Some subgroup embeddings in finite groups:
A mini-review

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Abstract
In this survey paper several subgroup embedding properties related
to some types of permutability are introduced and studied.
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1 Introduction
All groups in the paper are finite.

The purpose of this survey paper is to show how the embedding of cer-
tain types of subgroups of a finite group $G$ can determine the structure
of $G$. The types of subgroup embedding properties we consider include: S-
permutability, S-semipermutability, semipermutability, primitivity, and qua-
sipermutability.

A subgroup $H$ of a group $G$ is said to permute with a subgroup $K$ of $G$
if $HK$ is a subgroup of $G$. $H$ is said to be permutable in $G$ if $H$ permutes
with all subgroups of $G$. A less restrictive subgroup embedding property is
the S-permutability introduced by Kegel and defined in the following way:

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Definition 1. A subgroup $H$ of $G$ is said to be $S$-permutable in $G$ if $H$ permutes with every Sylow $p$-subgroup of $G$ for every prime $p$.

In recent years there has been widespread interest in the transitivity of normality, permutability and $S$-permutability.

Definition 2.  
1. A group $G$ is a $T$-group if normality is a transitive relation in $G$, that is, if every subnormal subgroup of $G$ is normal in $G$.

2. A group $G$ is a $PT$-group if permutability is a transitive relation in $G$, that is, if $H$ is permutable in $K$ and $K$ is permutable in $G$, then $H$ is permutable in $G$.

3. A group $G$ is a $PST$-group if $S$-permutability is a transitive relation in $G$, that is, if $H$ is $S$-permutable in $K$ and $K$ is $S$-permutable in $G$, then $H$ is $S$-permutable in $G$.

If $H$ is $S$-permutable in $G$, it is known that $H$ must be subnormal in $G$ ([1, Theorem 1.2.14(3)]). Therefore, a group $G$ is a PST-group (respectively a PT-group) if and only if every subnormal subgroup is $S$-permutable (respectively permutable) in $G$.

Note that T implies PT and PT implies PST. On the other hand, PT does not imply T (non-Dedekind modular $p$-groups) and PST does not imply PT (non-modular $p$-groups). The reader is referred to [1, Chapter 2] for basic results about these classes of groups. Other characterisations based on subgroup embedding properties can be found in [2].

Agrawal ([1, 2.1.8]) characterised soluble PST-groups. He proved that a soluble group $G$ is a PST-group if and only if the nilpotent residual in $G$ is an abelian Hall subgroup of $G$ on which $G$ acts by conjugation as power automorphisms. In particular, the class of soluble PST-groups is subgroup-closed.

Let $G$ be a soluble PST-group with nilpotent residual $L$. Then $G$ is a PT-group (respectively T-group) if and only if $G/L$ is a modular (respectively Dedekind) group ([1, 2.1.11]).

Definition 3 ([3]). A subgroup $H$ of a group $G$ is said to be semipermutable (respectively, $S$-semipermutable) provided that it permutes with every subgroup (respectively, Sylow subgroup) $K$ of $G$ such that $\gcd(|H|, |K|) = 1$.

An $S$-semipermutable subgroup of a group need not be subnormal. For example, a Sylow 2-subgroup of the nonabelian group of order 6 is semipermutable and $S$-semipermutable, but not subnormal.
Definition 4 (see [4]). A group $G$ is called a BT-group if semipermutability is a transitive relation in $G$.

L. Wang, Y. Li, and Y. Wang proved the following theorem which showed that soluble BT-groups are a subclass of PST-groups:

Theorem 5 ([4]). Let $G$ be a group with nilpotent residual $L$. The following statements are equivalent:

1. $G$ is a soluble BT-group;
2. every subgroup of $G$ of prime power order is $S$-semipermutable;
3. every subgroup of $G$ of prime power order is semipermutable;
4. every subgroup of $G$ is semipermutable;
5. $G$ is a soluble PST-group and if $p$ and $q$ are distinct primes not dividing the order of $L$ with $G_p$ a Sylow $p$-subgroup of $G$ and $G_q$ a Sylow $q$-subgroup of $G$, then $[G_p, G_q] = 1$.

Research papers on BT-groups include [4, 5, 6, 7].

