Consistent completion of incomplete judgments in decision making using AHP

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Abstract. Decision making (DM) processes are becoming increasingly complex. The reasons are manifold. DM usually involves many aspects; some are purely technical, while others are subjective and derived from social, political, and environmental factors, among others. As a result, they involve items that are not easily comparable under the same units of measurement. Problems are made even more complex by the fact that current governance processes tend to involve all the stakeholders in the DM process.

In this paper we consider the AHP methodology (analytic hierarchy process), which is used to build consistent aggregate results from data provided by decision makers. As some of the actors involved may not be completely familiar with all the criteria under consideration, it is common that the body of opinion, expressed in terms of pairwise comparison, is incomplete. To overcome this weakness, we propose a framework that enables users to provide data on their preferences in a partial and/or incomplete way and at different times. This article is an advance towards a dynamic model of AHP. The authors have addressed the problem of adding a new criterion or deleting obsolete criteria. Here, we address the consistent completion of a reciprocal matrix as a mechanism to obtain a consistent body of opinion issued in an incomplete manner by a specific actor. This feature is incorporated into a process of linearization previously introduced by the authors, which is concisely presented. Finally, we provide an application for leakage control in a water supply company. The adoption of suitable control leakage policies in water supply is a problem of enormous interest in the water industry, particularly in urban hydraulics.

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Keywords: analytic hierarchy process, dynamic decision-making, leakage control, water supply systems

1 Introduction

The alternatives may be varied in decision making (DM) processes, and the decision consists in choosing the most desirable alternative after considering a set of criteria. The decision process can be complicated for several reasons. One of these reasons arises from the fact that criteria are not often comparable using the same unit of measurement. Also, there is a tendency in current governance processes to involve all the stakeholders in the DM process. For these reasons the process of decision making can be very complex and so adequate tools are necessary to support the process.

AHP (analytic hierarchy process) was introduced by Saaty [24, 25, 26] and with appropriate modifications the process can be used to integrate all of these aspects. Its hierarchical structure is an effective framework for DM and organizes the problem in terms of objectives, criteria, and alternatives. Also, the evaluation of criteria and alternatives in pairs and the subsequent construction of comparison matrices have been shown to be effective mechanisms for the joint treatment of tangible and intangible objectives. The use of appropriate mathematical techniques enables the prioritization of heterogeneous –and often very different– elements. This is crucial in decision-making. We present the basic elements of AHP in Section 2.

There are several problems associated with this methodology. The main problem is the possible lack of consistency in comparison matrices, as comparative judgments are subjective since they are issued by experts and/or other actors in the decision process. These matrices should accommodate a chosen scale (see [12, 22, 18, 23], including, of course, the work by Saaty). Consequently, it is predictable –even reasonable– that the issued global opinion body lacks a minimum consistency, which is essential for the prioritization to be meaningful, reliable, and consequently, conducive to a sound decision. The literature contains numerous mechanisms to improve consistency, [11, 21, 8, 1, 20, 9, 13, 3, 4, 6] among many others, and any attempt to improve consistency results in a better quality decision [18]. Among the methods to improve consistency, we present in Section 3 the linearization method introduced by the authors [4] on which the main contribution of this article is built.

A major problem is caused by the growing necessity for all the actors to be involved in decision processes. This leads to a couple of challenges. Firstly, the design of appropriate mechanisms for achieving consensus on a final decision that integrates the different points of view, possibly conflicting, of the various actors. This is one of the challenges to which most effort is currently being devoted; see among others [19, 29, 2, 10, 15]. However, as a precondition, some actors may not be completely familiar with one or more of the elements about which they have to issue their judgment or opinion. The authors have addressed this issue in [5] for a specific scenario: the addition or deletion of a criterion. In leakage control, for example, only economic aspects have so far been considered. However, environmental concerns are becoming important, and more recently, social elements have begun to play an important role in decision-making policy on leakage control. It is natural that some of the actors involved are not familiar enough with all the issues to make appropriate comparison judgements. In this paper we address this scenario: that of stakeholders being consulted when they are not familiar with the effects that some elements may have in the problem. As a result, it is difficult to gather complete information about the preferences of such a decision maker at a given moment. It seems reasonable to allow such an actor to express their preferences several times at his or her own convenience. Meanwhile, partial results based on partial preference data may be generated from data collected at various times –and this data may eventually be consolidated when the information is complete.

