Some additive results on Drazin inverse

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Abstract. In this paper, we investigate additive results of the Drazin inverse of elements in a ring $\mathcal{R}$. Under the condition $ab = ba$, we show that $a + b$ is Drazin invertible if and only if $aa^D(a+b)$ is Drazin invertible, where the superscript D means the Drazin inverse. Furthermore we find an expression of $(a + b)^D$. As an application we give some new representations for the Drazin inverse of a 2×2 block matrix.

§1 Introduction and previous results

In this paper, $\mathcal{R}$ will denote a unital ring whose unity is $\mathbb{1}$. Let us recall that an element $a \in \mathcal{R}$ has a Drazin inverse [18] if there exists $b \in \mathcal{R}$ such that

$$bab = b, \qquad ab = ba, \qquad a - a^2b \text{ is nilpotent.}$$

The element b above is unique if it exists and is denoted by a^D. The nilpotency index of $a - a^2a^D$ is called the Drazin index of a, denoted by $\mathrm{ind}(a)$. The notation a^π means $\mathbb{1} - aa^D$ for any Drazin invertible element $a \in \mathcal{R}$. Observe that by the definition of the Drazin inverse, aa^π is nilpotent. The subset of $\mathcal{R}$ composed of Drazin invertible elements will be denote by $\mathcal{R}^D$.

Drazin proved, [18], that if $a, b \in \mathcal{R}^D$ and $ab = ba = 0$, then $a + b \in \mathcal{R}^D$ and $(a + b)^D = a^D + b^D$. In recent years, many papers focused on the problem under some weaker conditions. Hartwig et al., [19], expressed $(A+B)^D$ under the one-side condition $AB = 0$, where A and B are complex square matrices. This result was extended to bounded linear operators on an arbitrary complex Banach space by Djordjević and Wei in [15]. Again, it was extended for morphisms on arbitrary additive categories by Chen et al. in [8]. More results on the Drazin inverse or the generalized Drazin inverse can also be found in [3, 5, 6, 8, 9, 11, 12, 15]. In particular we must cite [13]: in this paper, the authors, under the commutative condition of $AB = BA$ (when A and B are Drazin invertible linear operators in Banach spaces), gave explicit representations of $(A + B)^D$ in term of A, A^D, B, and B^D.

In this paper, we assume that a and b are Drazin invertible elements which satisfy $ab = ba$ or $a^\pi b = 0$ and $a^n b = ba^n$ for some $n \in \mathbb{N}$, and we conclude that $a + b$ is Drazin invertible if and only if $aa^D(a + b)$ is Drazin invertible. Also we obtain an explicit expression for $(a + b)^D$. As an application, we give additive results of block matrices under some conditions.

We give now some previous results which will be useful in proving our results.

Lemma 1.1. *Let $a, x \in \mathcal{R}$. If $ax = xa$ and there exists $n \in \mathbb{N}$ such that $a^n = 0$, then $\mathbb{1} - xa$ is invertible and $(\mathbb{1} - xa)^{-1} = \sum_{i=0}^{n-1} x^i a^i$.*

Proof. Let $y = \sum_{i=0}^{n-1} x^i a^i$. It is enough to verify $(\mathbb{1} - xa)y = y(\mathbb{1} - xa) = \mathbb{1}$. □

Lemma 1.2. *Let x, y be two commuting nilpotent elements of $\mathcal{R}$. Then $x + y$ is nilpotent.*

Proof. It is enough to recall $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ for any $n \in \mathbb{N}$ since $xy = yx$. □

Next theorem was proved by Drazin [18, Th. 1].

Theorem 1.1. *Let $a \in \mathcal{R}^D$ and $b \in \mathcal{R}$. If $ab = ba$, then $a^D b = ba^D$.*

§2 Main results

Let us observe the expression for $(a-b)^D$ in [24, Th. 2.3]. If we assume that $w = aa^D(a+b)$ instead of $w = aa^D(a - b)bb^D$, we will get a much simpler expression for $(a + b)^D$.

