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Additional Information

std-convergence in fuzzy metric spaces

Valentín Gregori ^{a,*},¹, Juan-José Miñana ^{a,b},²

^a*Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, Camino de Vera s/n 46022 Valencia (SPAIN).*

^b*juamiapr@upvnet.upv.es*

Abstract

In this note we answer two recent questions posed by Morillas and Sapena [On Cauchy sequences in fuzzy metric spaces, Proceedings of the Conference in Applied Topology WiAT'13 101-108] related to standard convergence in fuzzy metric spaces in the sense of George and Veeramani. The obtained results lead us to establish what conditions must satisfy a concept about sequential convergence to be considered *compatible* with a concept of Cauchyness.

Key words: Fuzzy metric space; (*std*-) Cauchy sequence; (*std*-) convergent sequence.

1 Introduction

Kramosil and Michalek gave in [8] a concept of fuzzy metric space which is an extension of the concept of Menger space to the fuzzy setting. A more general version of this concept, denoted by *KM*-fuzzy metric space, was given later

* Corresponding author email: vgregori@mat.upv.es; telephone/fax number: +034 962849300

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4 ([2,3]). In this note we deal with the concept of fuzzy metric space due to
5 George and Veeramani (Definition 1) which is a modification of the concept
6 of KM -fuzzy metric space.
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9 A significant difference between fuzzy metric and metric is that the first one in-
10 cludes in its definition a t -parameter. From the mathematical point of view the
11 t -parameter allows to introduce novel (fuzzy) metric concepts with respect to
12 the classical metric ones (and sometimes the constructed fuzzy theory is more
13 general than the corresponding one in classical theory). For instance, several
14 well-motivated notions of convergence and Cauchyness related to sequences
15 can be found in the literature [2–4,9–11,13]. In particular, more recently a
16 stronger concept than convergence, called s -convergence, has been used to
17 characterize certain type of fuzzy metric spaces [5]. Then, for a concept of
18 convergence it is natural and interesting to study a concept of Cauchyness,
19 or *vice-versa*, such that both are *pairwise compatible*. This is not an original
20 idea. In fact, it was already suggested by D. Mihet in [9] where the author
21 defined a weaker concept than convergence, called p -convergence. Then, in [4]
22 the authors gave (an *appropriate*) concept of p -Cauchy sequence and also they
23 initiated the study on the relationship of the concept of p -convergence with
24 local bases. This study has been continued in [6].
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30 For establishing relationships between the theory of complete fuzzy metric
31 spaces and domain theory, Ricarte and Romaguera have introduced in [11] a
32 stronger concept than Cauchy sequence, called standard Cauchy, briefly *std*-
33 Cauchy. They have proved that the well-known theorem due to Edalat and
34 Heckmann [1] that characterizes complete metric spaces by means of continu-
35 ous domains can be obtained from their results in fuzzy metrics ([11], Corollary
36 1). Furthermore, the theory constructed in that paper cannot be obtained from
37 the metric case. Indeed, if M is a non-complete stationary fuzzy metric then it
38 is *std*-complete but the uniformity \mathcal{U}_M induced by M , see [7], is not complete
39 and so all metrics compatible with \mathcal{U}_M are not complete and then classical
40 theory cannot be applied on M .
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45 Inspired in the classical case the authors have introduced in [10], in a natural
46 way, the concept of standard convergence, briefly *std*-convergence, and they
47 have asked the following questions.
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50 $Q1$: Is every *std*-convergent sequence a *std*-Cauchy sequence?

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53 $Q2$: Let $\{x_n\}$ be a *std*-Cauchy and convergent sequence. Is $\{x_n\}$ *std*-convergent?

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55 In this note we give negative response to $Q1$ in Example 5 and then we con-
56 clude that the concept of *std*-convergence is not *appropriate*. Then, for avoid-
57 ing the proliferation of non-appropriate concepts related to convergence or
58 Cauchyness, we create a framework in which the study of the relationship be-
59 tween both concepts to be more useful. So, we establish in Definition 7 when
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a concept of convergence is *compatible* with a concept of Cauchyness, and *vice-versa*. Later, we give a concept of convergence which is *compatible* with *std*-Cauchy. Finally, we give a positive answer to Q2.

2 Preliminaries

Definition 1 (George and Veeramani [2]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (non-empty) set, $*$ is a continuous t -norm and M is a fuzzy set on $X \times X \times]0, \infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

- (GV1) $M(x, y, t) > 0$;
- (GV2) $M(x, y, t) = 1$ if and only if $x = y$;
- (GV3) $M(x, y, t) = M(y, x, t)$;
- (GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
- (GV5) $M(x, y, -) :]0, \infty[\rightarrow]0, 1[$ is continuous.