Several authors [17, 7] have addressed the problem of producing preference data generated from incomplete information using various techniques that mainly involve optimization applied to different objective functions. For example, in [7] an approximation to the priority vector is obtained from an incomplete judgement. In Section 4 we provide a full matrix termination mechanism for an incomplete comparison matrix produced by an actor. This mechanism uses an algebraic method (instead of optimization processes) to minimize a distance in a matrix set (see [5, Sec. 2.2] for a justification of this metric). As a result it is efficient, robust, and easy to use.

To conclude the paper, in Section 5 we present the case of a judgment made by an actor from a water supply system in relation to leakage management policies. We present some conclusions and comments on specific elements for future work in Section 6.

2 AHP basics

The AHP developed by Saaty [24] formalizes the intuitive understanding of complex problems by building a hierarchical model.

The purpose of the method is to allow the actor involved to visually structure a multicriteria problem in a hierarchical manner. This hierarchy consists of three levels: the highest level contains the goal, the middle level contains the criteria, and the lowest level presents alternatives. Once the hierarchical model is constructed, comparisons are made between pairs of criteria and also between pairs of alternatives for each criterion. The process typically concludes by providing a summary of results through a process of aggregation.

The entire process is based on the fact that it enables the assignation of numerical values to the judgments given by the actor, making it possible to measure how each element contributes to the level of the hierarchy that is immediately above. Use is made of a specific scale for these comparisons in terms of preference or importance. We use here the scale developed by Saaty [24], with the possibility of including intermediate numerical (decimal) values in the scale to model hesitation between two adjacent judgments [27].

In the first step, the expert makes a comparison between pairs of criteria. Based on the scale of values, a comparison matrix of criteria is built. It is a square matrix of order n, $A = [a_{ij}]$, $1 \leq i, j \leq n$, where n is the number of criteria considered. The element a_{ij} represents the comparison between element i and element j. Subsequently, a similar exercise comparing alternatives for each criterion is performed, thus building comparison matrices of alternatives.

We recall here the main facts about this type of matrix. Consider a real matrix of size $n \times n$. A is *positive* if $a_{ij} > 0$ for all i, j, A is *homogeneous* if A is positive and $a_{ii} = 1$ for all i, A is *reciprocal* if A is positive and $a_{ij} = 1/a_{ji}$, for all i, j. These are typical properties of comparison matrices commonly found in AHP. In addition, A is *consistent* if A is positive and $a_{ik} = a_{ij}a_{jk}$, for all i, j, k. We shall always consider vectors of $\mathbb{R}^n$ as column vectors. Among the various characterizations of consistent matrices, let us recall the following:

Proposition 1 (Theorem 1 of [6]). *A positive matrix A is consistent if and only if there is a vector $\mathbf{x}$ in $\mathbb{R}^n$ such that $A = \mathbf{x}J(\mathbf{x})^T$, where J is the map that associates a positive matrix $A = [a_{ij}]$ with the matrix whose entry (i, j) is $1/a_{ij}$.*

We also recall that if X is any matrix, X^T denotes the transpose matrix of X. The characterization given by Proposition 1 can be used to build the consistent matrix closest to a given comparison matrix, after obtaining the priority vector $\mathbf{x}$. This vector is closely related to the Perron vector of a positive matrix.