Theorem 2.1. *Let $a, b \in \mathcal{R}$ be Drazin invertible. If $ab = ba$, then $w = aa^D(a + b)$ is Drazin invertible if and only if $a + b$ is Drazin invertible. In this case, we have*

$$(a + b)^D = w^D + a^\pi(\mathbb{1} + b^D aa^\pi)^{-1} b^D = w^D + a^\pi \left(\sum_{i=0}^{\mathrm{ind}(a)-1} (-b^D a)^i \right) b^D. \tag{1}$$

Proof. Recall that aa^π is nilpotent and its index of nilpotency is the Drazin index of a. Let $r = \mathrm{ind}(a)$. Since $ab = ba$, by Theorem 1.1, $a^D b = ba^D$ and $ab^D = b^D a$. From $a^D b = ba^D$ we obtain $a^\pi b = ba^\pi$. Again by Theorem 1.1, a^π commutes with b^D. Therefore, $b^D a^\pi a = a^\pi ab^D$. By Lemma 1.1 we get that $\mathbb{1} + b^D aa^\pi$ is invertible and

$$(\mathbb{1} + b^D aa^\pi)^{-1} = \sum_{i=0}^{r-1} (-b^D aa^\pi)^i = \mathbb{1} + a^\pi \sum_{i=1}^{r-1} (-b^D a)^i.$$

In the rest of the proof, we will use frequently that $\{\mathbb{1}, a, b, a^D, b^D\}$ is a commutative family.

Assume that w is Drazin invertible and let us define

$$x = w^D + a^\pi(\mathbb{1} + b^D aa^\pi)^{-1} b^D.$$

From $ab = ba$ and $a^D b = ba^D$, we have $w(a + b) = aa^D(a + b)(a + b) = (a + b)w$. By Theorem 1.1, we obtain $w^D(a + b) = (a + b)w^D$. Since $r = \mathrm{ind}(a)$, then $(aa^\pi)^r = 0$, or equivalently,

$a^r a^\pi = 0$. We get

$$\begin{aligned}
&(a+b)a^\pi(\mathbb{1}+b^D a a^\pi)^{-1}b^D \\
&= (a+b)\left[\mathbb{1}+(-b^D a)a^\pi+(-b^D a)^2 a^\pi+\cdots+(-b^D a)^{r-1}a^\pi\right]b^D a^\pi \\
&= (a+b)\left[\mathbb{1}+(-b^D a)+(-b^D a)^2+\cdots+(-b^D a)^{r-1}\right]b^D a^\pi \\
&= \left[ab^D+a(-b^D a)b^D+a(-b^D a)^2 b^D+\cdots+a(-b^D a)^{r-1}b^D)\right]a^\pi \\
&\quad+\left[bb^D+b(-b^D a)b^D+b(-b^D a)^2 b^D+\cdots+b(-b^D a)^{r-1}b^D\right]a^\pi \\
&= \left[ab^D-(ab^D)^2+(ab^D)^3+\cdots+(-1)^{r-2}(ab^D)^{r-1}+(-1)^{r-1}(ab^D)^r\right]a^\pi \\
&\quad+\left[bb^D-ab^D+(ab^D)^2+\cdots+(-1)^{r-1}(ab^D)^{r-1}\right]a^\pi \\
&= bb^D a^\pi.
\end{aligned}$$

So, we get

$$(a+b)x = (a+b)\left(w^D+a^\pi(\mathbb{1}+b^D a a^\pi)^{-1}b^D\right) = (a+b)w^D + bb^D a^\pi. \tag{2}$$

Since $\{\mathbb{1}, a, b, a^D, b^D, w, w^D\}$ is a commutative family, we get $x(a+b) = (a+b)x$.