We next present an example of a soluble PST-group which is not a BT-group.

Example 6. Let $L$ be a cyclic group of order 7 and $A = C_3 \times C_2$ be the automorphism group of $L$. Here $C_3$ (respectively, $C_2$) is the cyclic group of order 3 (respectively, 2). Let $G = [L]A$ be the semidirect product of $L$ by $A$. Let $L = \langle x \rangle$, $C_3 = \langle y \rangle$ and $C_2 = \langle z \rangle$ and note that $[\langle y^2 \rangle, \langle z \rangle] \neq 1$. Now $G$ is a PST-group by Agrawal’s theorem, but $G$ is not a BT-group by Theorem 5.

A subclass of the class of soluble BT-groups is the class of soluble SST-groups, which has been introduced in [8].

Definition 7 (see [9]). A subgroup $H$ of a group $G$ is said to be SS-permutable (or SS-quasinormal) in $G$ if $H$ has a supplement $K$ in $G$ such that $H$ permutes with every Sylow subgroup of $K$.

Definition 8 (see [8]). We say that a group $G$ is an SST-group if SS-permutability is a transitive relation.

SS-permutability can be used to obtain a characterisation of soluble PST-groups.

Theorem 9 ([8]). Let $G$ be a group. Then the following statements are equivalent:
1. G is soluble and every subnormal subgroup of G is SS-permutable in G.

2. G is a soluble PST-group.

**Theorem 10** ([8]). A soluble SST-group G is a BT-group.

The following example shows that a soluble BT-group is not necessarily an SST-group.

**Example 11** ([8]). Let $G = \langle x, y \mid x^5 = y^4 = 1, x^y = x^2 \rangle$. The nilpotent residual of G is the Sylow 5-subgroup $\langle x \rangle$. By Theorem 5, G is a soluble BT-group. Let $H = \langle y \rangle$ and $M = \langle y^2 \rangle$. Suppose that $M$ is SS-permutable in $G$. Then $G$ is the unique supplement of $M$ in $G$. It follows that $M$ is S-permutable in $G$, and thus $M \leq O_2(G)$. This implies that either $O_2(G) = H$ or $O_2(G) = M$. Since $y^x = yx^{-1}$ and $(y^2)^x = y^2x^2$, neither $H$ nor $M$ are normal subgroups of $G$. This contradiction shows that $M$ is not SS-permutable in $G$. Since $M$ is SS-permutable in $\langle x, y^2 \rangle$ and this subgroup is SS-permutable in $G$, we obtain that the soluble group $G$ cannot be an SST-group.

A less restrictive class of groups is the class of T$_0$-groups which has been studied in [5, 7, 10, 11, 12].

**Definition 12.** A group $G$ is called a T$_0$-group if the Frattini factor group $G/\Phi(G)$ is a T-group.

**Theorem 13** ([11]). Let $L$ be the nilpotent residual of the soluble T$_0$-group. Then:

1. $G$ is supersoluble;

2. $L$ is a nilpotent Hall subgroup of $G$.

**Theorem 14** ([10]). Let $G$ be a soluble T$_0$-group. If all the subgroups of $G$ are T$_0$-groups, then $G$ is a PST-group.

A group $G$ is called an MS-group if the maximal subgroups of all the Sylow subgroups of $G$ are S-semipermutable.

**Theorem 15** ([13]). If $G$ is an MS-group, then $G$ is supersoluble.

**Theorem 16** ([7]). Let $L$ be the nilpotent residual of an MS-group $G$. Then:

1. $L$ is a nilpotent Hall subgroup of $G$;

2. $G$ is a soluble T$_0$-group.
We now provide three examples which illustrate several properties and differences of some of the classes presented in this paper. These examples are from [6, 7].

**Example 17.** Let \( C = \langle x \rangle \) be a cyclic group of order 7 and let \( A = \langle y \rangle \times \langle z \rangle \) be a cyclic group of order 6 with \( y \) an element of order 3 and \( z \) an element of order 2. Then \( A = \text{Aut}(C) \). Let \( G = [C]A \) be the semidirect product of \( C \) by \( A \). Then \( [(y)^2, z] \neq 1 \) and \( G \) is not a soluble BT-group. However, \( G \) is an MS-group.