Every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \epsilon, t) : x \in X, \epsilon \in]0, 1[, t > 0\}$, where $B_M(x, \epsilon, t) = \{y \in X : M(x, y, t) > 1 - \epsilon\}$ for all $x \in X, \epsilon \in]0, 1[, t > 0$.

Let (X, d) be a metric space and let M_d a function on $X \times X \times]0, \infty[$ defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

Then (X, M_d, \cdot) is a fuzzy metric space, [2], and M_d is called the *standard fuzzy metric* induced by d . The topology τ_{M_d} coincides with the topology $\tau(d)$ on X deduced from d .

Definition 2 (George and Veeramani [2]). A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called *Cauchy* if given $\epsilon \in]0, 1[$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. X is called *complete* if every *Cauchy* sequence in X is convergent.

Definition 3 (Ricarte and Romaguera [11]). A sequence $\{x_n\}$ is called *std-Cauchy* if given $\epsilon \in]0, 1[$ there exists $n_\epsilon \in \mathbb{N}$, depending on ϵ , such that $M(x_n, x_m, t) > \frac{t}{t + \epsilon}$, for all $n, m \geq n_\epsilon$ and for all $t > 0$. X is called *std-complete* if every *std-Cauchy* sequence in X is convergent.

Definition 4 (Morillas and Sapena [10]). A sequence $\{x_n\}$ in X is called *std-convergent* to $x_0 \in X$ if given $\epsilon \in]0, 1[$ there exists $n_\epsilon \in \mathbb{N}$, depending on ϵ , such that $M(x_n, x_0, t) > \frac{t}{t + \epsilon}$, for all $n \geq n_\epsilon$ and for all $t > 0$.

3 Results

The next example gives a negative response to the first question Q1.

Example 5 (A *std-convergent non-std-Cauchy* sequence). Let d be the usual metric on \mathbb{R} restricted to $[0, \infty[$ and consider the standard fuzzy metric induced by d . Let $X = [0, \infty[$. We define on $X \times X \times]0, \infty[$ the function

$$M(x, y, t) = \begin{cases} 1, & \text{if } x = y \\ M_d(x, 0, t) \cdot M_d(0, y, t), & \text{if } x \neq y \end{cases}$$

It is an easy exercise to prove that (X, M, \cdot) is a fuzzy metric space.

Now, consider the sequence $\{x_n\}$ in X , where $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. We claim that $\{x_n\}$ is *std-convergent* to 0. Indeed, take $\epsilon \in]0, 1[$, then we can find $n_\epsilon \in \mathbb{N}$ such that $n_\epsilon > \frac{1}{\epsilon}$ and hence $M(x_n, 0, t) = \frac{t}{t + \frac{1}{n}} > \frac{t}{t + \epsilon}$, for all $n \geq n_\epsilon$ and for all $t > 0$. So $\{x_n\}$ is *std-convergent* to 0.

We claim that $\{x_n\}$ is not *std-Cauchy*. Indeed, if we suppose that $\{x_n\}$ is *std-Cauchy*, then for each $\epsilon \in]0, 1[$ there exists $n_\epsilon \in \mathbb{N}$ such that

$$M(x_n, x_m, t) = \frac{t}{t + \frac{1}{n}} \cdot \frac{t}{t + \frac{1}{m}} > \frac{t}{t + \epsilon}$$

for all $n, m \geq n_\epsilon$ and $t > 0$. So, $\frac{t}{(t + \frac{1}{n_\epsilon})(t + \frac{1}{n_\epsilon})} > \frac{1}{t + \epsilon}$, for all $t > 0$.

Then, $\lim_{t \rightarrow 0} \frac{t}{(t + \frac{1}{n_\epsilon})(t + \frac{1}{n_\epsilon})} = 0 \geq \lim_{t \rightarrow 0} \frac{1}{t + \epsilon} = \frac{1}{\epsilon}$, a contradiction.

Remark 6 Attending to Definition 3 it is clear that a natural way of defining *std-convergence* is the one given by the authors in [10] (Definition 4). Unfortunately, as shows Example 5, this definition should be considered not appropriate.

Next we establish conditions under which a pair of concepts on convergence and Cauchyness, related to sequences, are considered *pairwise compatible*. These conditions have been chosen for preserving the natural structure among the concepts and also, for avoiding the unnecessary appearance of concepts or inner properties (which, finally, could distort the next diagrams).

