series of old and new examples.

3.1. **Reduction II.** Let $\Lambda$ be a subset of $\mathbb{N}_0^{(\mathbb{N})}$. Consider the Banach space

$$H_\infty^\Lambda(B_{c_0}) := \Big\{ f \in H_\infty(B_{c_0}) \;\Big|\; c_\alpha(f) \neq 0 \quad \text{only if} \quad \alpha \in \Lambda \Big\},$$

where as usual $H_\infty(B_{c_0})$ denotes the Banach space of all bounded holomorphic (= Fréchet differentiable) functions on the open unit ball $B_{c_0}$ of the Banach space of all null sequences $c_0$.

Now, the *Bohr radius* $K(\Lambda)$ is defined to be the best $r = r(\Lambda) \geq 0$ such that for every $f \in H_\infty^\Lambda(B_{c_0})$ we have

$$\sum_{\alpha \in \Lambda} |c_\alpha(f)| r^{|\alpha|} \leq \|f\|_\infty .$$

Note that, with this notation, the classical Bohr radius $K_n$ is just $K(\mathbb{N}_0^n)$.

The following result extends (2.3) to arbitrary index sets. Let us note that the proof of (2.3) was based on Bohr's fundamental lemma (2.2). We need, then, an extension of this. Inspired by an idea of Bohr and based on the fundamental theorem of arithmetic we here consider the following bijection:

$$\mathfrak{b} : \; \mathbb{N}_0^{(\mathbb{N})} \to \mathbb{N} \,, \;\; \mathfrak{b}(\alpha) = p^\alpha .$$

We denote now by $\mathcal{H}_\infty$ all Dirichlet series $\sum_n a_n n^{-s}$ defining a bounded holomorphic function on $[\operatorname{Re} s > 0]$; this vector space together with the sup norm on $[\operatorname{Re} s > 0]$ forms a Banach space. By [14, Lemma 2.3 and Theorem 3.1] (a fact also essentially due to Bohr [6]) there is a unique isometric and linear bijection $\Phi$ from $H_\infty(B_{c_0})$ onto $\mathcal{H}_\infty$ such $\Phi(z^\alpha) = n^{-s}$ with $\mathfrak{b}(\alpha) = n$:

$$H_\infty(B_{c_0}) \;=\; \mathcal{H}_\infty .$$

Using this general principle a simple translation argument from Dirichlet series into holomorphic functions, and vice versa gives the following result.

**Proposition 3.1.** *For each set $J \subset \mathbb{N}$ and $\Lambda \subset \mathbb{N}_0^{(\mathbb{N})}$ with $J = \mathfrak{b}(\Lambda)$*

$$K(\Lambda) = L(J) .$$

Our next device reduces the estimation of Dirichlet-Bohr radii of a given index set $J$ to the estimation of Dirichlet-Bohr radii of certain parts of $J$. Given $J \subseteq \mathbb{N}$ and $n, m \in \mathbb{N}$, the *$n$-dimensional kernel of $J$* is defined to be

$$J(n) = \left\{k \in J \;\middle|\; \forall j > n : p_j \nmid k\right\},$$

and its *$m$-homogeneous kernel*

$$J[m] = \left\{k \in J \;\middle|\; \Omega(k) = m\right\}.$$

Note that when $J = \mathbb{N}$, then the $n$-dimensional kernel consists of all the natural numbers that factor through the first $n$ primes and the $m$-homogeneous kernel consists of those which have precisely $m$ prime divisors (counted with multiplicities). In other words

$$\mathbb{N}(n) = \{p_1^{\alpha_1} \cdots p_n^{\alpha_n} \big| \alpha \in \mathbb{N}_0^n\},$$

$$\mathbb{N}[m] = \{p_1^{\alpha_1} \cdots p_k^{\alpha_k} \cdots \big| \alpha_1 + \cdots + \alpha_k + \cdots = m\}.$$

Then, clearly $J(n) = J \cap \mathbb{N}(n)$ and $J[m] = J \cap \mathbb{N}[m]$. We also have

$$\mathfrak{b}^{-1}(J(n)) = \left\{\alpha \in \mathbb{N}_0^n \,\middle|\, p^\alpha \in J\right\},$$

$$\mathfrak{b}^{-1}(J[m]) = \left\{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \,\middle|\, p^\alpha \in J \text{ with } |\alpha| = m\right\}.$$

In particular, $\mathfrak{b}^{-1}(\mathbb{N}(n)) = \mathbb{N}_0^n$ and $\mathfrak{b}^{-1}(\mathbb{N}[m]) = \left\{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \;\middle|\; |\alpha| = m\right\}$. Let us finally observe that

$$\mathbb{N}(n)[m] = \{p_1^{\alpha_1} \cdots p_n^{\alpha_n} \big| \alpha \in \mathbb{N}_0^n \text{ and } \alpha_1 + \cdots + \alpha_n = m\} = \mathbb{N}[m](n)$$

and from this $J(n)[m] = J \cap \mathbb{N}(n)[m] = J \cap \mathbb{N}[m](n) = J[m](n)$ for every $J \subseteq \mathbb{N}$ and every $n, m$. We can now give our announced reduction device.