Next, we give the proof of $x(a+b)x = x$. From (2) we can write $(a+b)x = x' + x''$, where $x' = w^D(a+b)$ and $x'' = b^D b a^\pi$. Observe that

$$w + a^\pi(a+b) = aa^D(a+b) + (\mathbb{1}-aa^D)(a+b) = a+b.$$

From $wa^\pi = (a+b)aa^D a^\pi = 0$ we get $w^D a^\pi = (w^D)^2 w a^\pi = 0$, hence

$$\begin{aligned}
xx' &= \left(w^D+a^\pi(\mathbb{1}+b^D a a^\pi)^{-1}b^D\right)w^D(a+b) \\
&= (w^D)^2(a+b) = w^D(a+b)w^D = w^D\left(w+a^\pi(a+b)\right)w^D = w^D
\end{aligned}$$

and

$$\begin{aligned}
xx'' &= \left(w^D+a^\pi(\mathbb{1}+b^D a a^\pi)^{-1}b^D\right)b^D b a^\pi \\
&= \left(a^\pi(\mathbb{1}+b^D a a^\pi)^{-1}b^D\right)b^D b a^\pi \\
&= (\mathbb{1}+b^D a a^\pi)^{-1}b^D a^\pi \\
&= x - w^D.
\end{aligned}$$

So, we get $x(a+b)x = x(x'+x'') = x$.

Now we will prove that $(a+b)-(a+b)^2 x$ is nilpotent. Since $a+b = w+a^\pi(a+b)$, $a^\pi w = 0$, and $a^\pi w^D = 0$, we have

$$\begin{aligned}
(a+b)^2 w^D &= \left(w+a^\pi(a+b)\right)^2 w^D \\
&= \left(w^2+2wa^\pi(a+b)+a^\pi(a+b)^2\right)w^D = w^2 w^D = w - ww^\pi.
\end{aligned} \tag{3}$$

Also we have

$$(a+b)b^D b a^\pi = (a+b)a^\pi(\mathbb{1}-b^\pi) = aa^\pi + ba^\pi - aa^\pi b^\pi - a^\pi bb^\pi. \tag{4}$$

From (2), (3), and (4) we get

$$\begin{aligned}&(a+b)-(a+b)^2x\\&=(a+b)-(a+b)\left(w^D(a+b)+bb^Da^\pi\right)\\&=(a+b)-(w-ww^\pi+aa^\pi+ba^\pi-aa^\pi b^\pi-a^\pi bb^\pi)\\&=(a+b)-\left[(a+b)aa^D+(a+b)a^\pi-aa^\pi b^\pi-a^\pi bb^\pi-ww^\pi\right]\\&=(a+b)-[(a+b)-aa^\pi b^\pi-a^\pi bb^\pi-ww^\pi]\\&=aa^\pi b^\pi+a^\pi bb^\pi+ww^\pi.\end{aligned}$$

Since aa^π, bb^π, and ww^π are nilpotent, and $\{aa^\pi, bb^\pi, ww^\pi\}$ is a commuting family, then by using Lemma 1.2 we get the nilpotency of $(a+b)-(a+b)^2x$. Therefore, we have proved $a+b\in\mathcal{R}^D$ and $(a+b)^D=x$, i.e., the expression (1).

Conversely, let us assume $a+b\in\mathcal{R}^D$. Let $y=aa^D(a+b)^D$. We will prove that $w=aa^D(a+b)\in\mathcal{R}^D$ and $w^D=y$. Observe that Theorem 1.1 implies that $\{a,b,a^D,b^D,(a+b)^D\}$ is a commuting family. Now, having in mind $(aa^D)^2=aa^D$, it is simple to prove $wy=yw=aa^D(a+b)(a+b)^D$, $y^2w=y$, and $w^2y-w=aa^D\left[(a+b)^2(a+b)^D-(a+b)\right]$, which leads to the nilpotency of w^2y-w. The proof is finished. □

Corollary 2.1. *Let $a,b\in\mathcal{R}$ be Drazin invertible. If $ab=ba$ and $baa^\pi=0$, then $w=aa^D(a+b)$ is Drazin invertible if and only if $a+b$ is Drazin invertible. In this case, we have*

$$(a+b)^D=w^D+a^\pi b^D.$$

Proof. From $baa^\pi=0$, we have $b^Daa^\pi=(b^D)^2baa^\pi=0$. It is enough apply Theorem 2.1 to prove this corollary. □