The principal eigenvalue of a comparison matrix and its associated eigenvector (Perron vector) provide information for complex decision-making: the normalized Perron eigenvector provides the priority vector sought [25, 26]. Generally, however, A is not consistent. The hypothesis that the estimates of these values are small perturbations of the 'correct' values also guarantees small perturbation to the eigenvalues (see, for example [28]). For non-consistent matrices, the problem to be solved is the eigenvalue problem $A\mathbf{w} = \lambda_{\max}\mathbf{w}$, where $\lambda_{\max}$ is the single largest eigenvalue of A that provides the Perron eigenvector as an estimate of the vector of priorities. As a measure of the inconsistency, Saaty proposes using the consistency index $\mathrm{CI} = (\lambda_{\max} - n)/(n - 1)$ and the consistency ratio $\mathrm{CR} = \mathrm{CI}/\mathrm{RI}$, where RI is an average consistency [26]. If $\mathrm{CR} < 0.1$, the estimate is accepted, otherwise, a new comparison matrix is requested until $\mathrm{CR} < 0.1$.

3 Consistency through linearization

There are several proposals in the literature to improve the consistency of a matrix. In this paper we use the technique called linearization [4]. The process begins with a comparison matrix provided by the actor who issued the judgment. Most of these matrices turn out to be inconsistent, even if issued by the best expert(s) in a topic. It is also highly probable that these matrices do not have acceptable consistency ratios. In this situation it is necessary to apply a method to improve the consistency, such as the linearization method concisely presented below.

Let $M_{m,n}$ be the set composed of $m \times n$ real matrices, and let $M^+_{m,n}$ be the subset of $M_{m,n}$ formed of matrices whose entries are positive. The symbol $\mathrm{tr}(A)$ denotes the trace of the square matrix A. Let us denote by R_n and C_n the sets of reciprocal and consistent, respectively, $n \times n$ matrices. It is evident that $C_n \subseteq R_n$. Furthermore, it is also simple to obtain that $C_2 = R_2$ and if $n > 2$, then $C_n \neq R_n$. In addition, $\mathbf{1}_n$ will denote the $n \times 1$ column vector all of whose components are 1. Let, finally, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$.

We shall use the nonlinear mappings L and its inverse E, which are defined now

$$L : M^+_{n,n} \to M_{n,n}, \qquad (L(A))_{ij} = \log(a_{ij}),$$

$$E : M_{n,n} \to M^+_{n,n}, \qquad (E(A))_{i,j} = \exp(a_{ij}).$$

As we have indicated, the following problem is significant: given $A \in R_n$, how can we find its closest matrix $B \in C_n$? Evidently, we have to specify the exact meaning of the words 'closest matrix' to answer this question. In other parlance, we have to define a distance in $M^+_{n,n}$. We shall utilize the distance in $M^+_{n,n}$

$$\mathrm{d}(A, B) = \|L(A) - L(B)\|_F, \qquad A, B \in M^+_{n,n}, \tag{1}$$

derived from the Frobenius norm $\|\cdot\|_F$, i.e., $\|X\|_F^2 = \mathrm{tr}(X^T X)$. This map, which is easily proven to be a distance, makes $M^+_{n,n}$ a complete metric space endowed with the distance the map defines. Furthermore, the use of this distance in AHP problems is natural – as justified in [5, Sec 2.2].

To solve the aforementioned problem, the following linear mapping will be useful

$$\phi_n : \mathbb{R}^n \to M_{n,n}, \qquad \phi_n(\mathbf{x}) = \mathbf{x}\mathbf{1}_n^T - \mathbf{1}_n\mathbf{x}^T. \tag{2}$$

The following results were given in [4]. We will consider $M_{n,n}$ as an Euclidean vector space endowed with the following inner product: $\langle A, B\rangle = \mathrm{tr}(A^T B)$ for $A, B \in M_{n,n}$. We shall also endow any $\mathbb{R}^p$ with the Euclidean norm: i.e., $\|\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^p$.

