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CO-EP BANACH ALGEBRA ELEMENTS

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Abstract. In this work, given a unital Banach algebra $\mathcal{A}$ and $a \in \mathcal{A}$ such that a has a Moore–Penrose inverse $a^\dagger$, it will be characterized when $aa^\dagger - a^\dagger a$ is invertible. A particular subset of this class of objects will also be studied. In addition, perturbations of this class of elements will be studied. Finally, the Banach space operator case will be also considered.

1. Introduction and preliminaries

A square matrix A is said to be EP if $\mathcal{N}(A) = \mathcal{N}(A^*)$. This notion was introduced in [34] and since then many authors have studied EP matrices. Since a necessary and sufficient condition for a matrix A to be EP is the fact that A commutes with its Moore–Penrose inverse, the notion under consideration has been extended to Hilbert space operators and C^*-algebra elements, see for example [14, 11, 18, 19, 20, 15, 16, 1].

In the context of Banach algebras the notion of Moore–Penrose inverse was introduced by V. Rakočević in [31] and its basic properties were studied in [31, 32, 26, 6]. In the recent past EP Banach space operators and EP Banach algebra elements, i.e., Moore–Penrose invertible operators or elements of a Banach algebra such that they commute with their Moore–Penrose inverse, were introduced and characterized, see [6, 7, 8, 27].

The main object of this work is to study a complementary class of object, the co-EP Banach algebra elements, i.e., the Moore–Penrose invertible elements

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$a \in \mathcal{A}$ such that $aa^\dagger - a^\dagger a$ is nonsingular, where $\mathcal{A}$ is a Banach algebra, $a \in \mathcal{A}$ and $a^\dagger$ denotes the Moore–Penrose inverse of a. This class of objects were studied for matrices ([2, 4]), for C^*-algebras ([3]) and for rings ([5]). In section 2, co-EP Banach algebra elements will be characterized. In addition, it will be also given necessary and sufficient conditions that ensure the nonsingularity of $aa^\dagger + a^\dagger a$, $a \in \mathcal{A}$, $\mathcal{A}$ a unital Banach algebra. Moreover, co-EP Banach space operators defined on finite and infinite dimensional Banach space will be also studied. On the other hand, in section 3 a particular set of co-EP Banach algebra elements will be characterized and in section 4 perturbations of co-EP elements will be considered.

It is worth noticing that since $aa^\dagger$ and $a^\dagger a$ are idempotents, characterizing co-EP elements is related to the following problem: given two idempotents p and q, find conditions that ensure the nonsingularity of $p-q$. This problem was studied in different frames and by many authors, see for example [36, 30, 12, 13, 21, 22, 23, 24].

It is also important to observe that due to the lack of an involution on a Banach algebra, and in particular on the Banach algebra of bounded and linear maps defined on a Banach space, the results presented in this work give a new insight into the cases where the involution does exist. In fact, the results and proofs presented do not depend on a particular norm, the Euclidean norn, but they hold for any norm. In particular, the results considered in this work also apply to co-EP matrices defined using an arbitrary norm on a finite dimensional vector space.

From now on $\mathcal{X}$ will denote a complex Banach space and $\mathcal{L}(\mathcal{X},\mathcal{Y})$ the Banach algebra of all bounded and linear maps defined on $\mathcal{X}$ with values in the Banach space $\mathcal{Y}$. As usual, when $\mathcal{X}=\mathcal{Y}$, $\mathcal{L}(\mathcal{X},\mathcal{Y})$ will be denoted by $\mathcal{L}(\mathcal{X})$. Let I denote the identity map of $\mathcal{L}(\mathcal{X})$. In addition, if $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, then $\mathcal{N}(T) \subseteq \mathcal{X}$ and $\mathcal{R}(T) \subseteq \mathcal{Y}$ will stand for the null space and the range of T respectively.

On the other hand, $\mathcal{A}$ will denote a unital complex Banach algebra with unit 1 and $\mathcal{A}^{-1}$ will stand for the set of all invertible elements of $\mathcal{A}$. If $a \in \mathcal{A}$, then $L_a\colon \mathcal{A} \to \mathcal{A}$ and $R_a\colon \mathcal{A} \to \mathcal{A}$ will denote the map defined by left and right multiplication respectively:

$$L_a(x) = ax, \qquad R_a(x) = xa, \qquad (x \in \mathcal{A}).$$

Moreover, the following notation will be used: $a^{-1}(0) = \mathcal{N}(L_a)$, $a\mathcal{A} = \mathcal{R}(L_a)$, $a_{-1}(0) = \mathcal{N}(R_a)$ and $\mathcal{A}a = \mathcal{R}(R_a)$. Observe that $a \in \mathcal{A}^{-1}$ if and only if $L_a \in \mathcal{L}(\mathcal{A})^{-1}$.

Recall that an element $a \in \mathcal{A}$ is called *regular,* if it has a *generalized inverse,* namely if there exists $b \in \mathcal{A}$ such that

$$a = aba.$$

Furthermore, a generalized inverse b of a regular element $a \in \mathcal{A}$ will be called *normalized,* if b is regular and a is a generalized inverse of b, equivalently,

$$a = aba, \qquad b = bab.$$

Note that if b is a generalized inverse of a, then bab is a normalized generalized inverse of a. What is more, when $b \in \mathcal{A}$ is a normalized generalized inverse of $a \in \mathcal{A}$, ab and ba are idempotents and the following identities hold:

$$ab\mathcal{A} = a\mathcal{A}, \qquad (ab)^{-1}(0) = b^{-1}(0), \qquad \mathcal{A}ab = \mathcal{A}b, \qquad (ab)_{-1}(0) = a_{-1}(0),$$
$$ba\mathcal{A} = b\mathcal{A}, \qquad (ba)^{-1}(0) = a^{-1}(0), \qquad \mathcal{A}ba = \mathcal{A}a, \qquad (ba)_{-1}(0) = b_{-1}(0).$$

Recall also that

$$(1-ab)\mathcal{A} = (ab)^{-1}(0), \qquad \mathcal{A}(1-ab) = (ab)_{-1}(0),$$
$$(1-ba)\mathcal{A} = (ba)^{-1}(0), \qquad \mathcal{A}(1-ba) = (ba)_{-1}(0).$$

Next follows the key notion in the definition of the Moore–Penrose inverse in the context of Banach algebras.

Definition 1.1. Given a unital Banach algebra $\mathcal{A}$, an element $a \in \mathcal{A}$ is said to be hermitian, if $\parallel \exp(ita) \parallel = 1$, for all $t \in \mathbb{R}$.

As regard equivalent definitions and the main properties of hermitian Banach algebra elements and Banach space operators, see [10, 17, 25, 28, 35]. In the conditions of the previous definition, recall that if $\mathcal{A}$ is a C^*-algebra, then $a \in \mathcal{A}$ is hermitian if and only if a is self-adjoint, see [10, Proposition 20, Section 12, Chapter I]. Furthermore, $\mathcal{H} = \{a \in \mathcal{A} \colon\ a \text{ is hermitian}\} \subseteq \mathcal{A}$ is a closed linear vector space over the real field ([35], [17, Theorem 4.4, Chapter 4]). Since $\mathcal{A}$ is unital, $1 \in \mathcal{H}$, which implies that $a \in \mathcal{H}$ if and only if $1 - a \in \mathcal{H}$.

Now the definition of Moore–Penrose invertible Banach algebra elements will be recalled.

