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Additional Information

On the product of two π -decomposable groups

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Abstract

The aim of this paper is to prove the following result: Let π be a set of odd primes. If the finite group $G = AB$ is a product of two π -decomposable subgroups $A = O_\pi(A) \times O_{\pi'}(A)$ and $B = O_\pi(B) \times O_{\pi'}(B)$, then $O_\pi(A)O_\pi(B) = O_\pi(B)O_\pi(A)$ and this is a Hall π -subgroup of G .

Keywords: finite groups; π -structure; π -decomposable groups; products of subgroups; Hall subgroups

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1 Introduction and statement of the main result

All groups considered are finite. Within the framework of factorized groups, a well known theorem by Kegel and Wielandt states the solubility of a group which is the product of two nilpotent subgroups. This theorem has been the starting point for a number of results on factorized groups and, in particular, by considering the case when one of the factors is π -decomposable for a set of primes π . A group X is said to be π -decomposable if $X = X_\pi \times X_{\pi'}$ is the direct product of a π -subgroup X_π and a π' -subgroup $X_{\pi'}$, where π' stands for the complementary of π in the set of all prime numbers. X_σ will always denote a Hall σ -subgroup of a group X , for any set of primes σ . For instance, different extensions of

the Kegel and Wielandt theorem for products of a 2-decomposable group and a group of odd order, with coprime orders, were obtained by Berkovich [5], Arad and Chillag [3], Rowley [20] and Kazarin [13].

The present paper contributes new progress to this investigation. More precisely we complete the study on products of π -decomposable groups carried out in [14] and [15] (see also [17]) and prove the following general result:

Main Theorem. *Let π be a set of odd primes. Let the group $G = AB$ be the product of two π -decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall π -subgroup of G .*

This result was announced as a conjecture in [15], [16] and [17], and also was mentioned in [4]. As approaches to the aimed result, we presented in [14] and [15] several particular positive cases, namely, when either one of the factors is a π -group ([14, Theorem 1, Lemma 1]), or they are soluble groups ([15, Theorem 2]), or when the factors have coprime orders ([15, Proposition 1]). These results largely extend the above mentioned known ones on products of 2-decomposable groups. Moreover, in [14] and [15] we obtained also some π -separability criteria for products of π -decomposable groups.

The next example, which appears in [14], shows that analogous results do not hold in general if the set of primes π contains the prime 2, although some related positive results were obtained in this case in [15]. Other examples in [14] and [15] give insight into occurring phenomena.

Example. Let G be a group isomorphic to $L_2(2^n)$ where n is a positive integer such that $2^n + 1$ is divisible by two different primes (this happens if $n \neq 3$ and $2^n + 1$ is not a Fermat prime). Let $q = 2^n$. Then $G = AB$ where $A \cong C_{q+1}$ is a cyclic group of order $q + 1$ and $B = N_G(G_2)$, with G_2 a Sylow 2-subgroup of G . Let r be a prime dividing $q + 1$ and take $\pi = \pi(N_G(G_2)) \cup \{r\}$. Clearly, $2 \in \pi$. Then $A = A_\pi \times A_{\pi'}$ and B is a π -group, but $A_\pi B$ is not a subgroup.

On the other hand, the paper [17] is devoted to give a complete description of a minimal counterexample of our Main Theorem. In particular, it is shown that such a minimal counterexample has to be an almost simple group. Hence, after providing in Section 2 some necessary preliminaries, mainly referred to finite simple groups, we will prove in Section 3 the Main Theorem by carrying out a case-by-case analysis of the simple groups involved as the socle of the minimal counterexample, leading to a final contradiction.

For notation, if n is an integer and p a prime number, n_p will denote the largest power of p dividing n and $\pi(n)$ the set of prime divisors of n . In particular, for the order $|G|$ of a group G we set $\pi(G) = \pi(|G|)$. Also, $\text{Syl}_p(G)$ will denote the set all Sylow p -subgroups of G .

2 Preliminaries

The following result on factorized groups will be freely used throughout the paper, usually without further reference.

Lemma 1. [2, Lemma 1.3.1] *Let the group $G = AB$ be the product of two subgroups A and B . If x, y are elements of G , then $G = A^x B^y$. Moreover, there exists an element z of G such that $A^x = A^z$ and $B^y = B^z$.*

The following basic lemma will also be used.

Lemma 2. *If G is a soluble group with an abelian Sylow r -subgroups R , for a prime number r , then $G = O_{r'}(G)N_G(R)$.*

Proof. It is well know (see, for example, [9, Theorem 6.3.2]) that in a soluble group $C_G(R \cap O_{r',r}(G)) \leq O_{r',r}(G)$, where $O_{r',r}(G)$ is the r -nilpotent radical of G . Then, when R is abelian, $RO_{r'}(G)$ is a normal subgroup of G and the result follows by applying the Frattini argument. \square

Next we introduce some arithmetical lemmas, which will be applied later in the paper.

Lemma 3 ((Zsigmondy [21])). *Let q and n be integers, $q, n \geq 2$. A prime number r is called primitive with respect to the pair (q, n) (or a primitive prime divisor of $q^n - 1$) if r divides $q^n - 1$ but r does not divide $q^i - 1$ for $i < n$. Then:*

- (1) *There exists a primitive prime divisor of $q^n - 1$ unless $n = 2$ and q is a Mersenne prime or $(q, n) = (2, 6)$.*
- (2) *If the prime r is a primitive prime divisor of $q^n - 1$, then $r - 1 \equiv 0 \pmod{n}$. In particular, $r \geq n + 1$.*

Lemma 4. *Let q and n be integers, $q, n \geq 2$. If an odd prime t divides $q^n + 1$ and is not primitive with respect to the pair $(q, 2n)$, then there exists j dividing n , $j \neq n$, such that t divides $q^j + 1$.*

Proof. Assume that t divides $q^n + 1$ and is not a primitive prime divisor of $q^{2n} - 1$. Then there exists $j < 2n$ such that t is a primitive prime divisor of $q^j - 1$. Since $(q^{2n} - 1, q^j - 1) = q^{(2n,j)} - 1$, it is clear that j divides $2n$. Assume first that j is odd. Since j divides $2n$, it follows that j divides n . Since $q^j \equiv 1 \pmod{t}$, this implies that $q^n \equiv 1 \pmod{t}$. But then t divides $(q^n - 1, q^n + 1)$ and so $t = 2$, a contradiction. So we may assume that j is even. Then $j = 2j_0$, for some j_0 such that j_0 divides n , $j_0 \neq n$. By the choice of j it follows that t divides $q^{j_0} + 1$ and we are done. \square

2.1 Preliminaries on finite simple groups

According to the classification theorem, the finite non-abelian simple groups are to be found among the following families: the alternating groups A_n , with $n \geq 5$; the finite simple groups of Lie type (classical and exceptional); the 26 sporadic groups. The book [11], by Gorenstein, Lyons and Solomon, can be taken as a general reference regarding the background on finite simple groups necessary for the paper. In particular, we will make extensive use along the paper of the detailed knowledge on the orders of the finite simple groups and their automorphisms groups. This can be found in [11] and also in the Atlas [6].

On the other hand, we will use information about the maximal factorizations of the finite simple groups and their automorphisms groups from [19]. In this reference also the orders of such groups are nicely collected in Table 2.1.