Theorem 1 (Theorems 2.2 and 2.4 of [4])**.** *Let $\mathscr{L}_n$ denote the set $\{L(A) : A \in C_n\}$. Then*

(i) *$\mathscr{L}_n$ is a linear subspace of $M_{n,n}$ whose dimension is $n-1$ and coincides with $\mathrm{Im}\,\phi_n$.*

(ii) *If $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ is an orthogonal basis of $(\mathrm{span}\{\mathbf{1}_n\})^\perp$, then $\{\phi_n(\mathbf{y}_1), \ldots, \phi_n(\mathbf{y}_{n-1})\}$ is an orthogonal basis of $\mathscr{L}_n$.*

Theorem 2 (Theorem 2.5 of [4])**.** *Let $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ be an orthogonal basis of $(\mathrm{span}\{\mathbf{1}_n\})^\perp$. If $A \in M^+_{n,n}$, then the following matrix*

$$X_A = \frac{1}{2n}\sum_{i=1}^{n-1} \frac{\mathrm{tr}\left(L(A)^T\phi_n(\mathbf{y}_i)\right)}{\|\mathbf{y}_i\|^2}\phi_n(\mathbf{y}_i)$$

is the orthogonal projection of $L(A)$ onto $\mathscr{L}_n$. In other words, the matrix $E(X_A)$ is the closest consistent matrix to A in the sense of the distance given in (1).

The utility of this result is that if $A \in M^+_{n,n}$ is reciprocal, then $E(X_A)$ is consistent and it is expected to be a good approximation of A.

4 Consistent completion of an incomplete comparison matrix

In this section we consider the scenario already stated in the introduction, in which the actor involved is not acquainted with the effects that some elements may have in the problem. The goal is to use (consistently) the partial information given by the actor, with which only an incomplete comparison matrix can be built, i.e., a matrix with some missing entries (located in symmetrical positions).

That is to say, consider the following problem.

Problem 1. *Let $A \in M^+_{n,n}$ be a reciprocal matrix with some unspecified symmetrical entries. How can it be completed consistently?*

In general, the answer to this problem is negative, as evidenced by the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & \star \\ 1/2 & 1 & 1 & 2 \\ 1/3 & 1 & 1 & \star \\ \star & 1/2 & \star & 1 \end{bmatrix}. \tag{3}$$

This matrix cannot be completed (assigning values to the asterisks) consistently since its rank is greater than or equal to 3 (there is a 3×3 nonsingular submatrix) taking into account that, as proven in [6], a consistent matrix is necessarily of rank 1.

In the light of this counterexample, we formulate the following problem.

Problem 2. *Let A be an $n \times n$ reciprocal matrix with some unspecified (symmetrical) entries. How can it be completed so that it is 'as much consistent as possible'?*

By using some logarithms this problem can be reformulated more accurately within the context of the linearization method described above.

Problem 3. *Let A be an $n \times n$ reciprocal and positive matrix with some unspecified (symmetrical) entries. How can it be completed so as to minimize $\mathrm{d}(A, C_n)$?*

In this statement $\mathrm{d}(\cdot, \cdot)$ represents the distance defined in (1).

This problem has an interesting added value. Indeed, it can also be used to provide consistency to a reciprocal matrix where some entries are considered untouchable, that is, whose values are not to be modified in the process of improving the consistency of a comparison matrix.

Let us think about the incomplete matrix A given in (3). Observe that $L(A)$ can be written as

$$\begin{aligned} B(\lambda,\mu) &= \begin{bmatrix} 0 & \log 2 & \log 3 & \lambda \\ -\log 2 & 0 & 0 & \log 2 \\ -\log 3 & 0 & 0 & \mu \\ -\lambda & -\log 2 & -\mu & 0 \end{bmatrix} \\ &= B_0 + \lambda \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} + \mu \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \\ &= B_0 + \lambda(\mathbf{e}_1\mathbf{e}_4^T - \mathbf{e}_4\mathbf{e}_1^T) + \mu(\mathbf{e}_3\mathbf{e}_4^T - \mathbf{e}_4\mathbf{e}_3^T), \end{aligned}$$

where λ and μ are unspecified real numbers and

$$B_0 = \begin{bmatrix} 0 & \log 2 & \log 3 & 0 \\ -\log 2 & 0 & 0 & \log 2 \\ -\log 3 & 0 & 0 & 0 \\ 0 & -\log 2 & 0 & 0 \end{bmatrix}.$$

From now on we will use the notation

$$B_{ij} = \mathbf{e}_i\mathbf{e}_j^T - \mathbf{e}_j\mathbf{e}_i^T, \qquad 1 \le i < j \le n. \tag{4}$$

In general, we can state the following simple but important fact.