Definition 1.2. Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}$. If there exists $x \in \mathcal{A}$ such that x is a normalized generalized inverse of a satisfying that ax and xa are hermitian, then x will be called the *Moore–Penrose inverse* of a, and it will be denoted by $a^\dagger$.

Recall that according to [31, Lemma 2.1], given $a \in \mathcal{A}$, there is at most one $x \in \mathcal{A}$ satisfying the conditions of Definition 1.2. In addition, recall that even for matrices with an arbitrary norm, the Moore–Penrose inverse could not exist (see [6, Remark 4]). Let $\mathcal{A}^\dagger$ denote the set of all Moore–Penrose invertible elements of $\mathcal{A}$. As regard the Moore–Penrose inverse in Banach algebra, see [6, 26, 31, 32]. For the original definition of the Moore–Penrose inverse for matrices, see [29].

Given $a \in \mathcal{A}$ such that $a^\dagger$ exists, $a \in \mathcal{A}$ is said to be *EP*, if $aa^\dagger = a^\dagger a$. EP elements in Banach algebras have been considered in [6, 7, 8, 9, 27]. Next the objects that will be studied in this article will be introduced.

Definition 1.3. Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}^\dagger$. The element a will be said to be *co-EP*, if $aa^\dagger - a^\dagger a \in \mathcal{A}^{-1}$.

Given a unital Banach algebra $\mathcal{A}$, the set of all co-EP elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{EP}_{co}$. Co-EP elements have been studied in [2, 3, 4].

2. Co-EP elements

In this section co-EP Banach algebra elements and Banach space operators will be characterized. Co-EP matrices were studied in [2, 4] and co-EP C^*-algebra elements in [3]. However, in order to characterize co-EP elements, some preparation is needed.

Proposition 2.1. *Let $\mathcal{A}$ be a unital Banach algebra, $a \in \mathcal{A}^\dagger$, and $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. The following statements hold.*

(i) $L_{\lambda a+\mu a^\dagger}((1-aa^\dagger)\mathcal{A}) = L_a((1-aa^\dagger)\mathcal{A}) \subseteq aa^\dagger\mathcal{A}$.

(ii) $L_{\lambda a+\mu a^\dagger}((1-a^\dagger a)\mathcal{A}) = L_{a^\dagger}((1-a^\dagger a)\mathcal{A}) \subseteq a^\dagger a\mathcal{A}$.

(iii) *The following statements are equivalent:*

(iii.a) $(1-aa^\dagger)\mathcal{A} \cap (1-a^\dagger a)\mathcal{A} = 0$;

(iii.b) $\mathcal{N}(L_{\lambda a+\mu a^\dagger}) \cap (1-aa^\dagger)\mathcal{A} = 0$;

(iii.c) $\mathcal{N}(L_{\lambda a+\mu a^\dagger}) \cap (1-a^\dagger a)\mathcal{A} = 0$.

(iv) *If* $a^\dagger \in (1-aa^\dagger)\mathcal{A} + (1-a^\dagger a)\mathcal{A}$, *then* $L_{\lambda a+\mu a^\dagger}((1-aa^\dagger)\mathcal{A}) = L_a((1-aa^\dagger)\mathcal{A}) = aa^\dagger\mathcal{A}$.

(v) *If* $a \in (1-aa^\dagger)\mathcal{A} + (1-a^\dagger a)\mathcal{A}$, *then* $L_{\lambda a+\mu a^\dagger}((1-a^\dagger a)\mathcal{A}) = L_{a^\dagger}((1-a^\dagger a)\mathcal{A}) = a^\dagger a\mathcal{A}$.

Proof. (i). Let $x \in \mathcal{A}$. The equalities $(\lambda a+\mu a^\dagger)(1-aa^\dagger)x = \lambda a(1-aa^\dagger)x$ and $a(1-aa^\dagger)x = aa^\dagger a(1-aa^\dagger)x$ prove (i).

(ii). Apply an argument similar to the one used in the proof of statement (i).

(iii.a) $\Rightarrow$ (iii.b) follows from $(1-aa^\dagger)\mathcal{A} = (aa^\dagger)^{-1}(0) = (a^\dagger)^{-1}(0)$ and $(1-a^\dagger a)\mathcal{A} = (a^\dagger a)^{-1}(0) = a^{-1}(0)$.

(iii.b) $\Rightarrow$ (iii.a). Let $x \in (1-aa^\dagger)\mathcal{A} \cap (1-a^\dagger a)\mathcal{A}$. Since $x \in a^{-1}(0) \cap (a^\dagger)^{-1}(0)$, $x \in \mathcal{N}(L_{\lambda a+\mu a^\dagger}) \cap (1-aa^\dagger)\mathcal{A} = 0$.

The equivalence (iii.a) $\Leftrightarrow$ (iii.c) can be proved in a similar way.

(iv). Let c, $d \in \mathcal{A}$ such that $a^\dagger = (1-aa^\dagger)c + (1-a^\dagger a)d$. In particular, if $x \in \mathcal{A}$, then $aa^\dagger x = a(1-aa^\dagger)cx = (\lambda a+\mu a^\dagger)(1-aa^\dagger)c(\lambda^{-1}x)$.

(v). Apply an argument similar to the one used in the proof of statement (iv). □

In the following theorem, co-EP Banach algebra elements will be characterized. Note that this theorem is stronger than [5, Theorem 2].

Theorem 2.2. *Let $\mathcal{A}$ be a unital Banach algebra, $a \in \mathcal{A}^\dagger$ and $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. The following statements are equivalent.*

(i) $aa^\dagger - a^\dagger a \in \mathcal{A}^{-1}$;

(ii) $\mathcal{A} = a\mathcal{A} \oplus a^\dagger\mathcal{A}$ *and* $\mathcal{A} = \mathcal{A}a \oplus \mathcal{A}a^\dagger$;

(iii) $\lambda a + \mu a^\dagger \in \mathcal{A}^{-1}$ *and* $a\mathcal{A} \cap a^\dagger\mathcal{A} = 0$;

(iv) $\lambda a + \mu a^\dagger \in \mathcal{A}^{-1}$ *and exists and idempotent h such that* $ha = a$, $ha^\dagger = 0$;

(v) $L_{\lambda a+\mu a^\dagger}$, $R_{\lambda a+\mu a^\dagger} \in \mathcal{L}(\mathcal{A})$ *are right invertible and* $a\mathcal{A} \cap a^\dagger\mathcal{A} = 0$;

(vi) $aa^\dagger + a^\dagger a \in \mathcal{A}^{-1}$ *and* $a\mathcal{A} \cap a^\dagger\mathcal{A} = 0$;

(vii) $\lambda a + \mu a^\dagger \in \mathcal{A}^{-1}$ *and exists and idempotent k such that* $ak = a$, $a^\dagger k = 0$;

(viii) $aa^\dagger + a^\dagger a \in \mathcal{A}^{-1}$ *and* $\mathcal{A}a \cap \mathcal{A}a^\dagger = 0$;

(ix) $\lambda a + \mu a^\dagger \in \mathcal{A}^{-1}$ *and* $\mathcal{A}a \cap \mathcal{A}a^\dagger = 0$.