Theorem 2.2. *Let $a,b\in\mathcal{R}$ be Drazin invertible, $a^\pi b=0$ and $a^nb=ba^n$ for some $n\in\mathbb{N}$. Then $a+b$ is Drazin invertible if and only if $w=aa^D(a+b)$ is Drazin invertible. In this case, we have*

$$(a+b)^D=w^D.$$

Proof. From $a\in\mathcal{R}^D$, it is simple to prove that $a^n\in\mathcal{R}^D$ and $(a^n)^D=(a^D)^n$. In addition, $(a^n)^\pi=\mathbb{1}-a^n(a^n)^D=\mathbb{1}-(aa^D)^n=\mathbb{1}-aa^D=a^\pi$. Since $a^nb=ba^n$, by Theorem 1.1 we get $(a^n)^Db=b(a^n)^D$, and therefore, $a^\pi b=ba^\pi$ and $aa^Db=baa^D$. Also, the following equality will be useful:

$$w+a^\pi(a+b)=aa^D(a+b)+(\mathbb{1}-aa^D)(a+b)=a+b. \tag{5}$$

Since aa^D commutes with a and b, we get $wa^\pi=a^\pi w=0$.

Assume that w is Drazin invertible. We will prove that w^D is the Drazin inverse of $a+b$, i.e., we will prove $w^D(a+b)=(a+b)w^D$, $(w^D)^2(a+b)=w^D$, and $(a+b)^2-w^D$ is nilpotent.

Since $aa^Db=baa^D$, we get

$$w(a+b)=aa^D(a+b)(a+b)=(a+b)aa^D(a+b)=(a+b)w.$$

By Theorem 1.1 we obtain $w^D(a+b)=(a+b)w^D$.

From $wa^\pi=0$ we get $w^Da^\pi=(w^D)^2wa^\pi=0$. By using $w^Da^\pi=0$ and (5) we have

$$(w^D)^2(a+b)=(w^D)^2(w+a^\pi(a+b))=(w^D)^2w+(w^D)^2a^\pi(a+b)=w^D.$$

Since $a+b = w + a^\pi(a+b)$ and $a^\pi w = wa^\pi = 0$, we have

$$(a+b)^2 = (w + a^\pi(a+b))^2 = w^2 + a^\pi(a+b)^2.$$

Hence from $a^\pi w^D = a^\pi w(w^D)^2 = 0$ we obtain

$$\begin{aligned}(a+b)^2 w^D &= (w^2 + a^\pi(a+b)^2)w^D = w^2 w^D = w - ww^\pi \\ &= aa^D(a+b) - ww^\pi = (\mathbb{1} - a^\pi)(a+b) - ww^\pi \\ &= a + b - a^\pi a - a^\pi b - ww^\pi.\end{aligned}$$

From $a^\pi b = 0$, we have $a + b - (a+b)^2 w^D = a^\pi a + w^\pi w$.

From $a^\pi w = wa^\pi$, we have $a^\pi w^D = w^D a^\pi$, so, we get

$$a^\pi w^\pi = a^\pi(\mathbb{1} - ww^D) = (\mathbb{1} - ww^D)a^\pi = w^\pi a^\pi.$$

From $wa^\pi = a^\pi w = 0$ we obtain $(aa^\pi)(ww^\pi) = 0$ and $(ww^\pi)(aa^\pi) = 0$. Hence for any $k \in \mathbb{N}$ we have

$$\left(a + b - (a+b)^2 w^D\right)^k = (a^\pi a + w^\pi w)^k = (a^\pi a)^k + (w^\pi w)^k.$$

Since aa^π and ww^π are nilpotent, it follows that $(a+b) - (a+b)^2 w^D$ is nilpotent. We have just proved that $a+b \in \mathcal{R}^D$ and $(a+b)^D = w^D$.

Assume that $a + b \in \mathcal{R}^D$. We will prove that $w = aa^D(a+b) \in \mathcal{R}^D$ and the Drazin inverse of $a+b$ is w^D, i.e., $(a+b)^D w = w(a+b)^D$, $\left((a+b)^D\right)^2 w = (a+b)^D$, and $w^2(a+b)^D - w$ is nilpotent.

Since aa^D commutes with a and b we have $(a+b)w = w(a+b)$. By Theorem 1.1, one gets $(a+b)w^D = w^D(a+b)$.