Proposition 2. *Let $A \in M_{n,n}^+$ be a reciprocal matrix with some unspecified (symmetrical) entries. Then there exist a skew-Hermitian matrix $B_0 \in M_{n,n}$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that*

$$L(A) = B_0 + \sum_{r=1}^{k} \lambda_r B_{i_r j_r}.$$

From now on, $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ will denote any orthogonal basis of $(\mathrm{span}\{\mathbf{1}_n\})^\perp$. Let us observe that Theorem 2.6 of [4] enables us to find an orthonormal basis of $(\mathrm{span}\{\mathbf{1}_n\})^\perp$ without any computation. Taking into account this result, Theorem 1, and Proposition 2, Problem 3 can be stated equivalently as

Problem 4. *Let $B_0 \in M_{n,n}$ be a skew-Hermitian matrix and $1 \le i_1, j_1, \ldots, i_k, j_k \le n$ be indices with $i_r < j_r$, $r = 1, \ldots, k$. Find $\lambda_1, \ldots, \lambda_k$ and $\mu_1, \ldots, \mu_{n-1}$ such that*

$$\left\| B_0 + \sum_{r=1}^{k} \lambda_r B_{i_r j_r} - \sum_{s=1}^{n-1} \mu_s \phi_n(\mathbf{y}_s) \right\|_F \le \left\| B_0 + \sum_{r=1}^{k} \lambda'_r B_{i_r j_r} - \sum_{s=1}^{n-1} \mu'_s \phi_n(\mathbf{y}_s) \right\|_F \tag{5}$$

for all $\lambda'_1, \ldots, \lambda'_k, \mu'_1, \ldots, \mu'_{n-1} \in \mathbb{R}$.

To solve Problem 4 we shall apply the following result whose proof is included for the sake of completeness.

Theorem 3. *Let U, V be linear subspaces of an Euclidean vector space E, $\mathbf{p}, \mathbf{q} \in E$, and $\mathbf{u} \in U$, $\mathbf{v} \in V$. The following affirmations are equivalent.*

(i) $\|\mathbf{p}+\mathbf{u}-(\mathbf{q}+\mathbf{v})\| \leq \|\mathbf{p}+\mathbf{u}'-(\mathbf{q}+\mathbf{v}')\|$ *for all* $(\mathbf{u}', \mathbf{v}') \in U \times V$.

(ii) $\mathbf{p}+\mathbf{u}-(\mathbf{q}+\mathbf{v}) \in U^{\perp} \cap V^{\perp}$.

Proof. (i) $\Rightarrow$ (ii): Let $\mathbf{x} = \mathbf{p}+\mathbf{u}-(\mathbf{q}+\mathbf{v})$. Take $\mathbf{y}$ an arbitrary vector of U. By hypothesis, we obtain for any $\lambda \in \mathbb{R}$

$$\|\mathbf{x}\|^2 \leq \|\mathbf{p}+\mathbf{u}+\lambda\mathbf{y}-(\mathbf{q}+\mathbf{v})\|^2 = \|\mathbf{x}+\lambda\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \lambda^2\|\mathbf{y}\|^2 + 2\lambda\langle\mathbf{x},\mathbf{y}\rangle.$$

Therefore,

$$0 \leq \lambda^2\|\mathbf{x}\|^2 + 2\lambda\langle\mathbf{x},\mathbf{y}\rangle. \tag{6}$$

