std-convergence in fuzzy metric spaces

Valentín Gregori\textsuperscript{a,\textdagger,1}, Juan-José Miñana\textsuperscript{a,\textasteriskcommand,\textdagger,2}

\textsuperscript{a}Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera s/n 46022 Valencia (SPAIN).
\textsuperscript{b}juamiapr@upvnet.upv.es

Abstract

In this note we answer two recent questions posed by Morillas and Sapena [On Cauchy sequences in fuzzy metric spaces, Proceedings of the Conference in Applied Topology WiAT’13 101-108] related to standard convergence in fuzzy metric spaces in the sense of George and Veeramani. The obtained results lead us to establish what conditions must satisfy a concept about sequential convergence to be considered compatible with a concept of Cauchyness.

Key words: Fuzzy metric space; (std−) Cauchy sequence; (std−) convergent sequence.

1 Introduction

Kramosil and Michalek gave in [8] a concept of fuzzy metric space which is an extension of the concept of Menger space to the fuzzy setting. A more general version of this concept, denoted by $KM$-fuzzy metric space, was given later

\textsuperscript{*} Corresponding author email: vgregori@mat.upv.es; telephone/fax number: +034 962849300
\textsuperscript{1} Valentín Gregori acknowledges the support of Spanish Ministry of Education and Science under Grant MTM 2012-37894-C02-01 and the supports of Universitat Politècnica de València under Grant PAID-05-12 SP20120696 and under Grant PAID-06-12 SP20120471.
\textsuperscript{2} Juan José Miñana acknowledges the support of Conselleria de Educación, Formación y Empleo (Programa Vali+d para investigadores en formación) of Generalitat Valenciana, Spain and the support of Universitat Politècnica de València under Grant PAID-06-12 SP20120471.

Preprint submitted to Elsevier 6 May 2014
((2,3)). In this note we deal with the concept of fuzzy metric space due to
George and Veeramani (Definition 1) which is a modification of the concept
of \(KM\)-fuzzy metric space.

A significant difference between fuzzy metric and metric is that the first one in-
cludes in its definition a \(t\)-parameter. From the mathematical point of view the
\(t\)-parameter allows to introduce novel (fuzzy) metric concepts with respect to
the classical metric ones (and sometimes the constructed fuzzy theory is more
general than the corresponding one in classical theory). For instance, several
well-motivated notions of convergence and Cauchyness related to sequences
can be found in the literature [2–4,9–11,13]. In particular, more recently a
stronger concept than convergence, called \(s\)-convergence, has been used to
characterize certain type of fuzzy metric spaces [5]. Then, for a concept of
convergence it is natural and interesting to study a concept of Cauchyness,
or \(vice-versa\), such that both are \(pairwise\) \(compatible\). This is not an original
idea. In fact, it was already suggested by D. Miheţ in [9] where the author
defined a weaker concept than convergence, called \(p\)-convergence. Then, in [4]
the authors gave (an \(appropriate\)) concept of \(p\)-Cauchy sequence and also they
initiated the study on the relationship of the concept of \(p\)-convergence with
local bases. This study has been continued in [6].

For establishing relationships between the theory of complete fuzzy metric
spaces and domain theory, Ricarte and Romaguera have introduced in [11] a
stronger concept than Cauchy sequence, called standard Cauchy, briefly \(std-
Cauchy\). They have proved that the well-known theorem due to Edalat and
Heckmann [1] that characterizes complete metric spaces by means of continu-
ous domains can be obtained from their results in fuzzy metrics ([11], Corollary
1). Furthermore, the theory constructed in that paper cannot be obtained from
the metric case. Indeed, if \(M\) is a non-complete stationary fuzzy metric then it
is \(std\)-complete but the uniformity \(U_M\) induced by \(M\), see [7], is not complete
and so all metrics compatible with \(U_M\) are not complete and then classical
theory cannot be applied on \(M\).