Since a is Drazin invertible, we can write $a = a_1 + a_2$ (this is the core-nilpotent decomposition of a, see e.g [16, Ch. 2]), where $a_1 \in aa^D\mathcal{R}aa^D$ and $a_2 \in a^\pi\mathcal{R}a^\pi$ is nilpotent. From $a^\pi b = ba^\pi = 0$ we obtain $b \in aa^D\mathcal{R}aa^D$. Hence $a+b$ can be decomposed as

$$a + b = (a_1 + b) + a_2, \qquad a_1 + b \in aa^D\mathcal{R}aa^D,\ a_2 \in a^\pi\mathcal{R}a^\pi. \tag{6}$$

From $(a+b)aa^D = aa^D(a+b)$ and Theorem 1.1 we get $(a+b)^D aa^D = aa^D(a+b)^D$, and therefore,

$$(a+b)^D = aa^D(a+b)^D aa^D + aa^D(a+b)^D a^\pi + a^\pi(a+b)^D aa^D + a^\pi(a+b)^D a^\pi$$

can be also decomposed as

$$(a+b)^D = u + v, \qquad u \in aa^D\mathcal{R}aa^D,\ v \in a^\pi\mathcal{R}a^\pi. \tag{7}$$

From the definition of the Drazin inverse and (6), (7) we have that $a_1 + b, a_2 \in \mathcal{R}^D$ and $(a_1+b)^D = u$, $a_2^D = v$. But, $a_2^D = 0$ because a_2 is nilpotent. Therefore, $(a+b)^D = (a_1+b)^D \in aa^D\mathcal{R}aa^D$. Now

$$\begin{aligned}\left((a+b)^D\right)^2 w &= \left((a_1+b)^D\right)^2 aa^D(a+b) \\ &= \left((a_1+b)^D\right)^2 (a+b) = \left((a+b)^D\right)^2 (a+b) = (a+b)^D.\end{aligned}$$

Now, let us prove that $w^2(a+b)^D - w$ is nilpotent. We have proved that aa^D commutes with $a+b$. Since aa^D is an idempotent,

$$\begin{aligned} w^2(a+b)^D - w &= \left[aa^D(a+b)\right]^2 (a+b)^D - aa^D(a+b) \\ &= aa^D(a+b)^2(a+b)^D - aa^D(a+b) \\ &= aa^D \left[(a+b)^2(a+b)^D - (a+b)\right]. \end{aligned}$$

Since aa^D commutes with $a+b$ and $(a+b)^D$, and $(a+b)^2(a+b)^D - (a+b)$ is nilpotent, then $w^2(a+b)^D - w$ is nilpotent. Therefore, $w \in \mathcal{R}^D$ and $w^D = (a+b)^D$. The proof is finished. □

If $(\mathcal{R}, \cdot)$ is a ring with a unity $\mathbb{1}$, then we can define a new multiplication in $\mathcal{R}$ by $a \odot b = ba$. With this multiplication, $(\mathcal{R}, \odot)$ becomes a ring with the same unity $\mathbb{1}$. We can apply Theorem 2.2 to $(\mathcal{R}, \odot)$ obtaining a dual result.

§3 Applications

In this section, we give some formulas for the Drazin inverse of a 2×2 block matrix under some conditions. Let $\mathbb{C}^{m\times n}$ be the set of all the $m \times n$ matrices over the complex field.

Let M be a matrix of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \qquad A \in \mathbb{C}^{m\times m},\ D \in \mathbb{C}^{n\times n}. \tag{8}$$

Campbell and Meyer, [2, Ch. 7] proposed the (until now open) problem to find an explicit formula of the Drazin inverse of M in terms of the blocks of M. Several authors have investigated this problem and they were able to find some partial answers (imposing some conditions on the blocks of M). Here we write an exemplary list.