If we take $\lambda > 0$, then $0 \leq \lambda\|\mathbf{x}\|^2 + 2\langle\mathbf{x},\mathbf{y}\rangle$, which by making $\lambda \to 0^+$ reduces to $0 \leq \langle\mathbf{x},\mathbf{y}\rangle$. If we take $\lambda < 0$, then (6) implies $0 \geq \lambda\|\mathbf{x}\|^2 + 2\langle\mathbf{x},\mathbf{y}\rangle$, which by making $\lambda \to 0^-$ reduces to $0 \geq \langle\mathbf{x},\mathbf{y}\rangle$. Therefore, $0 = \langle\mathbf{x},\mathbf{y}\rangle$. Since $\mathbf{y}$ was an arbitrary vector of U, then $\mathbf{x} \in U^{\perp}$.

The proof of $\mathbf{x} \in V^{\perp}$ is similar.

(ii) $\Rightarrow$ (i): Observe that $\mathbf{p}+\mathbf{u}-(\mathbf{q}+\mathbf{v}) \in U^{\perp} \cap V^{\perp} = (U+V)^{\perp}$. By the Pythagorean theorem, if $(\mathbf{u}', \mathbf{v}') \in U \times V$, then

$$\|\mathbf{p}+\mathbf{u}'-(\mathbf{q}+\mathbf{v}')\|^2 = \|[\mathbf{p}+\mathbf{u}-(\mathbf{q}+\mathbf{v})]+[\mathbf{u}'-\mathbf{u}+\mathbf{v}-\mathbf{v}']\|^2 = \|\mathbf{p}+\mathbf{u}-(\mathbf{q}+\mathbf{v})\|^2+\|\mathbf{u}'-\mathbf{u}+\mathbf{v}-\mathbf{v}'\|^2.$$

Now, it is evident $\|\mathbf{p}+\mathbf{u}'-(\mathbf{q}+\mathbf{v}')\| \geq \|\mathbf{p}+\mathbf{u}-(\mathbf{q}+\mathbf{v})\|$. □

Observe that if $\mathbf{p}+U$ and $\mathbf{q}+V$ are linear manifolds of an Euclidean space, then by decomposing $\mathbf{p}-\mathbf{q} = \mathbf{x}-\mathbf{u}+\mathbf{v}$, where $\mathbf{x} \in (U+V)^{\perp}$, $\mathbf{u} \in U$, $\mathbf{v} \in V$, and using Theorem 3, we can assure that there is always a solution for the following problem: find $\mathbf{u}, \mathbf{v} \in U \times V$ such that

$$\|\mathbf{p}+\mathbf{u}-(\mathbf{q}+\mathbf{v})\| \leq \|\mathbf{p}+\mathbf{u}'-(\mathbf{q}+\mathbf{v}')\| \qquad \forall\ (\mathbf{u}', \mathbf{v}') \in U \times V.$$

This former result will be applied to solve Problem 4 when the Euclidean vector space is $M_{n,n}$ endowed with this inner product: $\langle X, Y\rangle = \mathrm{tr}(X^T Y)$ for $X, Y \in M_{n,n}$.

We need the following result, which may be obtained from Equation (5) of [4]: If $\{\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}\}$ is an orthogonal basis of $(\mathrm{span}\{\mathbf{1}_n\})^{\perp}$, then

$$\|\phi_n(\mathbf{y}_r)\|_F^2 = 2n\|\mathbf{y}_r\|^2, \qquad r = 1, \ldots, n-1, \tag{7}$$

where the mapping ϕ_n is defined in (2).

Several elementary and known properties of the trace operator are collected in the following result for the convenience of the reader.

Proposition 3. *Let* $A \in M_{n,m}$*,* $B \in M_{n,m}$*, and* $C \in M_{m,n}$*. Then*

(i) $\operatorname{tr}(A) = \operatorname{tr}(A^T)$.