Inspired in the classical case the authors have introduced in [10], in a natural
way, the concept of standard convergence, briefly \(std\)-convergence, and they
have asked the following questions.

\(Q1\) : Is every \(std\)-convergent sequence a \(std\)-Cauchy sequence?

\(Q2\) : Let \(\{x_n\}\) be a \(std\)-Cauchy and convergent sequence. Is \(\{x_n\}\) \(std\)-convergent?

In this note we give negative response to \(Q1\) in Example 5 and then we con-
clude that the concept of \(std\)-convergence is not \(appropriate\). Then, for avoid-
ing the proliferation of non-appropriate concepts related to convergence or
Cauchyness, we create a framework in which the study of the relationship be-
tween both concepts to be more useful. So, we establish in Definition 7 when
a concept of convergence is compatible with a concept of Cauchyness, and vice-versa. Later, we give a concept of convergence which is compatible with std-Cauchy. Finally, we give a positive answer to Q2.

2 Preliminaries

Definition 1 (George and Veeramani [2]). A fuzzy metric space is an ordered triple \((X, M, \ast)\) such that \(X\) is a (non-empty) set, \(\ast\) is a continuous \(t\)-norm and \(M\) is a fuzzy set on \(X \times X \times ]0, \infty[\) satisfying the following conditions, for all \(x, y, z \in X\), \(s, t > 0\):

\[
\begin{align*}
(GV1) & \quad M(x, y, t) > 0; \\
(GV2) & \quad M(x, y, t) = 1 \text{ if and only if } x = y; \\
(GV3) & \quad M(x, y, t) = M(y, x, t); \\
(GV4) & \quad M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s); \\
(GV5) & \quad M(x, y, \cdot) : [0, \infty] \rightarrow [0, 1] \text{ is continuous.}
\end{align*}
\]

Every fuzzy metric \(M\) on \(X\) generates a topology \(\tau_M\) on \(X\) which has as a base the family of open sets of the form \(\{B_M(x, \epsilon, t) : x \in X, \epsilon \in ]0, 1[\}, t > 0\), where \(B_M(x, \epsilon, t) = \{y \in X : M(x, y, y) > 1 - \epsilon\}\) for all \(x \in X\), \(\epsilon \in ]0, 1[\), \(t > 0\).

Let \((X, d)\) be a metric space and let \(M_d\) a function on \(X \times X \times ]0, \infty[\) defined by

\[
M_d(x, y, t) = \frac{t}{t + d(x, y)}
\]

Then \((X, M_d, \cdot)\) is a fuzzy metric space, [2], and \(M_d\) is called the standard fuzzy metric induced by \(d\). The topology \(\tau_{M_d}\) coincides with the topology \(\tau(d)\) on \(X\) deduced from \(d\).

Definition 2 (George and Veeramani [2]). A sequence \(\{x_n\}\) in a fuzzy metric space \((X, M, \ast)\) is called Cauchy if given \(\epsilon \in ]0, 1[\) and \(t > 0\) there exists \(n_0 \in \mathbb{N}\) such that \(M(x_n, x_m, t) > 1 - \epsilon\) for all \(n, m \geq n_0\). \(X\) is called complete if every Cauchy sequence in \(X\) is convergent.

Definition 3 (Ricarte and Romaguera [11]). A sequence \(\{x_n\}\) is called std-Cauchy if given \(\epsilon \in ]0, 1[\) there exists \(n_\epsilon \in \mathbb{N}\), depending on \(\epsilon\), such that \(M(x_n, x_m, t) > \frac{t}{t + \epsilon}\), for all \(n, m \geq n_\epsilon\) and for all \(t > 0\). \(X\) is called std-complete if every std-Cauchy sequence in \(X\) is convergent.