Proof. (i) $\Leftrightarrow$ (ii). According to [22, Theorem 3.2], statement (i) is equivalent to $\mathcal{A} = aa^\dagger\mathcal{A} \oplus a^\dagger a\mathcal{A} = a\mathcal{A} \oplus a^\dagger\mathcal{A}$ and $\mathcal{A} = \mathcal{A}aa^\dagger \oplus \mathcal{A}a^\dagger a = \mathcal{A}a^\dagger \oplus \mathcal{A}a$.
(ii) $\Rightarrow$ (iii). According to [22, Lemma 2.1], $\mathcal{A} = (1 - aa^\dagger)\mathcal{A} \oplus (1 - a^\dagger a)\mathcal{A}$. Since $A = aa^\dagger\mathcal{A} \oplus a^\dagger a\mathcal{A}$, according to the proof of Proposition 2.1(i)-(ii),

$$L_{\lambda a+\mu a^\dagger} \colon (1 - aa^\dagger)\mathcal{A} \oplus (1 - a^\dagger a)\mathcal{A} \to aa^\dagger\mathcal{A} \oplus a^\dagger a\mathcal{A},$$

has the following matricial form: $\begin{pmatrix} L_{\lambda a} & 0 \\ 0 & L_{\mu a^\dagger} \end{pmatrix}$. Therefore, according to Proposition 2.1(iii)-(v), $L_{\lambda a+\mu a^\dagger}$ is invertible, equivalently, $\lambda a+\mu a^\dagger \in A^{-1}$. The remaining identity is clear.
(iii) $\Rightarrow$ (iv). Let $x = (\lambda a + \mu a^\dagger)^{-1}$. Since $1 = a(\lambda x) + a^\dagger(\mu x)$, $\mathcal{A} = a\mathcal{A} + a^\dagger\mathcal{A}$ and by hypothesis, $\mathcal{A} = a\mathcal{A} \oplus a^\dagger\mathcal{A}$. Now, (iv) follows from [22, Lemma 2.1].
(iv) $\Rightarrow$ (ii). As in the previous implication, the condition $\lambda a + \mu a^\dagger \in \mathcal{A}^{-1}$ implies $\mathcal{A} = a\mathcal{A} + a^\dagger\mathcal{A} = \mathcal{A}a + \mathcal{A}a^\dagger$. Consider any $x \in a\mathcal{A} \cap a^\dagger A$. There exist $u, v \in \mathcal{A}$ such that $x = au = a^\dagger v$, which leads to $hau = ha^\dagger v$, hence $au = 0$, and thus, $x = 0$. It has been proved that $a\mathcal{A} \cap a^\dagger A = 0$. Now consider $y \in \mathcal{A}a \cap \mathcal{A}a^\dagger$, i.e., $y = wa = ta^\dagger$ for some $w, t \in \mathcal{A}$. Since $\lambda a + \mu a^\dagger \in \mathcal{A}^{-1}$, exists $z \in \mathcal{A}$ such that $(\lambda a + \mu a^\dagger)z = 1$ (in fact, z is the standard inverse of $\lambda a + \mu a^\dagger$). Now, $h = h(\lambda a + \mu a^\dagger)z = \lambda az$, and thus $\mu a^\dagger z = 1 - h$. Since $yz = waz = ta^\dagger z$, $yz = \lambda^{-1}wh = \mu^{-1}t(1 - h)$, which by postmultiplying by h leads to $yz = 0$. By using $z \in \mathcal{A}^{-1}$, $y = 0$. As a result, $\mathcal{A}a \cap \mathcal{A}a^\dagger = 0$.
(iii) $\Rightarrow$ (v). Clear.
(v) $\Rightarrow$ (iii). If $L_{\lambda a+\mu a^\dagger}$ and $R_{\lambda a+\mu a^\dagger} \in \mathcal{L}(\mathcal{A})$ are right invertible, then $\lambda a + \mu a^\dagger \in \mathcal{A}^{-1}$.
(ii) $\Rightarrow$ (vi). Recall that $\mathcal{A} = (1 - aa^\dagger)\mathcal{A} \oplus (1 - a^\dagger a)\mathcal{A}$ ([22, Lemma 2.1]) and consider $L_{aa^\dagger+a^\dagger a} \colon \mathcal{A} \to \mathcal{A}$. Let $x \in \mathcal{A}$ such that $(aa^\dagger + a^\dagger a)x = 0$. As a result, $aa^\dagger x \in a\mathcal{A} \cap a^\dagger\mathcal{A} = 0$. Similarly, $a^\dagger ax = 0$. Consequently, $x \in (aa^\dagger)^{-1}(0) \cap (a^\dagger a)^{-1}(0) = (1 - aa^\dagger)\mathcal{A} \cap (1 - a^\dagger a)\mathcal{A} = 0$, i.e., $\mathcal{N}(L_{aa^\dagger+a^\dagger a}) = 0$. On the other hand, given $x \in \mathcal{A}$, there exist y, $z \in \mathcal{A}$ such that $x = (1 - aa^\dagger)y + (1 - a^\dagger a)z$. Then,

$$aa^\dagger x = aa^\dagger(1 - aa^\dagger)y + aa^\dagger(1 - a^\dagger a)z = aa^\dagger(1 - a^\dagger a)z = (aa^\dagger + a^\dagger a)(1 - a^\dagger a)z.$$

As a result, $aa^\dagger\mathcal{A} \subseteq \mathcal{R}(L_{aa^\dagger+a^\dagger a})$. Similarly,

$$a^\dagger ax = a^\dagger a(1 - aa^\dagger)y + a^\dagger a(1 - a^\dagger a)z = a^\dagger a(1 - aa^\dagger)y = (aa^\dagger + a^\dagger a)(1 - aa^\dagger)y,$$

which implies that $a^\dagger a\mathcal{A} \subseteq \mathcal{R}(L_{aa^\dagger+a^\dagger a})$. Since $A = aa^\dagger\mathcal{A} \oplus a^\dagger a\mathcal{A}$, $\mathcal{R}(L_{aa^\dagger+a^\dagger a}) = \mathcal{A}$. Therefore, $aa^\dagger + a^\dagger a \in \mathcal{A}^{-1}$.
(vi) $\Rightarrow$ (ii). If $aa^\dagger + a^\dagger a \in \mathcal{A}^{-1}$, then $\mathcal{A} = a\mathcal{A} + a^\dagger\mathcal{A}$ and $\mathcal{A} = \mathcal{A}a + \mathcal{A}a^\dagger$. By hypothesis, $\mathcal{A} = a\mathcal{A} \oplus a^\dagger A$. Now, [22, Lemma 2.1] assures the existence of an idempotent h such that $ha = a$ and $ha^\dagger = 0$. To prove (ii), it is sufficient to prove $\mathcal{A}a \cap \mathcal{A}a^\dagger = 0$ (this is rather similar to the proof of (iv) $\Rightarrow$ (ii)). If $y \in \mathcal{A}a \cap \mathcal{A}a^\dagger$, there exist $w, t \in \mathcal{A}$ such that $y = wa = ta^\dagger$. Let $z = (aa^\dagger + a^\dagger a)^{-1}$. Since $h = h(aa^\dagger z + a^\dagger az) = aa^\dagger z$ and $(aa^\dagger + a^\dagger a)z = 1$, $a^\dagger az = 1 - h$. Therefore, $a^\dagger h = a^\dagger z$ and $az = a(1 - h)$. Since $y = wa = ta^\dagger$, $yz = waz = ta^\dagger z$, and thus $yz = wa(1 - h) = ta^\dagger h$. Thus, $yz = 0$ and using the invertibility of z, $y = 0$.