We will deal mainly in the paper with classical simple groups of Lie type (for exceptional groups we will use a different strategy (see Lemma 11 below)). The definition and basic properties of such groups can be found in Carter's book [7] and also in [11, Chapters 2, 3, 4]. Moreover, the survey [18] is also a good source for this topic. We collect next the notation and fundamental facts that will be used later to prove our Main Theorem.

Let $L = G(q)$ be a classical finite group of Lie type over a finite field of characteristic p , where q is a power of p . The base field will in most cases be $GF(q)$, the finite field of q elements, except for some twisted groups (see [6, Chapter 3], or [7, Section 14.1]).

Denote by Φ the root system corresponding to the group L , let $\Pi = \{r_1, \dots, r_l\}$ be the set of all fundamental roots and $\Phi^+ \supseteq \Pi$ be the set of all positive roots. The integer l is called the *Lie rank* of L . Denote by X_r the root subgroup corresponding to the root r . In the case when L is a group of untwisted type ($A_l(q), B_l(q), C_l(q)$ or $D_l(q)$), it holds that $X_r = \{x_r(t) | t \in GF(q)\}$. In the remaining cases (twisted groups of types ${}^2A_l(q), {}^2D_l(q)$) we use the description of the root subgroups of the corresponding groups in [7, Chapter 13]. The structure of such subgroups can also be found in [11, Theorem 2.4.1].

Let U be the unipotent subgroup $\langle X_r \mid r \in \Phi^+ \rangle$ (a Sylow p -subgroup) of L . Let B be the *Borel subgroup* containing U , that is, the normalizer in L of U . Then we have that $B = UH$, where $U \cap H = 1$ and H is a *Cartan subgroup* of L . The normalizer of H in L contains a subgroup N such that $N/H \cong W$, the *Weyl group* of L (associated with Φ). The subgroups B and N form a so-called (B, N) *pair* with Weyl group W (see [7, Sections 8.3 and 13.5] or [11, Theorem 2.3.1]). Any subgroup which contains some conjugate of the Borel subgroup B is called a *parabolic subgroup*. A subgroup X of L is called *p -local* if $X = N_L(Q)$ for some non-trivial p -subgroup Q of L . For each $w \in W$, we choose a coset representative $n_w \in N$. We gather in addition the following properties:

- (P1) Each element $g \in L$ can be expressed in the form $g = bn_wu$ with $b \in B = UH$, $n_w \in N$ and $u \in U$ (see [7, Theorem 8.4.3 and Proposition 13.5.3] or [11, Theorem 2.3.5]).
- (P2) $U = \prod_{r \in \Phi^+} X_r$, where the product is taken over all positive roots in an arbitrary ordering (see [7, Theorems 5.3.3 and 13.6.1] or [11, Theorem 2.3.7]).
- (P3) Any p -local subgroup of L is contained in some parabolic subgroup of L . Moreover, if Q is a non-trivial p -subgroup of L , then there exists a parabolic subgroup P such that $Q \leq O_p(P)$ and $N_L(Q) \leq P$ (see [11, Theorem 3.1.3]).
- (P4) The main properties of parabolic subgroups can be found in [11, Section 2.6] and [7, Section 8.3]. In particular, if P is a parabolic subgroup of L , then $C_L(O_p(P)) \leq O_p(P)$ and $|O_p(P)|$ is a power of q . Moreover, P has a *Levi decomposition* $P = O_p(P)H\langle X_r, X_{-r} \mid r \in J \rangle$ for some set of fundamental roots $J \subseteq \Pi$ (see [11, 2.6.5, 2.6.6] or [7, Section 8.5]).
- (P5) The order of a Sylow p -subgroup of the centralizer of a p' -element in the simple group L of Lie rank at least 2 is at most $|U|q^{-2}$ (see [8, Propositions 7-12]).

The structure of the automorphism groups of the groups of Lie type is described thoroughly in [11, Section 2.5] and [7, Chapter 12]. Moreover, we will use also the information about the centralizers of non-inner automorphisms of prime order of such groups which can be found in [10, 9.1] (see also [11, Chapter 4]).

The following results on simple groups of Lie type will be essential for the proof of our Main Theorem.

Lemma 5. *Let $L = G(q)$ be a classical simple group of Lie type over the field $GF(q)$ of characteristic p . Then $|Out(L)|_p \leq q$ and equality holds only when $q \in \{2, 3, 4\}$. Moreover, if $q = 3$, the only possible case when $|Out(L)|_p = q$ appears when $L \cong P\Omega_8^+(q)$. In particular, it holds that $|Out(L)|_p < q^2$ for any classical simple group of Lie type.*

Proof. In [6, Table 5] (see also [19] Table 2.1.A), we find the order of $Out(L)$ when L is a classical group of Lie type. From this table it follows that $|Out(L)|_p \leq 2\log_p(q)$ if $p = 2$ and $|Out(L)|_p \leq \log_p(q)$ if $p \neq 2$ and $L \not\cong P\Omega_8^+(q)$ with $q = p = 3$. Hence $|Out(L)|_p \leq q$ and equality holds only in the asserted cases. \square

Lemma 6. *Let $L = G(q)$ be a classical simple group of Lie type over the field $GF(q)$ of characteristic p of Lie rank at least 2. Let U be a Sylow p -subgroup of L and $S \neq 1$ be a subgroup of U such that $|U : S| < q^2$. Then $C_L(S)$ is a p -group.*

Moreover, if $L \leq G \leq Aut(L)$, then $C_G(S)$ is a p -group.

Proof. We will use the notation and properties (P1)-(P5) of the simple groups of Lie typed described above.

Take a subgroup $S \neq 1$ of U such that $|U : S| < q^2$ and assume that $C_L(S)$ is not a p -group. Then there exists a p' -element g of prime order r in $C_L(S)$. We claim first that r divides $q^2 - 1$.

By (P3), there exists a parabolic subgroup P of L such that $S \leq O_p(P)$ and $N_L(S) \leq P$. Without loss of generality we may assume that $B \leq P$. Let $D = O_p(P)$. By [7], $\pi(H) \subseteq \pi(q^2 - 1)$. Hence, by (P1), $g = bn_w u \in P$, where $b \in B$, $1 \neq w \in W$ and $u \in U$. Now the fact that $|U : S| < q^2$ and (P4) imply that $P = B \cup Bn_w B = DH\langle X_\gamma, X_{-\gamma} \rangle$ for some fundamental root $\gamma \in \Pi$, $w = w_\gamma$ and $|X_\gamma| = q$. But then we obtain that the subgroup $\langle X_\gamma, X_{-\gamma} \rangle$ is isomorphic to $SL_2(q)$ or $L_2(q)$ and hence r divides $|SL_2(q)|$. Therefore, r divides $q^2 - 1$. But applying (P5) we get a contradiction which allows us to deduce that $C_L(S)$ is a p -group.

Now assume that $L \leq G \leq Aut(L)$. Using the information about the centralizers of non-inner automorphisms of prime order of groups of Lie type in [10, 9.1] (see also [11, Chapter 4]), it can be deduced that $C_G(S)$ is also a p -group. \square

We will need later the following lemma on sporadic simple groups.