- $B = 0$ (or $C = 0$). See [2, Ch. 7] or [23].
- $BC = 0$, $DC = 0$ (or $BD = 0$), and D is nilpotent. See [20].
- $BCA = 0$, $BD = 0$, and $DC = 0$ (or BC is nilpotent). See [4].
- $BCA = 0$, $BCB = 0$, $DCA = 0$, and $DCB = 0$. See [25].
- $BC = 0$, $BD = 0$ and $DC = 0$. See [14].
- $BC = 0$ and $DC = 0$. See [10].
- $BCA = 0$, $BCB = 0$, $ABD = 0$, and $CBD = 0$. See [22];
- $BC = 0$ and $BD = 0$. See [17].

We will find several expressions for M^D under some conditions involving the blocks A, B, C, D, and the Drazin inverses of A and D. Let us recall that the Drazin inverse of any square comples matrix always exists (see e.g., [1, Ch. 4])

First, we will state some auxiliary lemmas.

Lemma 3.1. (See [1, Ch. 4] or [2, Th. 7.8.4]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$. Then $(AB)^D = A[(BA)^D]^2B$.*

Lemma 3.2. (See [7] or [21]). *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{n\times m}$. Then*

$$\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}^D = \begin{bmatrix} 0 & (AB)^DA \\ (BA)^DB & 0 \end{bmatrix}.$$

Lemma 3.3. (See [2, Ch. 7] or [23]). *Let M_1 and M_2 be of a form*

$$M_1 = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}, \qquad M_2 = \begin{bmatrix} B & C \\ 0 & A \end{bmatrix}.$$

If $r = \mathrm{ind}(A)$ and $s = \mathrm{ind}(B)$, then

$$M_1^D = \begin{bmatrix} A^D & 0 \\ S & B^D \end{bmatrix}, \qquad M_2^D = \begin{bmatrix} B^D & S \\ 0 & A^D \end{bmatrix},$$

where

$$S = \left[\sum_{i=0}^{r-1}(B^D)^{i+2}CA^i\right]A^\pi + B^\pi\left[\sum_{i=0}^{s-1}B^iC(A^D)^{i+2}\right] - B^DCA^D. \tag{9}$$

Let M be a 2×2 block matrix represented as in (8). Let $r = \mathrm{ind}(A)$ and $s = \mathrm{ind}(D)$. To state next lemma, we define the following matrices, being k a nonnegative integer.

$$\Sigma_k = (D^D)^2\sum_{i=0}^{r-1}(D^D)^{i+k}CA^iA^\pi + D^\pi\sum_{i=0}^{s-1}D^iC(A^D)^{i+k}(A^D)^2 - \sum_{i=0}^{k}(D^D)^{i+1}C(A^D)^{k-i+1}. \tag{10}$$

Lemma 3.4. (See [17]). *Let M be a matrix of a form (8). If $BC = 0$ and $BD = 0$, then*

$$M^D = \begin{bmatrix} A^D & (A^D)^2B \\ \Sigma_0 & D^D + \Sigma_1 B \end{bmatrix},$$

where Σ_0 and Σ_1 are defined in (10).

Lemma 3.5. *Let $X \in \mathbb{C}^{n\times n}$. Then $(X^2X^D)^D = X^D$, $(X^2X^D)^\pi = X^\pi$, and $\mathrm{ind}(X^2X^D) = 1$.*

Proof. The Jordan canonical form of X permits write $X = S(C\oplus N)S^{-1}$, where S and C are nonsingular, and N is nilpotent. Evidently, $X^D = S(C^{-1}\oplus 0)S^{-1}$. Now, it is evident $X^2X^D = S(C\oplus 0)S^{-1}$, which leads to the affirmations of this lemma. □

Using Theorem 2.1 and the previous lemmas, we get the following results.

Theorem 3.1. *Let M be given by (8) and let $r = \mathrm{ind}(A)$.*

(i) *If $AB = BD$, $DC = CA$, and $BD^D = 0$, then*

$$M^D = \begin{bmatrix} A^D & (A^D)^2B \\ \Phi_0 & D^D + \Phi_1AA^DB \end{bmatrix} + \sum_{i=0}^{r-1}\begin{bmatrix} 0 & (BC)^DB \\ (CB)^DC & 0 \end{bmatrix}^i \begin{bmatrix} (-A)^iA^\pi & 0 \\ 0 & (-D)^iD^\pi \end{bmatrix},$$

where

$$\Phi_0 = (D^D)^2CA^\pi - D^DCA^D$$

and

$$\Phi_1 = (D^D)^3CA^\pi - D^DC(A^D)^2 - (D^D)^2CA^D.$$

(ii) *If $AB = BD$, $DC = CA$, and $BC = 0$, then*

$$M^D = \begin{bmatrix} A^D & -(A^D)^2 B \\ -(D^D)^2 C & D^D + (D^D)^3 CB \end{bmatrix}.$$