(i) $\Leftrightarrow$ (vi), (vii), (viii) or (ix). Condition (i) is invariant under the reversal of the algebra multiplication. Precisely speaking, apply each of these conditions to the algebra $(\mathcal{A}, \circ)$, where $x \circ y = yx$, and then reinterpret the results in the original algebra. Conditions (iii), (iv), (vi) and (ii) yield that (i) is equivalent to any of the conditions (vi), (vii), (viii) or (ix) respectively. $\square$

The next result deals with a weaker condition than the invertibility of $aa^\dagger + a^\dagger a$ when $a \in \mathcal{A}^\dagger$. It is noteworthy that $aa^\dagger - a^\dagger a \in \mathcal{A}^{-1}$ implies $aa^\dagger + a^\dagger a \in \mathcal{A}^{-1}$ (see Theorem 2.2 or [21, Theorem 3.5]). Also, in [21, Theorem 3.3] the authors characterized the invertibility of $aa^\dagger + a^\dagger a$. Furthermore, for $b \in \mathcal{A}$, observe that L_b is injective if and only if the condition $bx = 0$ implies $x = 0$ when $x \in \mathcal{A}$ (i.e., b satisfies a kind of left cancellation property).

Proposition 2.3. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}^\dagger$. The following statements are equivalent.*

(i) *$L_{aa^\dagger + a^\dagger a}$ is injective.*
(ii) *$a^\dagger a\mathcal{A} \cap aa^\dagger(1 - a^\dagger a)\mathcal{A} = 0$ and $a^{-1}(0) \cap (a^\dagger)^{-1}(0) = 0$.*

Proof. (i) $\Rightarrow$ (ii). Let $x \in a^\dagger a\mathcal{A} \cap aa^\dagger(1 - a^\dagger a)\mathcal{A}$. There exist $u, v \in \mathcal{A}$ such that

$$x = a^\dagger u = aa^\dagger(1 - a^\dagger a)v. \tag{2.1}$$

Now,

$$(aa^\dagger+a^\dagger a)x = aa^\dagger x+a^\dagger ax = aa^\dagger aa^\dagger(1-a^\dagger a)v+a^\dagger aa^\dagger u = aa^\dagger(1-a^\dagger a)v+a^\dagger u = 2x.$$

Hence

$$2(aa^\dagger + a^\dagger a)(1 - a^\dagger a)v = 2aa^\dagger(1 - a^\dagger a)v = 2x = (aa^\dagger + a^\dagger a)x.$$

The hypothesis implies $2(1 - a^\dagger a)v = x$, and thus $2aa^\dagger(1 - a^\dagger a)v = aa^\dagger x$. By using (2.1), $2x = x$, hence $x = 0$.

Let $y \in a^{-1}(0)\cap(a^\dagger)^{-1}(0)$, i.e., $ay = a^\dagger y = 0$. It is evident that $(aa^\dagger+a^\dagger a)y = 0$, and the hypothesis leads to $y = 0$.

(ii) $\Rightarrow$ (i). It is evident that $L_{aa^\dagger+a^\dagger a}((1 - aa^\dagger)\mathcal{A}) \subseteq a^\dagger\mathcal{A}$ and $L_{aa^\dagger+a^\dagger a}((1 - a^\dagger a)\mathcal{A}) \subseteq a\mathcal{A}$. Thus, if L_1 and L_2 are the restrictions of $L_{aa^\dagger+a^\dagger a}$ to $(1 - aa^\dagger)\mathcal{A}$ and $(1 - a^\dagger a)\mathcal{A}$ respectively, then

$$L_1 : (1 - aa^\dagger)\mathcal{A} \to a^\dagger\mathcal{A} \qquad \text{and} \qquad L_2 : (1 - a^\dagger a)\mathcal{A} \to a\mathcal{A}.$$

Let $x \in \mathcal{N}(L_{aa^\dagger+a^\dagger a})$. Since $x = x - aa^\dagger x - a^\dagger ax = (1 - aa^\dagger)x + (1 - a^\dagger a)x - x$, if $u = (1 - aa^\dagger)x$ and $v = (1 - a^\dagger a)x$, then $2x = u + v$, and thus $0 = L_1(u) + L_2(v)$. Moreover,

$$L_2(v) = (aa^\dagger + a^\dagger a)(1 - a^\dagger a)x = aa^\dagger(1 - a^\dagger a)x \in aa^\dagger(1 - a^\dagger a)\mathcal{A}$$

and

$$L_2(v) = -L_1(u) \in a^\dagger\mathcal{A} = a^\dagger a\mathcal{A}.$$

By hypothesis, $L_2(v) = 0$, and thus $L_1(u) = 0$. Therefore,

$$0 = (aa^\dagger + a^\dagger a)v = (aa^\dagger + a^\dagger a)(1 - a^\dagger a)x = aa^\dagger(1 - a^\dagger a)x = aa^\dagger v,$$

which by a premultiplication by $a^\dagger$ leads to $0 = a^\dagger v$. In other words, $v \in (a^\dagger)^{-1}(0)$. It is evident that $av = a(1 - a^\dagger a)x = 0$. Thus, $v \in a^{-1}(0) \cap (a^\dagger)^{-1}(0) = 0$. In a similar way it is possible to prove that $u = 0$. Since $2x = u + v$, $x = 0$. $\square$

In the following proposition the surjectivity of the operator $L_{aa^\dagger+a^\dagger a}\colon \mathcal{A}\to\mathcal{A}$ will be characterized.

Proposition 2.4. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a\in\mathcal{A}^\dagger$. Then, the operator $L_{aa^\dagger+a^\dagger a}\in\mathcal{L}(\mathcal{A})$ is surjective if and only if $aa^\dagger+a^\dagger a$ is regular and $(aa^\dagger+a^\dagger a)_{-1}(0)=0$.*

Proof. Assume that $L_{aa^\dagger+a^\dagger a}$ is surjective. Since $1\in L_{aa^\dagger+a^\dagger a}(\mathcal{A})$, there exists $u\in\mathcal{A}$ such that $(aa^\dagger+a^\dagger a)u=1$. This, obviously, implies that $aa^\dagger+a^\dagger a$ is regular. Also, if $x\in\mathcal{A}$ satisfies $0=x(aa^\dagger+a^\dagger x)$, then by a postmultiplication by u, $0=x$.

To prove the converse, consider $b\in\mathcal{A}$ such that $(aa^\dagger+a^\dagger)b(aa^\dagger+a^\dagger a)=aa^\dagger+a^\dagger a$. Since $[(aa^\dagger+a^\dagger a)b-1](aa^\dagger+a^\dagger a)=0$, $(aa^\dagger+a^\dagger a)b-1\in(aa^\dagger+a^\dagger a)_{-1}(0)=0$. Hence $(aa^\dagger+a^\dagger a)b=1$. Now, if $y\in\mathcal{A}$, then $L_{aa^\dagger+a^\dagger a}(by)=y$. This proves the surjectivity of $L_{aa^\dagger+a^\dagger a}$. □

As a corollary of Propositions 2.3 and 2.4, conditions that characterize the invertibility of $aa^\dagger+a^\dagger a$ will be given.