**Proposition 3.2.** *Let $J$ be a subset of $\mathbb{N}$. Then*

*(i)* $L(J) = \inf_n L(J(n))$

*(ii)* $\frac{1}{3} \inf_m L(J[m]) \leq L(J) \leq \inf_m L(J[m])$

*Proof.* The proof of the second statement follows from a word by word copy of the proof of Proposition 2.2. The argument of the first statement is easy after a translation to holomorphic functions by Proposition 3.1. □

Of course, (i) and (ii) can be combined to show that the infimum over $\big(L(J[m](n))\big)_{m,n}$ and $\big(L(J(n)[m])\big)_{m,n}$, respectively, up to the constant $1/3$ equals $L(J)$.

3.2. **Examples.** We first recover with this systematic language the fundamental examples (1.5) that were already mentioned in the introduction.

**Example 3.3.**

(i) $L\big(\mathbb{N}[1]\big) = L\big(\{p \,|\, p \text{ prime }\}\big) = 1$

(ii) $L\big(\mathbb{N}(1)\big) = L\big(\{2^k \,|\, k \in \mathbb{N}\}\big) = \frac{1}{3}$

We remark that (i) here is nothing else than Bohr's inequality (1.1), whereas (ii) is just a reformulation via Proposition 3.1 of Bohr's power series theorem (1.2) (see also (1.3)). Basically, these and the one in the following example are the only precise values of Dirichlet-Bohr radii we know.

**Example 3.4.** $L\Big(\{p_\ell^k \,|\, k, \ell \in \mathbb{N}\}\Big) = \dfrac{1}{3}.$

This turns out to be an immediate consequence of the following more general result. Given a subset $A$ of $\mathbb{N}$, we will denote its cardinal number by $|A|$.

**Proposition 3.5.** *Let $P_k$ , $k \in \mathbb{N}$, be disjoint sets of primes such that*

$$n = \max_k |P_k| < \infty\,.$$

*Define $J_{P_k}$ to be the set of all natural numbers which are finite products of primes in $P_k$, that is*

$$J_{P_k} = \big\{p^\alpha \,|\, \alpha_j = 0, \textit{ if } p_j \notin P_k\big\}\,.$$

*Then*

$$L\Big(\bigcup_k J_{P_k}\Big) = L\big(\mathbb{N}(n)\big)\,.$$

Clearly, Example 3.4 is an immediate consequence of this result: put $P_k = \{p_k\}$ (the $k$-th prime) and apply Example 3.3 together with Proposition 3.5.

*Proof.* Define the sets $\Lambda_k = \mathfrak{b}^{-1}(J_{P_k}) \subset \mathbb{N}_0^{(\mathbb{N})}$. Looking at Proposition 3.1, since $\mathbb{N}_0^n = \mathfrak{b}^{-1}(\mathbb{N}(n))$, it suffices to prove that

$$K\big(\bigcup_k \Lambda_k\big) = K\big(\mathbb{N}_0^n\big)\,.$$

Let $I_k = \bigcup_{\alpha \in \Lambda_k} \operatorname{supp} \alpha \subset \mathbb{N}$ be the support of $\Lambda_k$. Clearly, we have $n_k := |I_k| = |P_k|$ for all $k$. We identify $\operatorname{span}\{e_i : i \in I_k\}$ with $\mathbb{C}^{n_k}$ .
By considering bounded holomorphic functions with support in any $I_k$ of length $n$, we get that $K\big(\bigcup_k \Lambda_k\big) \le K\big(\mathbb{N}_0^n\big)$. We have to prove now the reverse inequality

$$K\big(\mathbb{N}_0^n\big) \;\le\; K\big(\bigcup_k \Lambda_k\big)\,. \tag{3.1}$$

Now, we want to show that

$$\sum_{\alpha\in\bigcup_k\Lambda_k} |a_\alpha|\, K\big(\mathbb{N}_0^n\big)^{|\alpha|} \le \sup_{z\in B_{c_0}} \Big|\sum_{\alpha\in\bigcup_k\Lambda_k} a_\alpha z^\alpha\Big|$$

for every function $\sum_{\alpha\in\bigcup_k\Lambda_k} a_\alpha z^\alpha \in H_\infty(B_{c_0})$. Since the $\Lambda_k$'s are disjoint, we have

$$\sup_{z\in B_{c_0}} \Big|\sum_{\alpha\in\bigcup_{k=1}^N\Lambda_k} a_\alpha z^\alpha\Big| \le \sup_{z\in B_{c_0}} \Big|\sum_{\alpha\in\bigcup_k\Lambda_k} a_\alpha z^\alpha\Big|$$

for all $N$, and then it will be enough to show that

$$\sum_{\alpha\in\bigcup_{k=1}^N} |a_\alpha|\, K\big(\mathbb{N}_0^n\big)^{|\alpha|} \le \sup_{z\in B_{c_0}} \Big|\sum_{\alpha\in\bigcup_{k=1}^N\Lambda_k} a_\alpha z^\alpha\Big|. \tag{3.2}$$