Lemma 7. *Assume that N is an sporadic simple group which is isomorphic to one in the following set: $\{M_{22}, M_{23}, M_{24}, HS, He, Ru, Suz, Fi_{22}, Co_1\}$. If s is the largest prime dividing $|N|$, then $C_{Aut(N)}(S)$ is an s -group, for any $S \in Syl_s(N)$.*

Proof. The result follows from a case-by-case analysis of the orders of the centralizers of Sylow s -subgroups in each case (see [6] or [11] for the details). \square

3 Proof of the Main Theorem

In this section we assume that G is a counterexample of minimal order to our Main Theorem. The main result in [17] gives a precise description of the structure of such a group:

Theorem 1. [17, Theorem 3] *Let π be a set of odd primes. Assume that the group $G = AB$ is the product of two π -decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ and G is a counterexample of minimal order to the assertion $A_\pi B_\pi = B_\pi A_\pi$.*

Then G has a unique minimal normal subgroup N , which is a non-abelian simple group, so that $N \trianglelefteq G \leq \text{Aut}(N)$.

Moreover, the following properties hold:

- (i) $G = AN = BN = AB$; in particular, $|N||A \cap B| = |G/N||N \cap A||N \cap B|$.
- (ii) $(|A_{\pi'}|, |B_{\pi'}|) \neq 1$, $A_{\pi'} \cap B_{\pi'} = 1$ and $A \cap B$ is a π -group.
- (iii) Neither A nor B is a π -group or a π' -group.
- (iv) $\pi(G) = \pi(N) \geq 5$.
- (v) If, in addition, N is a simple group of Lie type of characteristic p and $p \notin \pi$, then $A \cap B = 1$.

For such a group G and its unique minimal normal subgroup N we have the following results:

Lemma 8. *Assume that $S \leq X$ and S is a s -group for $X \in \{A, B\}$ and a prime number $s \in \sigma$, with $\sigma \in \{\pi, \pi'\}$. Then $\pi(|X : C_X(S)|) \subseteq \sigma$. In particular, $C_X(S)$ is not an s -group.*

Proof. The first part is clear since $X_{\sigma'} \leq C_X(S)$. Consequently, if $C_X(S)$ were a s -group, X would be a σ -group, a contradiction. \square

Lemma 9. *N is not a sporadic simple group.*

Proof. By [19, Theorem C] if N is a sporadic simple group, $N \trianglelefteq G \leq \text{Aut}(N)$ and G is factorized, we have that

$$N \in \{M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, J_2, HS, He, Ru, Suz, Fi_{22}, Co_1\}.$$

Note also that by Theorem 1(iv) the cases $N \cong M_{11}$, $N \cong M_{12}$ and $N \cong J_2$ are not possible. Then Lemmas 7 and 8 provide the contradiction. \square

Lemma 10. *N is not an alternating group of degree $n \geq 5$.*

Proof. First note that, by Theorem 1(iv), we may assume that $N \cong A_n$ with $n \geq 11$. By [19, Theorem D], if $N \trianglelefteq G \leq \text{Aut}(N)$, the only factorizations $G = AB$ where A and B are subgroups of G not containing N verify that $A_{n-k} \triangleleft A \leq S_{n-k} \times S_k$ for some $1 \leq k \leq 5$. Since A_{n-k} is a simple group, because $n - k \geq 5$, and $2 \in \pi(A_{n-k})$, it follows that A_{n-k} is a π' -group. But then $A \leq S_{n-k} \times S_k$ is also a π' -group, which is a contradiction. \square

Lemma 11. *N is not an exceptional group of Lie type.*

Proof. By [19, Theorem B], if N is an exceptional group of Lie type, $N \trianglelefteq G \leq \text{Aut}(N)$ and G is factorized, then

$$N \in \{G_2(q), q = 3^c; F_4(q), q = 2^c; G_2(4)\}.$$

We check next that each of the possibilities for the group N leads to a contradiction. Recall that $\pi(G) = \pi(N)$.

Case $N \cong G_2(4)$. In this case $|N| = 2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$ and $|\text{Out}(N)| = 2$. Since a Sylow 13-subgroup of N is self-centralizing in $\text{Aut}(N)$ (see [6]), we get a contradiction by Lemma 8.

Case $N \cong G_2(q), q = 3^c$. In this case all possible factorizations $G = AB$ (not only the maximal ones) with subgroups A, B not containing N verify $A \cap N \in \{SL_3(q), SL_3(q).2\}$, either $B \cap N \in \{SU_3(q), SU_3(q).2\}$ or $B \cap N = {}^2G_2(q)$ in the case when c is odd, and $N = (A \cap N)(B \cap N)$. Since 2 divides $(|A \cap N|, |B \cap N|)$ and each of the subgroups has a Sylow 3-subgroup containing its centralizer in the corresponding subgroup, we deduce that all these are π' -groups and hence N is a π' -group, a contradiction.

Case $N \cong F_4(q), q = 2^c$. In this case all possible factorizations $G = AB$ (not only the maximal ones) with subgroups A, B not containing N are as follows: $A \cap N = Sp_8(q)$ and $B \cap N \in \{{}^3D_4(q), {}^3D_4(q).3\}$ and $N = (A \cap N)(B \cap N)$. Since 2 divides $(|A \cap N|, |B \cap N|)$ and each of these subgroups has a Sylow 2-subgroup containing its centralizer in the corresponding subgroup, it follows that N is a π' -group, which is a contradiction. \square

From now on we assume that $N = G(q)$ is a classical simple group of Lie type over a field $GF(q)$ of prime characteristic p , with $q = p^e$.

Lemma 12. *Assume that N is of Lie rank $l > 1$, then $(|A \cap N|, |B \cap N|) \equiv 0 \pmod{p}$.*

Proof. From the fact that $G = AN = BN = AB$, it follows that

$$\frac{|N|_p |A \cap B|_p}{|N \cap A|_p |N \cap B|_p} = |G/N|_p,$$

which divides $|\text{Out}(N)|_p$. Suppose that $|B \cap N|$ is not divisible by p . It follows that $|N|_p / |N \cap A|_p$ divides $|G/N|_p$ and, in particular, $|N|_p / |N \cap A|_p \leq |\text{Out}(N)|_p \leq q$ by Lemma 5. For $S \in \text{Syl}_p(N \cap A)$ we deduce from Lemma 6 that $C_G(S)$ is a p -group, and so we have a contradiction by Lemma 8. \square

Lemma 13. *Let $a \in X$ and $b \in Y$, for any $X, Y \in \{A, B\}$, be elements of prime orders $r = o(a)$ and $s = o(b)$, respectively (eventually $a = b$). Assume that $C_N(a)$ and $C_N(b)$ are p' -groups. Then:*

- (i) *If $(|A \cap N|, |B \cap N|) \equiv 0 \pmod{p}$, then $\{p, s, r\} \subseteq \sigma$.*
- (ii) *If, in addition, $a \in A, b \in B$ and $C_N(a), C_N(b)$ are soluble, then $\{p, s, r\} \subseteq \pi'$. In particular, $A \cap B = 1$.*

Proof. (i) Observe that here $p \in \pi(|X : C_X(a)|) \cap \pi(|X : C_X(b)|)$ and the conclusion follows from Lemma 8.