Theorem 2.5. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a\in\mathcal{A}^\dagger$. Then, necessary and sufficient for $aa^\dagger+a^\dagger a\in\mathcal{A}^{-1}$ is that $aa^\dagger+a^\dagger a$ is regular, $(aa^\dagger+a^\dagger a)_{-1}(0)=0$, $a^\dagger a\mathcal{A}\cap aa^\dagger(1-a^\dagger a)\mathcal{A}=0$ and $a^{-1}(0)\cap(a^\dagger)^{-1}(0)=0$.*

Proof. Apply Propositions 2.3 and 2.4. □

Next co-EP Banach space operators will be characterized.

Proposition 2.6. *Let $\mathcal{X}$ be a Banach space, $T\in\mathcal{L}(\mathcal{X})$ be Moore–Penrose invertible, and $\lambda,\mu\in\mathbb{C}\setminus\{0\}$. The following statements are equivalent.*

(i) *T is co-EP;*
(ii) *$\lambda T+\mu T^\dagger\in\mathcal{L}(\mathcal{X})$ is invertible and $\mathcal{R}(T)\cap\mathcal{R}(T^\dagger)=0$;*
(iii) *$\lambda T+\mu T^\dagger\in\mathcal{L}(\mathcal{X})$ is invertible and there exists an idempotent $P\in\mathcal{L}(\mathcal{X})$ such that $\mathcal{R}(T)\subseteq\mathcal{R}(P)$ and $\mathcal{R}(T^\dagger)\subseteq\mathcal{N}(P)$;*
(iv) *$TT^\dagger+T^\dagger T\in\mathcal{L}(\mathcal{X})$ is invertible and $\mathcal{R}(T)\cap\mathcal{R}(T^\dagger)=0$;*
(v) *$\lambda T+\mu T^\dagger\in\mathcal{L}(\mathcal{X})$ is invertible and there exists an idempotent $Q\in\mathcal{L}(\mathcal{X})$ such that $\mathcal{R}(I-Q)\subseteq\mathcal{N}(T)$ and $\mathcal{R}(Q)\subseteq\mathcal{N}(T^\dagger)$;*
(vi) *$TT^\dagger+T^\dagger T\in\mathcal{L}(\mathcal{X})$ is invertible and $\mathcal{R}(I-TT^\dagger)\cap\mathcal{R}(I-T^\dagger T)=0$;*
(vii) *$\lambda T+\mu T^\dagger\in\mathcal{L}(\mathcal{X})$ is invertible and $\mathcal{R}(I-TT^\dagger)\cap\mathcal{R}(I-T^\dagger T)=0$.*

Proof. Observe that under any of the hypotheses we have $\mathcal{L}(\mathcal{X})T+\mathcal{L}(\mathcal{X})T^\dagger=\mathcal{L}(\mathcal{X})$ and $T\mathcal{L}(\mathcal{X})+T^\dagger\mathcal{L}(\mathcal{X})=\mathcal{L}(\mathcal{X})$. Note that according to [22, Lemma 5.1], $TT^\dagger\mathcal{L}(\mathcal{X})\cap T^\dagger T\mathcal{L}(\mathcal{X})=T\mathcal{L}(\mathcal{X})\cap T^\dagger\mathcal{L}(\mathcal{X})=0$ if and only if $\mathcal{R}(T)\cap\mathcal{R}(T^\dagger)=0$. Moreover, according to [22, Lemma 5.1] and [22, Lemma 2.1], an equivalent condition for $\mathcal{L}(\mathcal{X})TT^\dagger\cap\mathcal{L}(\mathcal{X})T^\dagger T=\mathcal{L}(\mathcal{X})T^\dagger\cap\mathcal{L}(\mathcal{X})T=0$ is that $\mathcal{R}(I-TT^\dagger)\cap\mathcal{R}(I-T^\dagger T)=0$. To conclude the proof, apply Theorem 2.2. □

To end this section, co-EP Banach space operators defined on finite dimensional Banach spaces will be considered.

Remark 2.7. Let $\mathcal{X}$ be a finite dimensional Banach space and consider $T \in \mathcal{L}(\mathcal{X})$. If T is co-EP, then $\dim \mathcal{X} = 2 \dim \mathcal{R}(T) = 2 \dim \mathcal{N}(T)$. In fact, since T is Moore–Penrose invertible, $\dim \mathcal{N}(T) + \dim \mathcal{R}(T^\dagger) = \dim \mathcal{X} = \dim \mathcal{N}(T^\dagger) + \dim \mathcal{R}(T)$. In addition, since $\dim \mathcal{N}(T) + \dim \mathcal{R}(T) = \dim \mathcal{X}$, $\dim \mathcal{N}(T) = \dim \mathcal{N}(T^\dagger)$ and $\dim \mathcal{R}(T) = \dim \mathcal{R}(T^\dagger)$. Now well, according to [22, Theorem 5.2], $\dim \mathcal{X} = 2 \dim \mathcal{R}(T) = 2 \dim \mathcal{N}(T)$.

Proposition 2.8. *Let $\mathcal{X}$ be a finite dimensional Banach space and consider $T \in \mathcal{L}(\mathcal{X})$ such that T is Moore–Penrose invertibe. The following statements are equivalent.*

(i) *T is co-EP;*
(ii) *$\mathcal{R}(T) \cap \mathcal{R}(T^\dagger) = 0$ and $\mathcal{N}(T) \cap \mathcal{N}(T^\dagger) = 0$;*
(iii) *$\mathcal{X} = \mathcal{R}(T) + \mathcal{R}(T^\dagger)$ and $\mathcal{X} = \mathcal{N}(T) + \mathcal{N}(T^\dagger)$.*

Proof. (i) $\Rightarrow$ (ii). According to Proposition 2.6, $\mathcal{R}(T) \cap \mathcal{R}(T^\dagger) = 0$. In addition, since $T - T^\dagger \in \mathcal{L}(\mathcal{X})$ is invertible, $\mathcal{N}(T) \cap \mathcal{N}(T^\dagger) = 0$.
(ii) $\Rightarrow$ (i). Let $x \in \mathcal{N}(TT^\dagger - T^\dagger T)$, i.e., $TT^\dagger(x) = T^\dagger T(x)$. Thus, $z = TT^\dagger(x) = T^\dagger T(x) \in \mathcal{R}(T) \cap \mathcal{R}(T^\dagger) = 0$. As a result, $x \in \mathcal{N}(T) \cap \mathcal{N}(T^\dagger) = 0$.
(i) $\Rightarrow$ (iii). Apply [22, Theorem 5.2].
(iii) $\Rightarrow$ (ii). Since $T^\dagger$ is Moore–Penrose invertible, $\dim \mathcal{N}(T) = \dim \mathcal{N}(T^\dagger)$ and $\dim \mathcal{R}(T) = \dim \mathcal{R}(T^\dagger)$. Let $n = \dim \mathcal{X}$, $r = \dim \mathcal{R}(T)$, $s_1 = \dim[\mathcal{R}(T) \cap \mathcal{R}(T^\dagger)]$ and $s_2 = \dim[\mathcal{N}(T) \cap \mathcal{N}(T^\dagger)]$. According to the hypotheses, $n = 2r - s_1$ and $n = 2(n-r) - s_2$. Hence $s_1 + s_2 = 0$, which leads to $s_1 = s_2 = 0$. $\square$

3. Hermitian co-EP elements

Given a unital Banach algebra $\mathcal{A}$, if $a \in \mathcal{A}$ is co-EP, then according to [22, Theorem 3.2], there exist $h, k \in \mathcal{A}$, $h = h^2$ and $k = k^2$, such that $h\mathcal{A} = a\mathcal{A}$, $(1-h)\mathcal{A} = a^\dagger\mathcal{A}$, $\mathcal{A}k = \mathcal{A}a^\dagger$ and $\mathcal{A}(1-k) = \mathcal{A}a$ (the idempotents h and k are unique). Next a particular class of co-EP elements, the ones that their idempotents h and k are hermitian, will be introduced.