Definition 7 Suppose it is given a sequential stronger (weaker, respectively) concept than *Cauchy*, say *s-Cauchy* (*w-Cauchy*, respectively). A concept on convergence, say *s-convergence* (*w-convergence*, respectively), is said to be compatible with *s-Cauchy* (*w-Cauchy*, respectively), and vice-versa, if the di-

agram of implications below on the left (on the right, respectively) is fulfilled

$$\begin{array}{ccc}
 s - \text{convergence} & \rightarrow & \text{convergence} & \rightarrow & w - \text{convergence} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 s - \text{Cauchy} & \rightarrow & \text{Cauchy} & & \text{Cauchy} & \rightarrow & w - \text{Cauchy}
 \end{array}$$

and there is not any other implication, in general, among these concepts.

So, by Example 5 we can assert that the concept of *std*-convergence is not compatible with *std*-Cauchy. After the next remark we give a concept of convergence which is compatible with *std*-Cauchy.

Remark 8 (*Existence of pairwise compatible s-concepts*). Suppose that a concept of *s*-Cauchyness which is stronger than Cauchy, is given. Also, suppose that there is not any implication between convergence and *s*-Cauchyness. Then, there exists a concept of *s*-convergence compatible with *s*-Cauchy if and only if *s*-Cauchy and convergence are non-mutually exclusive concepts. Indeed, in a such case we can give the next definition: A sequence $\{x_n\}$ is called *s**-convergent if it is convergent and *s*-Cauchy. Obviously, this concept of *s**-convergence is compatible with *s*-Cauchy. Further, any concept of *s*-convergence which is compatible with *s*-Cauchy is stronger than *s**-convergence.

Now, since every *std*-convergent sequence is convergent, [10], then Example 5 provides an example of a convergent sequence which is not *std*-Cauchy. On the other hand if (X, M_d, \cdot) is a standard fuzzy metric then a sequence in X is *std*-Cauchy if and only if it is Cauchy. Hence, in a non-complete standard fuzzy metric space we can find *std*-Cauchy sequences which are not convergent. Further, every convergent sequence in (X, M_d, \cdot) is *std*-Cauchy. Thus, by the last remark we can introduce the following definition of convergence which is compatible with *std*-Cauchy.

Definition 9 A sequence is called *std**-convergent if it is convergent and *std*-Cauchy.

Remark 10 (*Existence of pairwise compatible w-concepts*). Suppose that a concept of *w*-convergence which is weaker than convergence is given. Also, suppose that there is not any implication between *w*-convergence and Cauchy. Then, we always can find concepts of Cauchyness compatible with *w*-convergence. Indeed, in a such case we can give the next definition: $\{x_n\}$ is called *w**-Cauchy if $\{x_n\}$ is Cauchy or *w*-convergent. Clearly, *w**-Cauchy is compatible with *w*-convergence. Further, any other concept of *w*-Cauchy which is compatible with *w*-convergence is weaker than *w**-Cauchy.

Finally, in the next proposition we response in a positive way to Question Q2.

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4 **Proposition 11** *Let $(X, M, *)$ be a fuzzy metric space and let $\{x_n\}$ be a *std*-*
5 *Cauchy convergent sequence. Then $\{x_n\}$ is *std*-convergent.*
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9 **Proof.**

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11 Let $\{x_n\}$ be a *std*-Cauchy convergent sequence. Fix $\epsilon \in]0, 1[$ and $t > 0$.
12 Suppose that $\{x_n\}$ converges to x_0 . Since $M(x, y, _)$ is continuous for all
13 $x, y \in X$, by Corollary 7.2 of [3] (or using Proposition 1 of [12]) we have
14 that $\lim_m M(x_n, x_m, t) = M(x_n, x_0, t)$ for all $n \in \mathbb{N}$.
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17 On the other hand, since $\{x_n\}$ is *std*-Cauchy we have that for $\delta \in]0, \epsilon[$ there
18 exists $n_\delta \in \mathbb{N}$ such that
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$$20 \quad M(x_n, x_m, t) > \frac{t}{t + \delta} > \frac{t}{t + \epsilon}, \text{ for all } n, m \geq n_\delta \text{ and all } t > 0.$$

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24 Then

$$25 \quad M(x_n, x_0, t) = \lim_m M(x_n, x_m, t) \geq \frac{t}{t + \delta} > \frac{t}{t + \epsilon}, \text{ for all } n \geq n_\delta \text{ and all } t > 0$$

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28 and so $\{x_n\}$ is *std*-convergent.
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38 The authors are grateful to the referees for their valuable suggestions.
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