- (ii) Assume now that $a \in A, b \in B$ and $C_N(a), C_N(b)$ are soluble. If $\{p, s, r\} \subseteq \pi$, then $A_{\pi'} \cap N \leq C_N(a), B_{\pi'} \cap N \leq C_N(b)$ and so $A_{\pi'} \cap N$ and $B_{\pi'} \cap N$ are soluble groups. Since G/N is soluble, this means that $A_{\pi'}$ and $B_{\pi'}$ are soluble. Since $2 \notin \pi$, A_{π} and B_{π} are also soluble groups, and we conclude that both A and B are soluble, which is a contradiction to [15, Theorem 2]. Hence we have proved that $\{p, s, r\} \subseteq \pi'$. The assertion $A \cap B = 1$ follows from Theorem 1(v). □

Recall that q is a prime power, $q = p^e$, p a prime and e a positive integer. Also let $n \geq 3$ and $(q, n) \neq (2, 6), (4, 3)$. In the sequel we will denote by q_n any primitive prime divisor of $p^{en} - 1$, i.e. primitive with respect to the pair (p, ne) (so that $q_n | p^{en} - 1$ but $q_n \nmid p^i - 1$ for $i < en$). Note that if r is a primitive prime divisor of $q^{2k} - 1$ for some $k \geq 2$, then r divides $q^k + 1$.

Lemma 14. *For $N = G(q)$ a classical group of Lie type of characteristic p and $q = p^e$, there exist primes $r, s \in \pi(N) \setminus \pi(G/N)$ and maximal tori T_1 and T_2 of N as stated in Table 1.*

Moreover, except for the case denoted by (\star) in Table 1, for any element $a \in N$ of order r and any element $b \in N$ of order s we may assume that $C_N(a) \leq T_1$ and $C_N(b) \leq T_2$, and these are abelian p' -groups.

On the other hand, there is neither a field automorphism nor a graph-field automorphism of N centralizing elements of N of order r or s (except for the triality automorphism in the case $P\Omega_8^+(q)$).

Proof. This can be derived from the information about the maximal tori in these groups (see, for instance, [8]). The information about the centralizers of non-inner automorphisms of prime order of groups of Lie type can be found in [10, 9.1]. □

Whenever Lemma 14 will be applied, we will keep the same notation for the primes r and s and for the elements $a \in N$ and $b \in N$. Since $|N||A \cap B| = |G/N||N \cap A||N \cap B|$ and $r, s \notin \pi(G/N)$, we note that r , and also s , divides either $|N \cap A|$ or $|N \cap B|$. In particular, we may consider either $a \in A \cap N$ or $a \in B \cap N$, and the same for $b \in N$.

In the sequel we will use the notation and the main results in [19], where the maximal factorizations of the almost simple groups are described. More exactly, factorizations $G = XY$ where X and Y are maximal subgroups of the group G with $N \trianglelefteq G \leq \text{Aut}(N)$, not containing N , are described in [19, Tables 1-5].

N	r	s	$ T_1 $	$ T_2 $	Remarks
$L_n(q)$ $n \geq 4$	q_n	q_{n-1}	$\frac{q^n-1}{(n,q-1)(q-1)}$	$\frac{q^{n-1}-1}{(n,q-1)}$	$(n, q) \neq (6, 2)$ $(n-1, q) \neq (6, 2)$
$PSp_{2n}(q)$ $P\Omega_{2n+1}(q)$ $n \geq 3$	q_{2n} q_{2n}	$q_{2(n-1)}$ q_n	$\frac{q^n+1}{(2,q-1)}$ $\frac{q^n+1}{(2,q-1)}$	$\frac{(q^{n-1}+1)(q+1)}{(2,q-1)}$ $\frac{(q^n-1)}{(2,q-1)}$	n even (\star) $(n, q) \neq (4, 2)$ n odd $(n, q) \neq (3, 2)$
$P\Omega_{2n}^-(q)$ $n \geq 4$	q_{2n}	$q_{2(n-1)}$	$\frac{q^n+1}{(4,q^n+1)}$	$\frac{(q^{n-1}+1)(q-1)}{(4,q^n+1)}$	$(n, q) \neq (4, 2)$
$P\Omega_{2n}^+(q)$ $n \geq 4$	$q_{2(n-1)}$ $q_{2(n-1)}$	q_{n-1} q_n	$\frac{(q^{n-1}+1)(q+1)}{(4,q^{n-1})}$ $\frac{(q^{n-1}+1)(q+1)}{(4,q^{n-1})}$	$\frac{(q^{n-1}-1)(q-1)}{(4,q^{n-1})}$ $\frac{q^n-1}{(4,q^{n-1})}$	n even $(n, q) \neq (4, 2)$ n odd

Table 1:

Lemma 15. N is not isomorphic to $L_n(q)$, $n \leq 3$.

Proof. If $N \cong L_2(q)$, apart from some exceptional cases that we will consider next, from [19] we know that possible factorizations $G = AB$ satisfy that A and B are soluble, so the result follows from [15, Theorem 2]. The remaining cases are excluded by Theorem 1(iv), except for $N \cong L_2(q)$ when either $q = 29$ or $q = 59$. Since in both cases a Sylow q -subgroup of N is self-centralizing in $Aut(N)$ and $|G|_q = |N|_q = q$, we get a contradiction from Lemma 8.

Assume now that $N \cong L_3(q)$, so $|Out(N)| = 2(q-1, 3)\log_p(q)$. Observe first that the cases $q \leq 8$ are excluded by Theorem 1(iv). From [19] we know that all factorizations $G = AB$ satisfy that for one of the factors, say A , $|N \cap A|$ divides $\frac{q^3-1}{q-1} \cdot 3$, which is not divisible by $p \neq 3$, a contradiction by Lemma 12. For the case $p = 3$, we would get that $|N : N \cap B|_3 \leq q/3 < q^2$, so $C_G(N \cap B)$ is a p -group because of Lemma 6, and this is a contradiction by Lemma 8. \square

Lemma 16. N is not isomorphic to $L_n(q)$, $n \geq 4$.

Proof. Recall that $|\pi(N)| > 4$ because of Theorem 1(iv).

Assume first that either $N \cong L_6(2)$ or $N \cong L_7(2)$. In both cases, if s is the largest prime number dividing $|N|$, then $|G|_s = |N|_s = s$ and a Sylow s -subgroup of G is self-centralizing in G , a contradiction by Lemma 8.

So we may assume that $N \cong L_n(q)$, with $n \geq 4$, $(n, q) \neq (6, 2)$, $(n, q) \neq (7, 2)$. Then, by Lemma 14, there exist tori T_1 and T_2 in N of the following orders:

$$|T_1| = \frac{q^n - 1}{(n, q-1)(q-1)}, \quad |T_2| = \frac{q^{n-1} - 1}{(n, q-1)}.$$

With the notation of Lemma 14, let $r = q_n$ and $s = q_{n-1}$. Take an element $a \in N$ of order r , and an element $b \in N$ of order s . Then $C_N(a) \leq T_1$, $C_N(b) \leq T_2$ and both subgroups are abelian p' -groups. Since $(|A \cap N|, |B \cap N|) \equiv 0 \pmod{p}$ by Lemma 12, it follows from Lemma 13(i) that $\{p, s, r\} \subseteq \sigma$.

Recall that r does not divide $|G/N|$. We may assume without loss of generality that $r \in \pi(A)$ and $a \in A \cap N$.

Assume first that $p \in \pi$ and so $\{p, s, r\} \subseteq \pi$. In this case $A_{\pi'} \cap N$ and hence A are soluble groups. Assume in addition that s does not divide $|B \cap N|$. Then, both r and s should divide the order of the soluble group $N \cap A$. By the proof of [1, Lemma 3.1] (see also [1, Lemmas 2.5 and 2.6], this can only happen if $n = s \geq 5$, $p = q$ and $|N \cap A|$ divides $s(q^s - 1)$. Therefore, applying the order formula of N , we get that s should divide also $|B \cap N|$, a contradiction. Hence, if $p \in \pi$, we deduce that s divides $|B \cap N|$ and we may assume that $b \in B \cap N$, but this contradicts Lemma 13(ii). Hence we conclude that $\{p, s, r\} \subseteq \pi'$.