Definition 3.1. Let $\mathcal{A}$ be a unital Banach algebra and let $a \in \mathcal{A}$ such that a is co-EP. Then, a will be said hermitian co-EP if the above considered idempotent h is hermitian.

To characterize hermitian co-EP Banach algebra elements some preparation is needed.

Remark 3.2. Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}$ a Moore–Penrose invertible element.
(i) Note that $a \in \mathcal{A}$ is hermitian co-EP if and only if $a^\dagger$ is hermitian co-EP.
(ii) Suppose that $a \in \mathcal{A}$ is co-EP and consider the aforementioned idempotents h, $k \in A$. In particular, $ha^\dagger = 0$, $ha = a$, $a^\dagger k = a^\dagger$ and $ak = 0$. Hence $h(aa^\dagger - a^\dagger a) = aa^\dagger$ and $(aa^\dagger - a^\dagger a)k = aa^\dagger$, which implies that $h = aa^\dagger(aa^\dagger - a^\dagger a)^{-1}$ and $k = (aa^\dagger - a^\dagger a)^{-1} aa^\dagger$.

In the following theorem hermitian co-EP Banach algebra elements will be characterized. Compare with [2, Theorem 2.9 and Corollary 2.10], [3, Section 3] and [5, Section 4]. Note that given a unital Banach algebra $\mathcal{A}$, $a \in \mathcal{A}$ will be said to be *bi-EP*, if the idempotents $aa^\dagger$ and $a^\dagger a$ commute, see for example [14].

Theorem 3.3. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}^\dagger$. Then, the following statements are equivalent.*
(i) *a is co-EP and h is hermitian;*
(ii) *a is co-EP and $h = aa^\dagger$;*
(iii) *$a^\dagger a + aa^\dagger = 1$;*
(iv) *$a\mathcal{A} = a^{-1}(0)$;*
(v) *$\mathcal{A}a = a_{-1}(0)$;*
(vi) *a is co-EP and $k = aa^\dagger$;*
(vii) *a is co-EP and k is hermitian;*
(viii) *a is co-EP and $k = h$;*
(ix) *a is co-EP and $a^2 = 0$;*
(x) *$a^\dagger\mathcal{A} = (a^\dagger)^{-1}(0)$;*
(xi) *$\mathcal{A}a^\dagger = (a^\dagger)_{-1}(0)$;*
(xii) *a is co-EP and bi-EP.*

Proof. (i) $\Rightarrow$ (ii). Consider L_h, $L_{aa^\dagger} \in \mathcal{L}(\mathcal{A})$. Since $h\mathcal{A} = a\mathcal{A} = aa^\dagger\mathcal{A}$, $\mathcal{R}(L_h) = \mathcal{R}(L_{aa^\dagger})$. Now well, since h and $aa^\dagger$ are hermitian, according to the proof of [6, Theorem 5 (ii)], L_h and $L_{aa^\dagger}$ are hermitian idempotents in $\mathcal{L}(\mathcal{A})$. Therefore, according to [28, Theorem 2.2], $h = aa^\dagger$.
(ii) $\Rightarrow$ (iii). Since $(1-h)\mathcal{A} = a^\dagger\mathcal{A} = a^\dagger a\mathcal{A}$, $\mathcal{R}(L_{1-h}) = \mathcal{R}(L_{a^\dagger a})$, where L_{1-h}, $L_{a^\dagger a} \in \mathcal{L}(\mathcal{A})$. However, according to the hypothesis and to the proof of [6, Theorem 5 (ii)], L_{1-h} and $L_{a^\dagger a}$ are hermitian idempotents in $\mathcal{L}(\mathcal{A})$. Consequently, according to [28, Theorem 2.2], $1 - aa^\dagger = 1 - h = a^\dagger a$.
(iii) $\Rightarrow$ (iv). According to the hypothesis,

$$a\mathcal{A} = aa^\dagger\mathcal{A} = (1 - a^\dagger a)\mathcal{A} = (a^\dagger a)^{-1}(0) = a^{-1}(0).$$

(iv) $\Rightarrow$ (v). Since $aa^\dagger$ and $a^\dagger a$ are idempotents and

$$aa^\dagger\mathcal{A} = a\mathcal{A} = a^{-1}(0) = (a^\dagger a)^{-1}(0) = (1 - a^\dagger a)\mathcal{A},$$

it is not dificult to prove that

$$aa^\dagger(1 - a^\dagger a) = 1 - a^\dagger a, \quad (1 - a^\dagger a)aa^\dagger = aa^\dagger,$$

which in turn implies that

$$a_{-1}(0) = (aa^\dagger)_{-1}(0) = (1 - a^\dagger a)_{-1}(0) = \mathcal{A}a^\dagger a = \mathcal{A}a.$$

(v) $\Rightarrow$ (vi). Since $(aa^\dagger - 1)a = 0$, $aa^\dagger - 1 \in a_{-1}(0) = \mathcal{A}a$. Hence, there exists $u \in \mathcal{A}$ such that $aa^\dagger - 1 = ua$, and thus, $1 = (-u)a + aa^\dagger \in \mathcal{A}a + \mathcal{A}a^\dagger$, which proves $\mathcal{A} = \mathcal{A}a + \mathcal{A}a^\dagger$. Now it will be proved that $\mathcal{A}a \cap \mathcal{A}a^\dagger = 0$. To this end, note that the hypothesis implies that $a^2 = 0$. If $x \in \mathcal{A}a \cap \mathcal{A}a^\dagger$, then there exist $y, z \in \mathcal{A}$ such that $x = ya = za^\dagger$. So, $0 = ya^2a^\dagger = za^\dagger aa^\dagger = za^\dagger = x$. Thus, $\mathcal{A} = \mathcal{A}a \oplus \mathcal{A}a^\dagger$.

Using an argument similar to the one in the proof of (iv) $\Rightarrow$ (v), it is not difficult to prove that $\mathcal{A}a^\dagger a = \mathcal{A}(1 - aa^\dagger)$, which leads to $a^\dagger a(1 - aa^\dagger) = a^\dagger a$ and $(1 - aa^\dagger)a^\dagger a = 1 - aa^\dagger$. In particular, $a^{-1}(0) = a\mathcal{A}$. As in the previous paragraph it is possible to prove that $\mathcal{A} = a\mathcal{A} \oplus a^\dagger\mathcal{A}$. By Theorem 2.2, a is co-EP.