Definition 4 (Morillas and Sapena [10]). A sequence \(\{x_n\}\) in \(X\) is called std-convergent to \(x_0 \in X\) if given \(\epsilon \in ]0, 1[\) there exists \(n_\epsilon \in \mathbb{N}\), depending on \(\epsilon\), such that \(M(x_n, x_0, t) > \frac{t}{t + \epsilon}\), for all \(n \geq n_\epsilon\) and for all \(t > 0\).
3 Results

The next example gives a negative response to the first question Q1.

**Example 5** (A std-convergent non-std-Cauchy sequence). Let $d$ be the usual metric on $\mathbb{R}$ restricted to $[0, \infty[$ and consider the standard fuzzy metric induced by $d$. Let $X = [0, \infty[$. We define on $X \times X \times [0, \infty[$ the function

$$M(x, y, t) = \begin{cases} 1, & \text{if } x = y \\ M_d(x, 0, t) \cdot M_d(0, y, t), & \text{if } x \neq y \end{cases}$$

It is an easy exercise to prove that $(X, M, \cdot)$ is a fuzzy metric space.

Now, consider the sequence $\{x_n\}$ in $X$, where $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. We claim that $\{x_n\}$ is std-convergent to 0. Indeed, take $\epsilon \in ]0, 1[$, then we can find $n_\epsilon \in \mathbb{N}$ such that $n_\epsilon > \frac{1}{\epsilon}$ and hence $M(x_n, 0, t) = \frac{t}{t + \frac{n_\epsilon}{t}} > \frac{t}{t + \epsilon}$, for all $n \geq n_\epsilon$ and for all $t > 0$. So $\{x_n\}$ is std-convergent to 0.

We claim that $\{x_n\}$ is not std-Cauchy. Indeed, if we suppose that $\{x_n\}$ is std-Cauchy, then for each $\epsilon \in ]0, 1[$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$M(x_n, x_m, t) = \frac{t}{t + \frac{1}{n}} \cdot \frac{t}{t + \frac{1}{m}} > \frac{t}{t + \epsilon}$$

for all $n, m \geq n_\epsilon$ and $t > 0$. So, $\frac{t}{(t + \frac{1}{n_\epsilon})(t + \frac{1}{n_\epsilon})} > \frac{1}{t + \epsilon}$, for all $t > 0$.

Then, $\lim_{t \to 0} \frac{t}{(t + \frac{1}{n_\epsilon})(t + \frac{1}{n_\epsilon})} = 0 \geq \lim_{t \to 0} \frac{1}{t + \epsilon} = \frac{1}{\epsilon}$, a contradiction.

**Remark 6** Attending to Definition 3 it is clear that a natural way of defining std-convergence is the one given by the authors in [10] (Definition 4). Unfortunately, as shows Example 5, this definition should be considered not appropriate.

Next we establish conditions under which a pair of concepts on convergence and Cauchyness, related to sequences, are considered pairwise compatible. These conditions have been chosen for preserving the natural structure among the concepts and also, for avoiding the unnecessary appearance of concepts or inner properties (which, finally, could distort the next diagrams).

**Definition 7** Suppose it is given a sequential stronger (weaker, respectively) concept than Cauchy, say s-Cauchy (w-Cauchy, respectively). A concept on convergence, say s-convergence (w-convergence, respectively), is said to be compatible with s-Cauchy (w-Cauchy, respectively), and vice-versa, if the di-
agram of implications below on the left (on the right, respectively) is fulfilled

\[
s - \text{convergence} \rightarrow \text{convergence} \quad \text{convergence} \rightarrow w - \text{convergence} \\
\downarrow \quad \downarrow \quad \downarrow \\
s - \text{Cauchy} \quad \rightarrow \quad \text{Cauchy} \quad \text{Cauchy} \quad \rightarrow \quad w - \text{Cauchy}
\]

and there is not any other implication, in general, among these concepts.

So, by Example 5 we can assert that the concept of \(\text{std}\)-convergence is not compatible with \(\text{std}\)-Cauchy. After the next remark we give a concept of convergence which is compatible with \(\text{std}\)-Cauchy.