Example 18 shows that the classes of MS- and \( T_0 \)-groups are not subgroup closed.

**Example 18.** Let \( H = \langle x, y \mid x^3 = y^3 = [x, y]^3 = [x, [x, y]] = [y, [x, y]] = 1 \rangle \) be an extraspecial group of order 27 and exponent 3. Then \( H \) has an automorphism \( a \) of order 2 given by \( x^a = x^{-1}, y^a = y^{-1} \) and \( [x, y]^a = [x, y] \). Put \( G = [H] \langle a \rangle \), the semidirect product of \( H \) by \( \langle a \rangle \). Let \( z = \langle x, y \rangle \). Then \( \Phi(G) = \Phi(H) = \langle z \rangle = Z(G) = Z(H) \). Note that \( G/\Phi(G) \) is a \( T \)-group so that \( G \) is a \( T_0 \)-group. The maximal subgroups of \( H \) are normal in \( G \) and it follows that \( G \) is an MS-group. Let \( K = \langle x, z, a \rangle \). Then \( \langle xz \rangle \) is a maximal subgroup of \( \langle x, z \rangle \), the Sylow 3-subgroup of \( K \). However, \( \langle xz \rangle \) does not permute with \( \langle a \rangle \) and hence \( \langle xz \rangle \) is not an S-semipermutable subgroup of \( K \). Therefore, \( K \) is not an MS-subgroup of \( G \). Also note that \( \Phi(K) = 1 \) and so \( K \) is not a T-subgroup of \( G \) and \( K \) is not a \( T_0 \)-subgroup of \( G \). Hence the class of soluble \( T_0 \)-groups is not closed under taking subgroups. Note that \( G \) is not a soluble PST-group.

Example 19 presents an example of a soluble PST-group which is not an MS-group.

**Example 19.** Let \( C = \langle x \rangle \) be a cyclic group of order 19\(^2\), \( D = \langle y \rangle \) a cyclic group of order 3\(^2\), and \( E = \langle z \rangle \) is a cyclic group of order 2 such that \( D \times E \leq \text{Aut}(C) \). Then \( G = [C] (D \times E) \) is a soluble PST-group and \( G \) is not an MS-group since \( [(y^3)^x, z] \neq 1 \).

The following notation is needed in the presentation of the next theorem which characterises MS-groups. Let \( G \) be a group whose nilpotent residual \( L \) is a Hall subgroup of \( G \). Let \( \pi = \pi(L) \) and let \( \theta = \pi' \), the complement of \( \pi \) in the set of all prime numbers. Let \( \theta_N \) denote the set of all primes \( p \) in \( \theta \) such that if \( P \) is a Sylow \( p \)-subgroup of \( G \), then \( P \) has at least two maximal subgroups. Further, let \( \theta_C \) denote the set of all primes \( q \) in \( \theta \) such that if \( Q \) is a Sylow \( q \)-subgroup of \( G \), then \( Q \) has only one maximal subgroup, or, equivalently, \( Q \) is cyclic.
Theorem 20 ([6]). Let $G$ be a group with nilpotent residual $L$. Then $G$ is an MS-group if and only if $G$ satisfies the following:

1. $G$ is a $T_0$-group.
2. $L$ is a nilpotent Hall subgroup of $G$.
3. If $p \in \pi$ and $P \in \text{Syl}_p(G)$, then a maximal subgroup of $P$ is normal in $G$.
4. Let $p$ and $q$ be distinct primes with $p \in \theta_N$ and $q \in \theta$. If $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$, then $|P,Q| = 1$.
5. Let $p$ and $q$ be distinct primes with $p \in \theta_C$ and $q \in \theta$. If $P \in \text{Syl}_p(G)$ and $Q \in \text{Syl}_q(G)$ and $M$ is the maximal subgroup of $P$, then $QM = MQ$ is a nilpotent subgroup of $G$.

Theorem 21 ([6]). Let $G$ be a soluble PST-group. Then $G$ is an MS-group if and only if $G$ satisfies 4 and 5 of Theorem 20.

Theorem 22 ([6]). Let $G$ be a soluble PST-group which is also an MS-group. If $\theta_C$ is the empty set, then $G$ is a BT-group.