Since $\mathcal{A}aa^\dagger = \mathcal{A}a^\dagger$, $\mathcal{A}(1-aa^\dagger) = a_{-1}(0) = \mathcal{A}a$ and the idempotent k is unique, $k = aa^\dagger$.
(vi) $\Rightarrow$ (vii). Clear
(vii) $\Rightarrow$ (viii). Since $\mathcal{A}k = \mathcal{A}a^\dagger = \mathcal{A}aa^\dagger$, an argument similar to the one used to prove that statement (i) implies statement (ii) proves that $k = aa^\dagger$. In addition, since k is hermitian and $\mathcal{A}(1-k) = \mathcal{A}a = \mathcal{A}a^\dagger a$, according [28, Theorem 2.2], $1 - aa^\dagger = 1-k = a^\dagger a$, equivalently, $aa^\dagger + a^\dagger a = 1$. Since $aa^\dagger\mathcal{A} = a\mathcal{A}$, $(1-aa^\dagger)\mathcal{A} = a^\dagger a\mathcal{A} = a^\dagger\mathcal{A}$ and the idempotent h is unique, $h = aa^\dagger = k$.
(viii) $\Rightarrow$ (ix). Since $a \in a\mathcal{A} = h\mathcal{A} = k\mathcal{A}$ and $a \in \mathcal{A}a = \mathcal{A}(1-k)$, there exist $u, v \in \mathcal{A}$ such that $a = ku$ and $a = v(1-k)$. Hence $a^2 = v(1-k)ku = 0$.
(ix) $\Rightarrow$ (iv). Since $a^2 = 0$, $a\mathcal{A} \subseteq a^{-1}(0)$. Let $x \in a^{-1}(0)$. Since a is co-EP, from Theorem 2.2, such x can be written as $x = au + a^\dagger v$. Thus, $0 = ax = a^2u + aa^\dagger v = aa^\dagger v$, hence $0 = a^\dagger v$. So, $x = au \in a\mathcal{A}$.
(iv) $\Rightarrow$ (i). Since (iv) $\Rightarrow$ (vii) $\Rightarrow$ (viii), h is hermitian.
(i) $\Leftrightarrow$ (x) (respectively (xi)). Since a is hermitian co-EP if and only if $a^\dagger$ is hermitian co-EP, the equivalence between statments (i) and (x) (respectively between statments (i) and (xi)) can be derived applying the Theorem to $a^\dagger$ using the equivalence between statments (i) and (iv) (respectively between statments (i) and (v)).
(viii) $\Leftrightarrow$ (xii) Note that according to Remark 3.2 (ii),

$$\begin{aligned} h = k \quad &\Longleftrightarrow \quad aa^\dagger(aa^\dagger - a^\dagger a)^{-1} = (aa^\dagger - a^\dagger a)^{-1}aa^\dagger \\ &\Longleftrightarrow \quad (aa^\dagger - a^\dagger a)aa^\dagger = aa^\dagger(aa^\dagger - a^\dagger a) \\ &\Longleftrightarrow \quad (aa^\dagger)(a^\dagger a) = (a^\dagger a)(aa^\dagger). \end{aligned}$$

□

Remark 3.4. Let $\mathcal{A}$ be a unital Banach algebra. Recall that an element $a \in \mathcal{A}^\dagger$ is said EP when $aa^\dagger - a^\dagger a = 0$ and a is said co-EP when $aa^\dagger - a^\dagger a \in \mathcal{A}^{-1}$. What is more, according to Theorem 3.3, $a \in \mathcal{A}$ is hermitian co-EP if $a^\dagger a + aa^\dagger = 1$. However, the situation $aa^\dagger - a^\dagger a = 1$ is impossible. In fact, assume $aa^\dagger - a^\dagger a = 1$. By postmultiplying it by a one gets $a^\dagger a^2 = 0$, hence $a^2 = 0$. By premultiplying $aa^\dagger - a^\dagger a = 1$ by a one has $a = 0$, which is unfeasible in view of $aa^\dagger - a^\dagger a = 1$.

4. Condition $(\mathcal{P})$ and the perturbations of co-EP elements

In this section the condition $(\mathcal{P})$ will be introduced and the perturbations of co-EP elements will be studied. The condition $(\mathcal{P})$ resembles the condition $(\mathcal{W})$ studied for perturbations of Drazin invertible elements ([33], [37]).

Let $a \in \mathcal{A}$ be Moore–Penrose invertible. The element $b \in \mathcal{A}$ will be said that *obeys the condition* $(\mathcal{P})$ *at* a if

$$b - a = aa^\dagger(b-a)a^\dagger a \quad \text{and} \quad \|a^\dagger(b-a)\| < 1.$$

Note that the condition

$$b - a = aa^\dagger(b-a)a^\dagger a$$

is equivalent to

$$b - a = aa^\dagger(b-a) = (b-a)a^\dagger a. \tag{4.1}$$

Basic auxiliary results are summarized in the following lemma.

Lemma 4.1. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}$, a Moore–Penrose invertible element. If $b \in \mathcal{A}$ obeys the condition $\mathcal{P}$ at a. Then*
(i) $b = a(1 + a^\dagger(b - a))$;
(ii) $b = (1 + (b - a)a^\dagger)a$;
(iii) $1 + a^\dagger(b - a)$ *and* $1 + (b - a)a^\dagger$ *are invertible, and*

$$(1 + a^\dagger(b - a))^{-1}a^\dagger = a^\dagger(1 + (b - a)a^\dagger)^{-1}. \tag{4.2}$$

Proof. To prove (i) and (ii) observe that by (4.1)

$$b = a + (b - a) = a + aa^\dagger(b - a) = a(1 + a^\dagger(b - a))$$

and

$$b = a + (b - a) = a + (b - a)a^\dagger a = (1 + (b - a)a^\dagger)a.$$

Clearly, the condition $\|a^\dagger(b - a)\| < 1$ implies that $1 + a^\dagger(b - a)$ is invertible, and so $1 + (b - a)a^\dagger$ is invertible. Finally, (4.2) follows by direct verification. □

Now the main results of this section will be proved.

Theorem 4.2. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}$, a Moore–Penrose invertible element. If $b \in \mathcal{A}$ obeys the condition $\mathcal{P}$ at a, then b is Moore–Penrose invertible, $bb^\dagger = aa^\dagger$, $b^\dagger b = a^\dagger a$ and $b^\dagger = (1 + a^\dagger(b - a))^{-1}a^\dagger = a^\dagger(1 + (b - a)a^\dagger)^{-1}$.*

Proof. By Lemma 4.1 (iii), $1 + a^\dagger(b - a)$ and $1 + (b - a)a^\dagger$ are invertible and

$$(1 + a^\dagger(b - a))^{-1}a^\dagger = a^\dagger(1 + (b - a)a^\dagger)^{-1}.$$

Set $\tilde{b} = (1 + a^\dagger(b - a))^{-1}a^\dagger = a^\dagger(1 + (b - a)a^\dagger)^{-1}$. Then, b is Moore–Penrose invertible and that $b^\dagger = \tilde{b}$. In fact, by Lemma 4.1 (i),

$$b\tilde{b} = a(1 + a^\dagger(b - a))(1 + a^\dagger(b - a))^{-1}a^\dagger = aa^\dagger \tag{4.3}$$

and by Lemma 4.1 (ii),

$$\tilde{b}b = a^\dagger(1 + (b - a)a^\dagger)^{-1}(1 + (b - a)a^\dagger)a = a^\dagger a. \tag{4.4}$$

Hence, by (4.3) and Lemma 4.1 (i),

$$b\tilde{b}b = (aa^\dagger)b = (aa^\dagger a)(1 + a^\dagger(b - a)) = a(1 + a^\dagger(b - a)) = b. \tag{4.5}$$

Furthermore, by (4.4),

$$\tilde{b}b\tilde{b} = (a^\dagger a)a^\dagger(1 + (b - a)a^\dagger)^{-1} = (a^\dagger aa^\dagger)(1 + (b - a)a^\dagger)^{-1} = a^\dagger(1 + (b - a)a^\dagger)^{-1} = \tilde{b}. \tag{4.6}$$

Finally, by (4.3), (4.4), (4.5) and (4.6), b is Moore–Penrose invertible with $b^\dagger = \tilde{b}$. □

The following corollary consists in a direct application of Theorem 4.2.