**Remark 8** (Existence of pairwise compatible \(s\)-concepts). Suppose that a concept of \(s\)-Cauchyness which is stronger than Cauchy, is given. Also, suppose that there is not any implication between convergence and \(s\)-Cauchyness. Then, there exists a concept of \(s\)-convergence compatible with \(s\)-Cauchy if and only if \(s\)-Cauchy and convergence are non-mutually exclusive concepts. Indeed, in a such case we can give the next definition: A sequence \(\{x_n\}\) is called \(s^*\)-convergent if it is convergent and \(s\)-Cauchy. Obviously, this concept of \(s^*\)-convergence is compatible with \(s\)-Cauchy. Further, any concept of \(s\)-convergence which is compatible with \(s\)-Cauchy is stronger than \(s^*\)-convergence.

Now, since every \(\text{std}\)-convergent sequence is convergent, [10], then Example 5 provides an example of a convergent sequence which is not \(\text{std}\)-Cauchy. On the other hand if \((X, M_d, \cdot)\) is a standard fuzzy metric then a sequence in \(X\) is \(\text{std}\)-Cauchy if and only if it is Cauchy. Hence, in a non-complete standard fuzzy metric space we can find \(\text{std}\)-Cauchy sequences which are not convergent. Further, every convergent sequence in \((X, M_d, \cdot)\) is \(\text{std}\)-Cauchy. Thus, by the last remark we can introduce the following definition of convergence which is compatible with \(\text{std}\)-Cauchy.

**Definition 9** A sequence is called \(\text{std}^*\)-convergent if it is convergent and \(\text{std}\)-Cauchy.

**Remark 10** (Existence of pairwise compatible \(w\)-concepts). Suppose that a concept of \(w\)-convergence which is weaker than convergence is given. Also, suppose that there is not any implication between \(w\)-convergence and Cauchy. Then, we always can find concepts of Cauchyness compatible with \(w\)-convergence. Indeed, in a such case we can give the next definition: \(\{x_n\}\) is called \(w^*\)-Cauchy if \(\{x_n\}\) is Cauchy or \(w\)-convergent. Clearly, \(w^*\)-Cauchy is compatible with \(w\)-convergence. Further, any other concept of \(w\)-Cauchy which is compatible with \(w\)-convergence is weaker than \(w^*\)-Cauchy.

Finally, in the next proposition we response in a positive way to Question Q2.
Proposition 11 Let \((X, M, *)\) be a fuzzy metric space and let \(\{x_n\}\) be a std-Cauchy convergent sequence. Then \(\{x_n\}\) is std-convergent.

Proof.

Let \(\{x_n\}\) be a std-Cauchy convergent sequence. Fix \(\epsilon \in ]0, 1[\) and \(t > 0\). Suppose that \(\{x_n\}\) converges to \(x_0\). Since \(M(x, y, )\) is continuous for all \(x, y \in X\), by Corollary 7.2 of [3] (or using Proposition 1 of [12]) we have that \(\lim_{m} M(x_n, x_m, t) = M(x_n, x_0, t)\) for all \(n \in \mathbb{N}\).

On the other hand, since \(\{x_n\}\) is std-Cauchy we have that for \(\delta \in ]0, \epsilon[\) there exists \(n_\delta \in \mathbb{N}\) such that

\[
M(x_n, x_m, t) > \frac{t}{t+\delta} > \frac{t}{t+\epsilon}, \text{ for all } n, m \geq n_\delta \text{ and all } t > 0.
\]

Then

\[
M(x_n, x_0, t) = \lim_{m} M(x_n, x_m, t) \geq \frac{t}{t+\delta} > \frac{t}{t+\epsilon}, \text{ for all } n \geq n_\delta \text{ and all } t > 0
\]

and so \(\{x_n\}\) is std-convergent.

Acknowledgements

The authors are grateful to the referees for their valuable suggestions.

References