Recall now that the field automorphisms of N do not centralize elements of order r or s . Moreover, there is no diagonal automorphism of N centralizing an element of order r . This implies that G/N is a π' -group.

If $s \in \pi(A)$, since $r, s \in \pi'$ we get $\pi \cap \pi(N \cap A) \subseteq \pi(C_N(a)) \cap \pi(C_N(b))$. Since $(|T_1|, |T_2|) = \left(\frac{q^n - 1}{(q-1)(n, q-1)}, \frac{q^{n-1} - 1}{(n, q-1)}\right) = 1$, this means that $A \cap N$ and hence A are π' -groups, a contradiction.

Therefore we may assume that $\{p, s, r\} \subseteq \pi'$, $a \in A \cap N$ and $b \in B \cap N$. Then $\pi \cap \pi(N \cap A) \subseteq \pi(T_1)$ and $\pi \cap \pi(N \cap B) \subseteq \pi(T_2)$, where $(|T_1|, |T_2|) = 1$. Therefore $A_\pi = A_\pi \cap N \leq T_1$, $B_\pi = B_\pi \cap N \leq T_2$ and both are Hall subgroups of N .

Assume first that there exists a prime divisor t of $|A_\pi|$ such that t is not primitive with respect to the pair (q, n) . Since t divides $q^n - 1$ but is not a primitive prime divisor, t divides $q^j - 1$ with j a divisor of n , $j \neq n$ (recall that $(q^n - 1, q^j - 1) = q^{(n, j)} - 1$). If $n = jk$, with $k > 1$ an integer, then it holds that N contains a subgroup of order $((q^j - 1)_t)^k$. But then, by checking the order formula of N , we deduce that t should divide $|B|$, a contradiction since $(|A_\pi|, |B_\pi|) = 1$.

So we may assume that any prime divisor of $|A_\pi|$ is primitive with respect to the pair (q, n) . Then, if we consider any element $x \in A_\pi \leq T_1$ of prime order, we have also that $C_N(x) \leq T_1$, but this means that $A \cap N \leq T_1$, which is the final contradiction since $p \in \pi(A \cap N)$ by Lemma 12. \square

Lemma 17. *N is not isomorphic to $U_n(q)$, $n \geq 3$.*

Proof. Assume that $N \cong U_n(q)$, $n \geq 3$. Suppose first that n is odd. From [19, Theorem A], the only groups G such that $N \leq G \leq \text{Aut}(N)$ and N is a unitary group of odd dimension which are factorizable appear for $N \cong U_3(3)$, $U_3(5)$, $U_3(8)$ or $U_9(2)$. Since in our case $|\pi(N)| \geq 5$, the only possible case would be $N \cong U_9(2)$. Note that in this case $\pi(N) = \{2, 3, 5, 7, 11, 17, 19, 43\}$ and $\text{Out}(N) \cong S_3$. By Lemma 12 we may assume that $p = 2$ divides $(|A \cap N|, |B \cap N|)$. This group N has maximal tori of orders $19 \cdot 3$ and $17 \cdot 5$. We may let $r = 17 \in \pi(A)$. Since the centralizer of an element of order 17 in N has odd order $17 \cdot 5$ and $2 \in \pi'$, we deduce that $r = 17 \in \pi'$, $5 \in \pi$ and $|A_\pi \cap N| = 5$,

so $|A_\pi|$ divides $5 \cdot 3$. On the other hand, an element of N of order $s = 19$ has a centralizer in N of order $19 \cdot 3$. Since $r \in \pi(A)$, we have that $s \notin \pi(A)$ and $s \in \pi' \cap \pi(B)$. This means that $|B_\pi|$ divides 3^2 . Since the order of a 5-Sylow subgroup of N is at least 25, this gives a contradiction.

Assume now that $n = 2m$ is even, $m \geq 2$. It follows from [19, Tables 1, 3] (and with the same notation) that one of the maximal subgroups in the factorization of G with $N \leq G \leq \text{Aut}(N)$, say X , has the property $X \cap N = N_1 \cong U_{2m-1}(q)$, unless $N \cong U_4(2)$ or $U_4(3)$. Since $|\pi(U_4(2))| < 5$ and $|\pi(U_4(3))| < 5$, these possibilities are excluded.

Apart from some exceptional cases that we will check later, any group H such that $N_1 \leq H \leq \text{Aut}(N_1)$ has no proper factorizations (in the sense that the factors do not contain N_1). Assume that $A \leq X$ and so $X = A(X \cap B)$. Now note that $X = N_G(N_1)$ and so $X/C_G(N_1)$ is isomorphic to a subgroup of $\text{Aut}(N_1)$ and then it has no proper factorizations. If $N_1 \cong N_1 C_G(N_1)/C_G(N_1)$ were contained either in $AC_G(N_1)/C_G(N_1)$ or in $(X \cap B)C_G(N_1)/C_G(N_1)$, which are π -decomposable groups, it would follow that $N_1 = X \cap N$ would be a π -group, a contradiction. This means that either $X = AC_G(N_1)$ or $X = (X \cap B)C_G(N_1)$. In the latter case we would have $G = AB = C_G(N_1)B$. But from the structure of $\text{Out}(N)$, it follows that $|C_G(N_1)|$ divides $q+1$ and this factorization would not be possible by order arguments. Now assume $X = AC_G(N_1)$. Since $A = A_\pi \times A_{\pi'}$, applying again that $X/C_G(N_1)$ has no proper factorizations, we get that either $X = A_\pi C_G(N_1)$ or $X = A_{\pi'} C_G(N_1)$. Since A_π is a soluble group and $X/C_G(N_1)$ contains a subgroup isomorphic to N_1 , the case $X = A_\pi C_G(N_1)$ cannot occur. Then $X = A_{\pi'} C_G(N_1)$, and $A_\pi \leq C_G(N_1)$ is of order dividing $q+1$. Then $|X|$ divides $|A_{\pi'}|(q+1)$. But, if $n = 2m > 4$, then $(q+1)^3$ divides $|N_1| = |U_{2m-1}(q)|$, and so $q+1$ divides $|A_{\pi'}|$, which means that A is a π' -group, a contradiction. Finally, if $n = 2m = 4$, then $(q+1)^2/(3, q+1)$ divides $|N_1| = |U_{2m-1}(q)|$, and so $\pi(X) \subseteq \pi(A_{\pi'}) \cup \{3\}$, but $|N_1|_3 > (q+1)_3$, so $3 \in \pi'$ and A is again a π' -group, a contradiction.