Corollary 4.3. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}^{EP}_{co}$. If $b \in \mathcal{A}$ obeys the condition $\mathcal{P}$ at a, then $b \in \mathcal{A}^{EP}_{co}$, $bb^\dagger = aa^\dagger$ and $b^\dagger b = a^\dagger a$.*

Corollary 4.4. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}$, a Moore–Penrose invertible element. If $b \in \mathcal{A}$ obeys the condition $\mathcal{P}$ at a, then b is Moore–Penrose invertible and*

$$\frac{\|b^\dagger - a^\dagger\|}{\|a^\dagger\|} \leq \frac{\|a^\dagger(b-a)\|}{1 - \|a^\dagger(b-a)\|}. \tag{4.7}$$

Proof. By Theorem 4.2

$$\begin{aligned} b^\dagger - a^\dagger &= (1 + a^\dagger(b-a))^{-1}a^\dagger - a^\dagger \\ &= [(1 + a^\dagger(b-a))^{-1} - 1]a^\dagger \\ &= -(1 + a^\dagger(b-a))^{-1}(a^\dagger(b-a))a^\dagger. \end{aligned}$$

Hence

$$\begin{aligned} \|b^\dagger - a^\dagger\| &\leq \|(1 + a^\dagger(b-a))^{-1}\|\|a^\dagger(b-a)\|\|a^\dagger\| \\ &\leq \frac{\|a^\dagger(b-a)\|}{1 - \|a^\dagger(b-a)\|}\|a^\dagger\|, \end{aligned}$$

and (4.7) holds. □

Corollary 4.5. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}$, a Moore–Penrose invertible element. If $b \in \mathcal{A}$ obeys the condition $\mathcal{P}$ at a, then $b \in \mathcal{A}$ is Moore–Penrose invertible and*

$$\frac{\|a^\dagger\|}{1 + \|a^\dagger(b-a)\|} \leq \|b^\dagger\| \leq \frac{\|a^\dagger\|}{1 - \|a^\dagger(b-a)\|}. \tag{4.8}$$

Proof. By Theorem 4.2,

$$b^\dagger = (1 + a^\dagger(b-a))^{-1}a^\dagger,$$

hence

$$a^\dagger = (1 + a^\dagger(b-a))b^\dagger.$$

Now

$$\|b^\dagger\| \leq \|(1 + a^\dagger(b-a))^{-1}\|\|a^\dagger\| \leq \frac{\|a^\dagger\|}{1 - \|a^\dagger(b-a)\|},$$

$$\|a^\dagger\| \leq \|(1 + a^\dagger(b-a))\|\|b^\dagger\| \leq (1 + \|a^\dagger(b-a)\|)\|b^\dagger\|,$$

which proves (4.8). □

Corollary 4.6. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}$, a Moore–Penrose invertible element. If $b \in \mathcal{A}$ obeys the condition $\mathcal{P}$ at a and $\|a^\dagger(b-a)\| < 1/2$, then b is Moore–Penrose invertible and a obeys the condition $\mathcal{P}$ at b.*

Proof. By Theorem 4.2, $bb^\dagger = aa^\dagger$ and $b^\dagger b = a^\dagger a$. Hence $a - b = bb^\dagger(a-b)bb^\dagger$. Again, by Theorem 4.2,

$$\begin{aligned}
\|b^\dagger(a-b)\| &= \|(1 + a^\dagger(b-a))^{-1}a^\dagger(a-b)\| \\
&\leq \|(1 + a^\dagger(b-a))^{-1}\| \, \|a^\dagger(a-b)\| \\
&\leq \frac{\|a^\dagger(a-b)\|}{1 - \|a^\dagger(b-a)\|} < 1.
\end{aligned}$$

This completes the proof. □

Corollary 4.7. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}$, a Moore–Penrose invertible element. If $b \in \mathcal{A}$ obeys the condition $\mathcal{P}$ at a and $\|a^\dagger\|\|b-a\| < 1$, then b is Moore–Penrose invertible and*

$$\frac{\|b^\dagger - a^\dagger\|}{\|a^\dagger\|} \leq \frac{k_\dagger(a)\|b-a\|/\|a\|}{1 - k_\dagger(a)\|(b-a)\|/\|a\|},$$

where

$$k_\dagger(a) = \|a\|\|a^\dagger\|$$

is defined as the condition number with respect to the Moore–Penrose inverse.

To finish this section, some algebraic properties of the Moore–Penrose inverse will be considered. Recall that if $a, b \in \mathcal{A}$ are Moore–Penrose invertible, then ab is not necessary Moore–Penrose invertible, and if ab is Moore–Penrose invertible, then in general $(ab)^\dagger \neq b^\dagger a^\dagger$. In the next proposition, it will be shown that if b obeys the condition $\mathcal{P}$ at a, then it is possible to give precise characterizations.

Proposition 4.8. *Let $\mathcal{A}$ be a unital Banach algebra and consider $a \in \mathcal{A}$, a Moore–Penrose invertible element. If $b \in \mathcal{P}$ obey the condition $\mathcal{P}$ at a, then the following statements hold.*
(i) *$ab^\dagger$ has Moore–Penrose inverse and $(ab^\dagger)^\dagger = ba^\dagger$; moreover $a \in \mathcal{A}^{EP}_{co}$ implies $ab^\dagger \in \mathcal{A}^{EP}_{co}$.*
(ii) *$b^\dagger a$ has Moore–Penrose inverse and $(b^\dagger a)^\dagger = a^\dagger b$; moreover $a \in \mathcal{A}^{EP}_{co}$ implies $b^\dagger a \in \mathcal{A}^{EP}_{co}$.*
(iii) $ba^\dagger$ has Moore–Penrose inverse and $(ba^\dagger)^\dagger = ab^\dagger$; moreover $a \in \mathcal{A}^{EP}_{co}$ implies $ba^\dagger \in \mathcal{A}^{EP}_{co}$.
(iv) $a^\dagger b$ has Moore–Penrose inverse and $(a^\dagger b)^\dagger = b^\dagger a$; moreover $a \in \mathcal{A}^{EP}_{co}$ implies $a^\dagger b \in \mathcal{A}^{EP}_{co}$.

Proof. It is enough to prove (i) and (ii). To prove (i) recall that by Theorem 4.2 $bb^\dagger = aa^\dagger$. Hence

$$(ab^\dagger)(ba^\dagger) = a(b^\dagger b)a^\dagger = a(a^\dagger a)a^\dagger = aa^\dagger \tag{4.9}$$

and

$$(ba^\dagger)(ab^\dagger) = b(a^\dagger a)b^\dagger = b(b^\dagger b)b^\dagger = bb^\dagger = aa^\dagger. \tag{4.10}$$

Further,

$$(ab^\dagger)(ba^\dagger)(ab^\dagger) = aa^\dagger(ab^\dagger) = (aa^\dagger a)b^\dagger = ab^\dagger. \tag{4.11}$$

Finally,

$$(ba^\dagger)(ab^\dagger)(ba^\dagger) = (ba^\dagger)aa^\dagger = b(a^\dagger aa^\dagger) = ba^\dagger. \tag{4.12}$$

Now by (4.9), (4.10), (4.11) and (4.12), (i) holds. Statement (ii) can be proved similarly. □

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