Proof. (i) We can split the matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}.$$

From $AB = BD$ and $DC = CA$, we have $PQ = QP$. Applying Theorem 1.1 and Theorem 2.1, we get

$$M^D = \left(PP^D(P+Q)\right)^D + \left[\sum_{i=0}^{r-1} (Q^D)^{i+1}(-P)^i\right] P^\pi. \tag{11}$$

Observe that

$$\left(PP^D(P+Q)\right)^D = \begin{bmatrix} A^2 A^D & AA^D B \\ DD^D C & D^2 D^D \end{bmatrix}^D.$$

From $BD^D = 0$, the matrix $PP^D(P+Q)$ satisfies Lemma 3.4. In view of Lemma 3.5 we get (recall that the index of matrices A^2A^D and D^2D^D is 1)

$$\left(PP^D(P+Q)\right)^D = \begin{bmatrix} A^D & (A^D)^2 B \\ \Phi_0 & D^D + \Phi_1 AA^D B \end{bmatrix},$$

where

$$\Phi_0 = (D^D)^2 CA^\pi - D^D CA^D$$

and

$$\Phi_1 = (D^D)^3 CA^\pi - D^D C(A^D)^2 - (D^D)^2 CA^D.$$

Also we have

$$\sum_{i=0}^{r-1} (Q^D)^{i+1}(-P)^i = \sum_{i=0}^{r-1} \begin{bmatrix} 0 & (BC)^D B \\ (CB)^D C & 0 \end{bmatrix}^i \begin{bmatrix} (-A)^i & 0 \\ 0 & (-D)^i \end{bmatrix}.$$

The proof of (i) is finished.

(ii) Now, we split the matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}. \tag{12}$$

From $AB = BD$ and $DC = CA$, we have $PQ = QP$. Hence we can use the expression (11); but now for the matrices P and Q defined in (12).

Since $BC = 0$, it is easy to get $P^3 = 0$. Therefore, $P^D = 0$ and (11) reduces to

$$M^D = Q^D - (Q^D)^2 P + (Q^D)^3 P^2.$$

Furthermore, we have

$$(Q^D)^2 P = \begin{bmatrix} (A^D)^2 & 0 \\ 0 & (D^D)^2 \end{bmatrix} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & (A^D)^2 B \\ (D^D)^2 C & 0 \end{bmatrix}.$$

and

$$(Q^D)^3 P^2 = \begin{bmatrix} (A^D)^3 & 0 \\ 0 & (D^D)^3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & CB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & (D^D)^3 CB \end{bmatrix}.$$

The proof is finished. □

Theorem 3.2. *Let M be given by* (8). *If $BC = 0$, $ABD^D = 0$, $CA^\pi B = 0$, and $AB = BD$, then*

$$M^D = \begin{bmatrix} A^D & (A^D)^2 B \\ \Sigma_0 & D^D + \Sigma_1 AA^D B - D^D \Sigma_0 A^\pi B \end{bmatrix},$$

where Σ_0 and Σ_1 are defined in (10).