(ii) *The trace is linear, i.e.,* $\operatorname{tr}(\alpha A + \beta B) = \alpha \operatorname{tr}(A) + \beta \operatorname{tr}(B)$ *for any* $\alpha, \beta \in \mathbb{R}$*.*

(iii) $\operatorname{tr}(AC) = \operatorname{tr}(CA)$.

The following technical lemmas will be used.

Lemma 1. *Let the matrices* B_{ij} *be defined as in (4). Then*

$$\operatorname{tr}(B_{pq}^T B_{rs}) = \begin{cases} 2 & \text{if } (p,q) = (r,s), \\ 0 & \text{if } (p,q) \neq (r,s). \end{cases}$$

PROOF: First we simplify $B_{pq}^T B_{rs}$:

$$\begin{aligned} B_{pq}^T B_{rs} &= [\mathbf{e}_p\mathbf{e}_q^T - \mathbf{e}_q\mathbf{e}_p^T]^T[\mathbf{e}_r\mathbf{e}_s^T - \mathbf{e}_s\mathbf{e}_r^T] \\ &= [\mathbf{e}_q\mathbf{e}_p^T - \mathbf{e}_p\mathbf{e}_q^T][\mathbf{e}_r\mathbf{e}_s^T - \mathbf{e}_s\mathbf{e}_r^T] \\ &= \mathbf{e}_q\mathbf{e}_p^T\mathbf{e}_r\mathbf{e}_s^T - \mathbf{e}_q\mathbf{e}_p^T\mathbf{e}_s\mathbf{e}_r^T - \mathbf{e}_p\mathbf{e}_q^T\mathbf{e}_r\mathbf{e}_s^T + \mathbf{e}_p\mathbf{e}_q^T\mathbf{e}_s\mathbf{e}_r^T. \end{aligned}$$

Observe that $\mathbf{e}_p^T\mathbf{e}_r$, $\mathbf{e}_p^T\mathbf{e}_s$, $\mathbf{e}_q^T\mathbf{e}_r$, and $\mathbf{e}_q^T\mathbf{e}_s$ are scalars and, consequently, commute with any matrix. Furthermore, since $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthonormal basis of $\mathbb{R}^n$, then $\mathbf{e}_i^T\mathbf{e}_j = \delta_{ij}$, where δ is used to denote the Kronecker's delta. Thus,

$$B_{pq}^T B_{rs} = \delta_{pr}\mathbf{e}_q\mathbf{e}_s^T - \delta_{ps}\mathbf{e}_q\mathbf{e}_r^T - \delta_{qr}\mathbf{e}_p\mathbf{e}_s^T + \delta_{qs}\mathbf{e}_p\mathbf{e}_r^T.$$

By using Proposition 3 one has

$$\begin{aligned} \operatorname{tr}(B_{pq}^T B_{rs}) &= \operatorname{tr}(\delta_{pr}\mathbf{e}_q\mathbf{e}_s^T - \delta_{ps}\mathbf{e}_q\mathbf{e}_r^T - \delta_{qr}\mathbf{e}_p\mathbf{e}_s^T + \delta_{qs}\mathbf{e}_p\mathbf{e}_r^T) \\ &= \delta_{pr}\operatorname{tr}(\mathbf{e}_s^T\mathbf{e}_q) - \delta_{ps}\operatorname{tr}(\mathbf{e}_r^T\mathbf{e}_q) - \delta_{qr}\operatorname{tr}(\mathbf{e}_s^T\mathbf{e}_p) + \delta_{qs}\operatorname{tr}(\mathbf{e}_r^T\mathbf{e}_p) \\ &= \delta_{pr}\mathbf{e}_s^T\mathbf{e}_q - \delta_{ps}\mathbf{e}_r^T\mathbf{e}_q - \delta_{qr}\mathbf{e}_s^T\mathbf{e}_p + \delta_{qs}\mathbf{e}_r^T\mathbf{e}_p \\ &= \delta_{pr}\delta_{sq} - \delta_{ps}\delta_{rq} - \delta_{qr}\delta_{sp} + \delta_{qs}\delta_{rp} \\ &= 2(\delta_{pr}\delta_{qs} - \delta_{ps}\delta_{qr}). \end{aligned}$$

If $(p,q) = (r,s)$, it is evident that $\delta_{pr} = \delta_{qs} = 1$. By definition (4), it follows that $p < q$ and $r < s$, hence $p = r < s$ and $r = p < q$, and therefore, $\delta_{ps} = \delta_{qr} = 0$.