We proceed now by induction on $N$. For $N=1$, (3.2) is just a consequence of the following: $K(\mathbb{N}_0^n) \le K(\mathbb{N}_0^{n_1}) = K(\Lambda_1)$. For the inductive step, we write

$$\sum_{\alpha\in\bigcup_{k=1}^N\Lambda_k} a_\alpha z^\alpha = a_0 + f_1(u_1) + \cdots + f_N(u_N)\,,$$

where $u_k$ is the projection of $z$ in the $\Lambda_k$-coordinates and

$$f_k(w) = \sum_{\substack{\alpha\in\mathbb{N}_0^{n_k}\\ |\alpha|\ge 1}} a_\alpha^k w^\alpha$$

for $w\in\mathbb{C}^{n_k}$. Note that $f_k(0)=0$ for every $k$. By inductive hypothesis we know that

$$|a_0| + \sum_{k=1}^{N-1}\sum_{\substack{\alpha\in\mathbb{N}_0^{n_k}\\ |\alpha|\ge 1}} |a_\alpha|\, K(\mathbb{N}_0^n)^{|\alpha|} \le \sup_{u_1\in\mathbb{D}^{n_1},\dots,u_{N-1}\in\mathbb{D}^{n_{N-1}}} \Big|a_0 + \sum_{k=1}^{N-1} f_k(u_k)\Big|. \tag{3.3}$$

Fix now $u_k\in\mathbb{D}^{n_k}$ for $k=1,\dots,N-1$ and set $\widetilde{a}_0 = a_0 + \sum_{k=1}^{N-1} f_k(u_k)$. Since $K(\mathbb{N}_0^n) \le K(\mathbb{N}_0^{n_N}) = K(\Lambda_N)$, we have

$$\big|\widetilde{a}_0\big| + \sum_{\substack{\alpha\in\mathbb{N}_0^{n_N}\\ |\alpha|\ge 1}} |a_\alpha^N|\, K(\mathbb{N}_0^n)^{|\alpha|} \le \sup_{u_N\in\mathbb{D}^{n_N}} \Big|\widetilde{a}_0 + f_N(u_N)\Big|,$$

which just means that

$$\Big|a_0+\sum_{k=1}^{N-1} f_k(u_k)\Big| + \sum_{\substack{\alpha\in\mathbb{N}_0^{n_N}\\ |\alpha|\ge 1}} |a_\alpha^N|\, K(\mathbb{N}_0^n)^{|\alpha|} \le \sup_{u_N\in\mathbb{D}^{n_N}} \Big|\Big(a_0+\sum_{k=1}^{N-1} f_k(u_k)\Big)+f_N(u_N)\Big|. \tag{3.4}$$

Combining (3.3) and (3.4) we obtain (3.2). □

In the following results we present asymptotically correct estimates on Dirichlet-Bohr radii.

**Example 3.6.**

(1) $\lim_n \frac{L\big(\mathbb{N}(n)\big)}{\sqrt{\frac{\log n}{n}}} = 1\,;$

(2) There is a constant $C > 1$ such that

$$C^{-m}\Big(\frac{m}{n}\Big)^{\frac{m-1}{2m}} \leq L\Big(\big(\mathbb{N}(n)\big)[m]\Big) \leq C^{m}\Big(\frac{m}{n}\Big)^{\frac{m-1}{2m}} \qquad \text{for} \quad n > m$$
$$C^{-m} \leq L\Big(\big(\mathbb{N}(n)\big)[m]\Big) \leq C^{m} \qquad \text{for} \quad n \leq m\,.$$

Both results follow from Proposition 3.1 and their counterparts for Bohr radii:

$$\lim_n \frac{K\big(\mathbb{N}_0^n\big)}{\sqrt{\frac{\log n}{n}}} = 1$$

and

$$C^{-m}\Big(\frac{m}{n}\Big)^{\frac{m-1}{2m}} \leq K\Big(\big\{\alpha \in \mathbb{N}_0^n \,\big|\, |\alpha| = m\big\}\Big) \leq C^{m}\Big(\frac{m}{n}\Big)^{\frac{m-1}{2m}} \qquad \text{for} \quad n > m$$
$$C^{-m} \leq K\Big(\big\{\alpha \in \mathbb{N}_0^n \,\big|\, |\alpha| = m\big\}\Big) \leq C^{m} \qquad \text{for} \quad n \leq m\,.$$

The first formula is due to Bayart, Pellegrino, and Seoane-Sepúlveda [4], who improve an earlier result from [10]. The upper estimate in the second result follows from [10], and the lower one is a consequence of the Kahane-Salem-Zygmund inequality (or [11, Lemma 2.1 and (4.4)]). It would be of particular interest to know the precise values of $L\big(\mathbb{N}(n)\big)$, $L\big(\mathbb{N}[m]\big)$ and $L\big(\big(\mathbb{N}(n)\big)[m]\big)$ for all/some $n, m > 1$.

If Example 3.6 is combined with Proposition 3.2, then we see the following examples.

**Example 3.7.**

(1) $L\big(\mathbb{N}\big) = 0 \quad ;$

(2) $L\big(\mathbb{N}[m]\big) = 0 \quad \text{for all} \quad m > 1\,.$

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