The exceptional cases when $N \cong U_{2m}(q)$ and $X \cap N = N_1 \cong U_{2m-1}(q)$ is factorized, appear when $N_1 \cong U_3(3)$, $U_3(5)$, $U_3(8)$ or $U_9(2)$, by [19, Table 3]. The case $N \cong U_4(3)$ corresponding to the first possibility is excluded since $|\pi(N)| \leq 4$. Hence we should study the cases $N \cong U_4(5)$, $U_4(8)$ and $U_{10}(2)$. In all these three cases there exist maximal tori T_1 and T_2 of orders

$$|T_1| = \frac{q^n - 1}{(n, q+1)(q+1)} \quad \text{and} \quad |T_2| = \frac{q^{n-1} + 1}{(n, q+1)}.$$

Take $r = q_n$ and $s = q_{2(n-1)}$, so s divides $q^{n-1} + 1$. It can be seen that:

$$(r, s) = (13, 7), |T_1| = 13 \cdot 2^2 \text{ and } |T_2| = 7 \cdot 3^2, \text{ for } U_4(5);$$

$$(r, s) = (17, 19), |T_1| = 5 \cdot 7 \cdot 13 \text{ and } |T_2| = 3^3 \cdot 19, \text{ for } U_4(8);$$

$$(r, s) = (31, 19), |T_1| = 11 \cdot 31 \text{ and } |T_2| = 19 \cdot 3^3, \text{ for } U_{10}(2).$$

Note also that p divides $(|A \cap N|, |B \cap N|)$ by Lemma 12. Moreover, if a and b are elements of orders r and s , respectively, we have here that $C_N(a) = T_1$ and $C_N(b) = T_2$. Since T_1 and T_2 are soluble p' -groups, we deduce that $\{p, s, r\} \subseteq \pi'$. Moreover, from [19, Table 1] we know that for one of the factors, say B , it holds that $|B \cap N|$ divides $|N_1| = |U_{n-1}(q)|$. By order arguments, we see in each case that r divides $|N \cap A|$ and s divides $|N \cap B|$, and in all cases the primes 2 and 3

divide both $|A \cap N|$ and $|B \cap N|$. On the other hand, $C_N(a) = T_1$ is a $3'$ -group, so $3 \in \pi'$ and this implies that G/N is a π' -group in all cases (recall that $2 \in \pi'$). But then $B_\pi = B_\pi \cap N \leq C_N(b)$ and this is a π' -group, which means that B is a π' -group, a contradiction. \square

Lemma 18. *N is not isomorphic to $PSp_4(q)$, $q = p^e$.*

Proof. Assume that $N \cong PSp_4(q)$ Then $|N| = \frac{1}{(2, q-1)} q^4 (q^4 - 1)(q^2 - 1)$ and $|Out(N)| = (2, q-1)(2, p)e$. Moreover, the cases $q \leq 7$ can be excluded by Theorem 1(iv).

There is a torus T in N of order $\frac{q^2+1}{(2, q-1)}$. Since q^2+1 is not divisible by 4, we have that $|T|$ is odd. Let $r \in \pi(T)$. Since $(\frac{q^2+1}{(2, q-1)}, q^2 - 1) = 1$, we can deduce that r is a primitive prime divisor of $q^4 - 1$ and any element of prime order in T acts irreducibly on the natural module of $Sp_4(q)$. Hence we have that $C_N(a) \leq T$ for any element $1 \neq a \in T$. Since T is a p' -group, applying Lemmas 12 and 13, we deduce that $\{p\} \cup \pi(T) \subseteq \sigma$, for some $\sigma \in \{\pi, \pi'\}$. Moreover, there is no field automorphism of N centralizing any element of T . Without loss of generality assume that $\pi(A) \cap \pi(T) \neq \emptyset$. Then it is easy to deduce that either A is a σ -group or $A = A_\pi \times A_2$ and A is soluble. In the latter case, looking at the orders of maximal soluble subgroups of N divisible by a primitive prime divisor of $q^4 - 1$ (see [1, Lemma 2.8]), we get that $|A \cap N| = |A_\pi \cap N|$ divides $q^2 + 1$. This contradicts Lemma 12 and concludes the proof, since A is not a σ -group. \square

Lemma 19. *N is neither isomorphic to $PSp_{2n}(q)$ nor to $P\Omega_{2n+1}(q)$, $q = p^e$, $n \geq 3$.*

Proof. Assume that N is isomorphic either to $PSp_{2n}(q)$ or to $P\Omega_{2n+1}(q)$, with $n \geq 3$. Then $|N| = \frac{1}{(2, q-1)} q^{n^2} (q^{2n} - 1)(q^{2n-2} - 1) \cdots (q^2 - 1)$ and $|Out(N)| = (2, q-1)e$.

We deal first with the cases (*) not considered in Lemma 14. If $n = 3$, $q = 2$, then $N \cong PSp_6(2) \cong \Omega_7(2)$ and, in this case, $|\pi(N)| = 4$, which contradicts Theorem 1(iv). If $n = 4$, $q = 2$, then $N \cong PSp_8(2) \cong \Omega_9(2)$ and this group has a self-centralizing Sylow subgroup of order 17, which is contained either in A or in B , a contradiction by Lemma 8.

For the cases $(n, q) \neq (3, 2)$ and $(n, q) \neq (4, 2)$, as stated in Lemma 14, N has tori T_1 and T_2 of the following orders:

(a) If n is even:

$$|T_1| = \frac{q^n + 1}{(2, q-1)}, \quad |T_2| = \frac{(q^{n-1} + 1)(q + 1)}{(2, q-1)}.$$

Denote here $r = q_{2n}$ and $s = q_{2n-2}$.

(b) If n is odd:

$$|T_1| = \frac{q^n + 1}{(2, q-1)}, \quad |T_2| = \frac{(q^n - 1)}{(2, q-1)}.$$

Denote here $r = q_{2n}$ and $s = q_n$.

In both cases we will denote by $a \in N$ an element of order r and by $b \in N$ an element of order s . We study these cases separately:

Case (a): n even.

Without loss of generality we may assume that $r \in \pi(A)$ and $a \in A \cap N$.

In this case $C_N(a) \leq T_1$ (and T_1 is abelian), and $C_N(b)/Z(C_N(b)) \cong C \times L$, with $C \leq C_{q^{n-1}+1}$ and $L \cong L_2(q)$. (Recall that $L_2(q) \cong PSp_2(q) \cong \Omega_3(q)$.)

Suppose first that $r \in \pi$. Since $C_N(a)$ is a p' -group, and p divides $(|N \cap A|, |N \cap B|)$ by Lemma 12, we deduce by Lemma 13 that $\{p, r\} \subseteq \pi \cap \pi(A)$ (recall also that r does not divide $|G/N|$). In this case $A_{\pi'} \cap N$ is a soluble group and hence A is a soluble group. By [1, Lemma 2.8], the order of $A \cap N$ divides either $2n(q^n + 1)$ or $16n^2(q - 1)r \log_2(2n)$. In the latter case we have $q = p$, $r = 2n + 1$ and n is a power of 2. Since s is a primitive prime divisor of $q^{2n-2} - 1$, we have that $s \geq 2n - 1$. Hence we deduce that $s \notin \pi(A \cap N)$ and so $s \in \pi(N \cap B)$. If $s \in \pi'$, by the order of $C_N(b)$ we deduce that a Sylow p -subgroup of $B \cap N$ has order at most q . Since $|N|_p \leq |G/N|_p |N \cap A|_p |N \cap B|_p$ we deduce that $q^{n^2} \leq \max\{(\log_p(q) \cdot \log_p(n))_p \cdot q, (\log_2(2n))_p \cdot q\}$ (recall that $p \neq 2$, since we are in the case $\{p, r\} \subseteq \pi$). This gives a contradiction since $n \geq 4$. Therefore we have $s \in \pi$, i. e. $\{p, r, s\} \subseteq \pi$.