Proof. We can split the matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} 0 & A^\pi B \\ 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} A & AA^D B \\ C & D \end{bmatrix}.$$

From $BC = 0$, $CA^\pi B = 0$, and $AB = BD$ we have $PQ = QP$. Moreover it is trivial to verify $P^2 = 0$, hence $P^D = 0$. Applying Theorem 2.1, we get

$$M^D = Q^D - (Q^D)^2 P. \tag{13}$$

Matrix Q satisfies Lemma 3.4, so we get

$$Q^D = \begin{bmatrix} A^D & (A^D)^2 AA^D B \\ \Sigma_0 & D^D + \Sigma_1 AA^D B \end{bmatrix}, \tag{14}$$

where Σ_0 and Σ_1 are defined in (10). Evidently, $(A^D)^2 AA^D B = (A^D)^2 B$. Now,

$$Q^D P = \begin{bmatrix} A^D & (A^D)^2 B \\ \Sigma_0 & D^D + \Sigma_1 AA^D B \end{bmatrix} \begin{bmatrix} 0 & A^\pi B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_0 A^\pi B \end{bmatrix}$$

because $A^D A^\pi = 0$. Therefore,

$$(Q^D)^2 P = \begin{bmatrix} A^D & (A^D)^2 B \\ \Sigma_0 & D^D + \Sigma_1 AA^D B \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_0 A^\pi B \end{bmatrix} = \begin{bmatrix} 0 & (A^D)^2 B \Sigma_0 A^\pi B \\ 0 & (D^D + \Sigma_1 AA^D B)\Sigma_0 A^\pi B \end{bmatrix}.$$

Observe that $A^D B D^D = (A^D)^2 ABD^D = 0$, which leads to

$$\begin{aligned} A^D B \Sigma_0 &= A^D B \left((D^D)^2 \sum_{i=0}^{r-1} (D^D)^i C A^i A^\pi + D^\pi \sum_{i=0}^{s-1} D^i C (A^D)^i (A^D)^2 - D^D C A^D \right) \\ &= A^D B D^\pi \sum_{i=0}^{s-1} D^i C (A^D)^i (A^D)^2 \\ &= A^D B D^\pi C (A^D)^2 \\ &= A^D B (I - DD^D) C (A^D)^2 \\ &= A^D BC (A^D)^2 = 0. \end{aligned}$$

Thus,

$$(Q^D)^2 P = \begin{bmatrix} 0 & 0 \\ 0 & D^D \Sigma_0 A^\pi B \end{bmatrix}. \tag{15}$$

To prove the theorem, it is enough consider (13), (14), and (15). □

Next result generalizes Lemma 3.3

Theorem 3.3. *Let M be a matrix written as in* (8). *If $BC = 0$, $CB = 0$, and $AB = BD$, then*

$$M^D = \begin{bmatrix} A^D & -B(D^D)^2 \\ S & D^D \end{bmatrix}.$$

where

$$S = \sum_{i=0}^{r-1} (D^D)^{i+2} C A^i A^\pi + \sum_{i=0}^{s-1} D^\pi D^i C (A^D)^{i+2} - D^D C A^D, \tag{16}$$

$r = \text{ind}(A)$, *and* $s = \text{ind}(D)$.

Proof. We split the matrix M as $M = P + Q$, where

$$P = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}.$$

From the hypotheses of the theorem we get $PQ = QP$. Since $P^2 = 0$, then $P^D = 0$ and $P^\pi = I$. Thus, Theorem 2.1 and Theorem 1.1 imply

$$M^D = Q^D - P(Q^D)^2. \tag{17}$$

By using Lemma 3.3 we can find an expression for Q^D:

$$Q^D = \begin{bmatrix} A^D & 0 \\ S & D^D \end{bmatrix}, \tag{18}$$

where S is defined in (16). Now we have

$$PQ^D = \begin{bmatrix} BS & BD^D \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad Q^D P = \begin{bmatrix} 0 & A^D B \\ 0 & SB \end{bmatrix}.$$

By Theorem 1.1, we get $BS = 0$ and $SB = 0$ (in addition, we get $BD^D = A^D B$, but this equlity will not be useful). Now,

$$P(Q^D)^2 = (PQ^D)Q^D = \begin{bmatrix} BD^D S & B(D^D)^2 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad Q^D(PQ^D) = \begin{bmatrix} 0 & A^D BD^D \\ 0 & SBD^D \end{bmatrix}.$$

As before, by Theorem 1.1, we get

$$P(Q^D)^2 = \begin{bmatrix} 0 & B(D^D)^2 \\ 0 & 0 \end{bmatrix}. \tag{19}$$

To prove the theorem, it is enough to consider (17), (18), and (19). □

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