Definition 23 ([14]). A subgroup $H$ of a group $G$ is called primitive if it is a proper subgroup in the intersection of all subgroups containing $H$ as a proper subgroup.

All maximal subgroups of $G$ are primitive. Some basic properties of primitive subgroups include:

Proposition 24. 1. Every proper subgroup of $G$ is the intersection of a set of primitive subgroups of $G$.

2. If $X$ is a primitive subgroup of a subgroup $T$ of $G$, then there exists a primitive subgroup $Y$ of $G$ such that $X = Y \cap T$.

Johnson [14] proved that a group $G$ is supersoluble if every primitive subgroup of $G$ has prime power index in $G$.

The next results on primitive subgroups of a group $G$ indicate how such subgroups give information about the structure of $G$.

Theorem 25 ([15]). Let $G$ be a group. The following statements are equivalent:

1. every primitive subgroup of $G$ containing $\Phi(G)$ has prime power index;
2. $G/\Phi(G)$ is a soluble PST-group.

**Theorem 26** ([16]). Let $G$ be a group. The following statements are equivalent:

1. every primitive subgroup of $G$ has prime power index;

2. $G = [L]M$ is a supersoluble group, where $L$ and $M$ are nilpotent Hall subgroups of $G$, $L$ is the nilpotent residual of $G$ and $G = LN_G(L \cap X)$ for every primitive subgroup $X$ of $G$. In particular, every maximal subgroup of $L$ is normal in $G$.

Let $\mathfrak{X}$ denote the class of groups $G$ such that the primitive subgroups of $G$ have prime power index. By Proposition 24 (1), it is clear that $\mathfrak{X}$ consists of those groups whose subgroups are intersections of subgroups of prime power indices.

The next example shows that the class $\mathfrak{X}$ is not subgroup closed.

**Example 27.** Let $P = \langle x, y \mid x^5 = y^5 = [x, y]^5 = 1 \rangle$ be an extraspecial group of order 125 and exponent 5. Let $z = [x, y]$ and note that $Z(P) = \Phi(P) = \langle z \rangle$. Then $P$ has an automorphism $a$ of order four given by $x^a = x^2$, $y^a = y^2$, and $z^a = z^4 = z^{-1}$. Put $G = [P] \langle a \rangle$ and note that $Z(G) = 1$, $\Phi(G) = \langle z \rangle$, and $G/\Phi(G)$ is a T-group. Thus $G$ is a soluble T$_0$-group. Let $H = \langle y, z, a \rangle$ and notice that $\Phi(H) = 1$. Then $H$ is not a T-group since the nilpotent residual $L$ of $H$ is $\langle y, z \rangle$ and $a$ does not act on $L$ as a power automorphism. Thus $H$ is not a T$_0$-group, and hence not a soluble PST-group. By Theorem 25, $G$ is an $\mathfrak{X}$-group and $H$ is not an $\mathfrak{X}$-group.

**Theorem 28** ([17]). Let $G$ be a group. The following statements are equivalent:

1. $G$ is a soluble PST-group;

2. every subgroup of $G$ is an $\mathfrak{X}$-group.

We bring the paper to a close with the quasipermutable embedding which is defined in the following way.

**Definition 29.** A subgroup $H$ is called quasipermutable in $G$ provided there is a subgroup $B$ of $G$ such that $G = N_G(H)B$ and $H$ permutes with $B$ and with every subgroup (respectively, with every Sylow subgroup) $A$ of $B$ such that $\gcd(|H|, |A|) = 1$.

Theorem 30 contains new characterisations of soluble PST-groups with certain Hall subgroups.
Theorem 30 ([18]). Let $D = G^{\text{nil}}$ be the nilpotent residual of the group $G$ and let $\pi = \pi(D)$. Then the following statements are equivalent:

1. $D$ is a Hall subgroup of $G$ and every Hall subgroup of $G$ is quasipermutable in $G$;
2. $G$ is a soluble PST-group;
3. every subgroup of $G$ is quasipermutable in $G$;
4. every $\pi$-subgroup of $G$ and some minimal supplement of $D$ in $G$ are quasipermutable in $G$.

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