Now note that the only non-soluble composition factors of $C_N(b)$ are isomorphic to $L_2(q)$. Since $B_{\pi'}$ is not soluble because of [15, Theorem 2] and its order is coprime with $p \in \pi$, by Dickson's theorem (see [12, II, 8.27]) we deduce that the order of a non-soluble subgroup of $N \cap B$ divides $|A_5|$ or $|S_5|$ and it holds that $q \equiv \pm 1 \pmod{5}$. In this case $5 \in \pi'$, $p \neq 5$ and $q^n + 1 \equiv 2 \pmod{5}$ (recall that n is even). In particular $|A \cap N|_5$ is either n_5 or $\log_2(2n)_5$. On the other hand, $|N \cap B|_5$ does not exceed $((q^{n-1} + 1)(q^2 - 1))_5$. Moreover, since there are no field automorphisms centralizing elements of order r , it follows that $\log_p(q)_5 = 1$. Hence $|N|_5 \leq \max\{n_5((q^{n-1} + 1)(q^2 - 1))_5, \log_2(2n)_5((q^{n-1} + 1)(q^2 - 1))_5\}$, which is a contradiction (recall that $n \geq 3$).

Therefore, we may assume $\{p, r\} \subseteq \pi'$. Suppose that $s \in \pi(A)$. Since $(|C_N(a)|, s) = 1$, this means that $s \in \pi'$. It follows that $\pi \cap \pi(A \cap N) \subseteq \pi(C_N(a)) \cap \pi(C_N(b)) \cap \pi$. But note that $\pi((q^n + 1, (q^{n-1} + 1)(q^2 - 1))) \subseteq \{2\}$ and so it follows that $\pi \cap \pi(A \cap N) = \emptyset$. This means that $A \cap N$ and so A are π' -groups, a contradiction (recall that there is no field automorphism centralizing elements of order r or s).

Thus we conclude that $s \in \pi(B \cap N)$. Assume first that $s \in \pi$. Since field automorphisms do not centralize elements of order $s \in \pi$, we may assume that $p \in \pi'$ does not divide $|G/N|$ (note that for $p = 2$, each outer automorphism of N is a field automorphism). Note also that $|N \cap B|_p \leq q$. Hence it follows from the order formula $|N|_p = |G/N|_p |N \cap A|_p |N \cap B|_p$, that $|N \cap A|_p \leq q^{n^2-1}$, and so $|N_p : (N \cap A)_p| \leq q$ (recall that $A \cap B = 1$, since $p \in \pi'$ by Theorem 1(v)). By Lemma 6, this means that $C_G((N \cap A)_p)$ is a p -group, so A is a π' -group, a contradiction.

Therefore we have that $\{p, r, s\} \subseteq \pi'$. Hence $\pi \cap \pi(N \cap A) \subseteq \pi(q^n + 1)$, $\pi \cap \pi(N \cap B) \subseteq \pi((q^{n-1} + 1)(q^2 - 1))$ and then $\pi \cap \pi(N \cap A) \cap \pi(N \cap B) = \emptyset$. On the other hand, since the field automorphisms of N do not centralize elements of order r or s , and $2 \in \pi'$, we deduce that $A_{\pi} \leq N$, $B_{\pi} \leq N$ and both are Hall subgroups of N .

Assume that there exists $t \in \pi \cap \pi(A)$ which is not a primitive prime divisor

of $q^{2n} - 1$, it follows from Lemma 4 that t divides $q^j + 1$, for some $j \neq 1$ dividing n . We claim that $n = lj$, with l odd and $l \geq 3$. Indeed, if l is even, since $q^j \equiv (-1) \pmod{t}$, we get $q^n = (q^j)^l \equiv 1 \pmod{t}$, a contradiction since t divides $q^n + 1$. Now, since N has a torus of order $(q^j + 1)^l$ which is not contained in $A_\pi = A_\pi \cap N \leq T_1$ and G/N is a π' -group, we get a contradiction with the fact that $(t, |N \cap B|) = 1$ (recall $n \geq 3$).

Hence we may assume that each prime in $\pi \cap \pi(A)$ is a primitive prime divisor of $q^{2n} - 1$. Then if we consider any element $x \in A_\pi \leq T_1$ of prime order we have also that $C_N(x) \leq T_1$, but this means that $A \cap N \leq T_1$, which is the final contradiction since $p \in \pi(A \cap N)$.

Case (b): n odd.

Without loss of generality we may assume that $r \in \pi(A)$. In this case $C_N(a) \leq T_1$, $C_N(b) \leq T_2$ and both centralizers are abelian. If $r \in \pi$, we have also $p \in \pi$, by Lemmas 12 and 13. In this case A is soluble and we deduce that $s = q_n \notin \pi(A)$ as in case (a). Hence $s \in \pi(B \cap N)$ and since p divides $|N \cap B|$ and $|C_N(b)|$ divides $q^n - 1$, we deduce that $s \in \pi$. In this case both subgroups $A \cap N$ and $B \cap N$ are soluble, so A and B are soluble and this gives a contradiction with [15, Theorem 2].

So we can assume that $r \in \pi'$, so that $p \in \pi'$ and $\pi \cap \pi(N \cap A) \subseteq \pi(C_N(a)) \subseteq \pi(q^n + 1)$. If $s \in \pi(A)$, we get $s \in \pi'$ by Lemma 13, and hence $\pi \cap \pi(N \cap A) \subseteq \pi(C_N(b)) \subseteq \pi(q^n - 1)$. Since $(q^n + 1, q^n - 1)_{2'} = 1$, this means that $A \cap N$ and hence A are π' -groups, a contradiction.

Now we may assume $s \in \pi(B \cap N) \cap \pi'$, because $p \in \pi'$. Again we have $\pi \cap \pi(N \cap A) \subseteq \pi(q^n + 1)$ and since the field automorphisms of N do not centralize an element of order r , it follows that $|G/N|$ is a π' -group and $A_\pi = A_\pi \cap N$. On the other hand, we deduce also that $\pi \cap \pi(B \cap N) \subseteq \pi(q^n - 1)$ and $B_\pi = B_\pi \cap N$. Since $(q^n + 1, q^n - 1)_{2'} = 1$, it turns out that A_π and B_π are Hall subgroups of N , and also of G . As in case (a) we deduce that for some prime divisor of $q^n + 1$, $t \in \pi$, we have $n = lj$ with $l \geq 3$ odd and $q^j + 1 \equiv 0 \pmod{t}$. We get a contradiction as in case (a), since $(q^j + 1)^l$ divides $|N|$.

□

Lemma 20. N is not isomorphic to $P\Omega_{2n}^+(q)$, $q = p^e$, $n \geq 4$.

Proof. Note that $P\Omega_6^+(q) \cong L_4(q)$ and this case has been studied in Lemma 16. Assume that $N \cong P\Omega_{2n}^+(q)$, $n \geq 4$. Then $|N| = \frac{1}{d}q^{n(n-1)}(q^{2n-2} - 1) \cdots (q^2 - 1)(q^n - 1)$, $d = (4, q^n - 1)$, and $|Out(N)| = 2de$ if $n \geq 5$ and $|Out(N)| = 6de$ if $n = 4$.

As stated in Lemma 14, N has tori T_1 and T_2 of the following orders:

(a) If n is even:

$$|T_1| = \frac{1}{d}(q^{n-1} + 1)(q + 1), \quad |T_2| = \frac{1}{d}(q^{n-1} - 1)(q - 1).$$

With the notation of Lemma 14, let $r = q_{2n-2}$ and $s = q_{n-1}$.

(b) If n is odd:

$$|T_1| = \frac{1}{d}(q^{n-1} + 1)(q + 1), \quad |T_2| = \frac{1}{d}(q^n - 1).$$

In this case let $r = q_{2n-2}$ and $s = q_n$.

If $n = 4$ and $q = 2$, $|\pi(P\Omega_8^+(2))| = 4$, hence $(n, q) \neq (4, 2)$ and, in particular, all such primitive prime divisors exist.

Let $a \in N$ be an element of order r and $b \in N$ an element of order s , $C_N(a) \leq T_1$, $C_N(b) \leq T_2$ and recall that these subgroups are abelian p' -groups. Since p divides $(|N \cap A|, |N \cap B|)$ we deduce that $\{p, r, s\} \subseteq \sigma$, for $\{\sigma, \sigma'\} = \{\pi, \pi'\}$.

Now note that, for $n > 4$, since a field or a graph-field automorphism do not centralize neither an element of order r nor an element of order s , it follows that $\pi(G/N) \setminus \{2\} \subseteq \sigma$ if $r, s \in \sigma$. In the case $n = 4$ there exist graph automorphisms of order 3 and $|G/N|_3 \leq 3 \cdot \log_p(q)_3$. We claim that in this case $\{r, s, 3\} \subseteq \sigma$ and so the previous conclusion for $\pi(G/N)$ remains valid when $n = 4$. Assume that $3 \in \sigma'$. If $r \in \pi(A)$ and $s \in \pi(B)$, then $|N \cap A|_3 |N \cap B|_3$ divides $((q^3 + 1)(q + 1))_3 ((q^3 - 1)(q - 1))_3$, which is not the case by checking $|N|_3$. Without loss of generality if $r, s \in \pi(A)$, then $\pi(A) \cap \sigma' \subseteq \pi(((q^3 + 1)(q + 1), (q^3 - 1)(q - 1))) \subseteq \{2\}$ and so $3 \notin \sigma'$, a contradiction which proves the claim.

Without loss of generality assume that $r \in \pi(A \cap N)$. Observe that in both considered cases (a) and (b), $|N_N(\langle a \rangle) / C_N(\langle a \rangle)|$ divides $2(n - 1)$ and $r \equiv 1 \pmod{2n - 2}$. Moreover, in case (a) it holds that $|N_N(\langle b \rangle) / C_N(\langle b \rangle)|$ divides $2(n - 1)$ and $s \geq n$. On the other hand, in case (b) we have that $|N_N(\langle b \rangle) / C_N(\langle b \rangle)|$ divides $2n$ and $s \geq n + 1$.

Assume that $\{p, r, s\} \subseteq \pi$. Since $A_{\pi'} \cap N \leq C_N(a) \leq T_1$, we deduce that $A_{\pi'} \cap N$ and hence A are soluble groups. Since a Sylow r -subgroup of A is cyclic, we have that $A \cap N = O_{r'}(A \cap N) N_{A \cap N}(\langle a \rangle)$ by Lemma 2. Moreover, $|N_{N \cap A}(\langle a \rangle) / C_{N \cap A}(\langle a \rangle)|$ divides $2n - 2$.

Suppose first that $s \in \pi(A)$ and $b \in N \cap A$. Since s does not divide $|T_1|$, nor $|C_{N \cap A}(a)|$, it follows that either s divides $2(n - 1)$ or $s \in \pi(O_{r'}(A \cap N))$. Since $s \geq n$, the first case cannot occur. Hence $s \in \pi(O_{r'}(A \cap N))$. Since Sylow s -subgroups of A are also cyclic, we have that $A \cap N = O_{s'}(A \cap N) N_{A \cap N}(\langle b \rangle)$. Observe that elements of order sr do not exist in N . Consequently, r divides $2n - 2$ or $2n$, which is not the case as $r \geq 2n - 1$. Hence $s \in \pi(B \cap N)$ and we may assume that $b \in N \cap B$. This contradicts that $\{p, r, s\} \subseteq \pi$ by Lemma 13(ii).

Hence we have $\{p, r, s\} \subseteq \pi'$ and so $\pi(G/N) \subseteq \pi'$. Suppose that $s \in \pi(A)$. Then we may deduce that $\pi \cap \pi(A) \subseteq \pi(C_N(a)) \cap \pi(C_N(b))$. But note that $\pi(|T_1|, |T_2|) \subseteq \{2\}$, in both cases (a) and (b), and so it follows that $\pi \cap \pi(A) = \emptyset$, which means that A is a π' -group, a contradiction.

Now we have that $s \in \pi(B \cap N)$. It follows that $A_{\pi} = A_{\pi} \cap N \leq T_1$ and $B_{\pi} = B_{\pi} \cap N \leq T_2$ are Hall subgroups of N , and also of G . Arguing as in cases $L_n(q)$ or $PSp_{2n}(q)$, by using the order formula of N , we get the final contradiction. \square

Lemma 21. N is not isomorphic to $P\Omega_{2n}^-(q)$, $q = p^e$, $n \geq 4$.

Proof. If $N \cong P\Omega_8^-(2)$, we may consider $r = 17$ and there exists a self-centralizing Sylow subgroup of this order, so we get a contradiction by Lemma 8.

Assume that $N \cong P\Omega_{2n}^-(q)$, $n > 4$. By Lemma 14 we can consider tori T_1 and T_2 of N of the following orders:

$$|T_1| = \frac{q^n + 1}{(4, q^n + 1)}, \quad |T_2| = \frac{(q^{n-1} + 1)(q - 1)}{(4, q^n + 1)},$$

primitive divisors $r = q_{2n}$, $s = q_{2n-2}$, and elements a and b of orders r and s , respectively, such that $C_N(a) \leq T_1$, $C_N(b) \leq T_2$, and these subgroups are abelian p' -groups. In particular, $\{p, r, s\} \subseteq \sigma$, for $\{\sigma, \sigma'\} = \{\pi, \pi'\}$, since $(|N \cap A|, |N \cap B|) \equiv 0 \pmod{p}$. Moreover, $\pi(\log_p(q)) \subseteq \sigma$ because field automorphisms of N do not centralize elements of order r or s .

Without loss of generality assume that $r \in \pi(A)$. Suppose first that $r \in \pi$. Then $A_{\pi'} \cap N$ and hence A are soluble groups. Moreover, by Lemma 13(ii) we deduce that $s \in \pi(A)$. Since Sylow r -subgroups of N and Sylow s -subgroups of N are cyclic, we may consider $A \cap N = O_{r'}(A \cap N)N_{A \cap N}(\langle a \rangle) = O_{s'}(A \cap N)N_{A \cap N}(\langle b \rangle)$. Observe that $|N_N(\langle a \rangle)/C_N(\langle a \rangle)|$ divides $2n$ and $|N_N(\langle b \rangle)/C_N(\langle b \rangle)|$ divides $2(n-1)$, where $r \geq 2n+1$ and $s \geq 2n-1$. Since there are no elements of order rs in N we deduce that $s \notin \pi(A)$, a contradiction.

Hence we may assume $\{p, r, s\} \subseteq \pi'$ and then $|G/N|$ is a π' -group. If $r, s \in \pi(A)$, the order of A_π would divide $(|T_1|, |T_2|)_{2'} = 1$ and so A would be a π' -group, a contradiction.

Therefore we have that $r \in \pi(A \cap N)$, $s \in \pi(B \cap N)$, and so $A_\pi = A_\pi \cap N \leq T_1$, $B_\pi = B_\pi \cap N \leq T_2$ are Hall subgroups of N and G . Arguing like in previous cases, using the order formula of $|N|$, we get the final contradiction. \square

The Main Theorem is proved.

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