If $p \neq r$, obviously $\delta_{pr} = 0$. Remember that we can use $p < q$ and $r < s$. We have two possibilities: $p < r$ or $r < p$. The former leads to $p < r < s$, hence $\delta_{ps} = 0$. The latter leads to $r < p < q$, therefore $\delta_{qr} = 0$.

If $q \neq s$, the reasoning is symmetrical as in the prior paragraph and we can conclude $\delta_{pr}\delta_{qs} - \delta_{ps}\delta_{qr} = 0$. ■

Lemma 2. *For any* $\mathbf{y} = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$ *and* $1 \leq i < j \leq n$ *one has* $\operatorname{tr}(B_{ij}^T \phi_n(\mathbf{y})) = 2(y_j - y_i)$*.*

PROOF: Observe that $\mathbf{e}_i^T \mathbf{1}_n = \mathbf{e}_j^T \mathbf{1}_n = 1$. Let us bear in mind that $y_i = \mathbf{e}_i^T \mathbf{y}$ and $y_j = \mathbf{e}_j^T \mathbf{y}$ are scalar and commute with any matrix. Thus

$$\begin{aligned} B_{ij}^T \phi_n(\mathbf{y}) &= \left(\mathbf{e}_i \mathbf{e}_j^T - \mathbf{e}_j \mathbf{e}_i^T\right)\left(\mathbf{y} \mathbf{1}_n^T - \mathbf{1}_n \mathbf{y}^T\right) \\ &= \mathbf{e}_i \mathbf{e}_j^T \mathbf{y} \mathbf{1}_n^T - \mathbf{e}_j \mathbf{e}_i^T \mathbf{y} \mathbf{1}_n^T - \mathbf{e}_i \mathbf{e}_j^T \mathbf{1}_n \mathbf{y}^T + \mathbf{e}_j \mathbf{e}_i^T \mathbf{1}_n \mathbf{y}^T \\ &= y_j \mathbf{e}_i \mathbf{1}_n^T - y_i \mathbf{e}_j \mathbf{1}_n^T - \mathbf{e}_i \mathbf{y}^T + \mathbf{e}_j \mathbf{y}^T. \end{aligned}$$

Using Proposition 3 leads to

$$\begin{aligned} \operatorname{tr}(B_{ij}^T \phi_n(\mathbf{y})) &= y_j \operatorname{tr}(\mathbf{e}_i \mathbf{1}_n^T) - y_i \operatorname{tr}(\mathbf{e}_j \mathbf{1}_n^T) - \operatorname{tr}(\mathbf{e}_i \mathbf{y}^T) + \operatorname{tr}(\mathbf{e}_j \mathbf{y}^T) \\ &= y_j \operatorname{tr}(\mathbf{1}_n^T \mathbf{e}_i) - y_i \operatorname{tr}(\mathbf{1}_n^T \mathbf{e}_j) - \operatorname{tr}(\mathbf{y}^T \mathbf{e}_i) + \operatorname{tr}(\mathbf{y}^T \mathbf{e}_j) \\ &= 2y_j - 2y_i. \end{aligned}$$

This finishes the proof of the Lemma. ■

If we define for $1 \leq i < j \leq n$

$$\mathbf{d}_{ij} = \mathbf{e}_j - \mathbf{e}_i,$$

the conclusion of Lemma 2 can be rewritten as $\operatorname{tr}(B_{ij}^T \phi_n(\mathbf{y})) = 2\, \mathbf{d}_{ij}^T \mathbf{y}$.

It is evident that any skew-Hermitian matrix $B_0 \in M_n$ can be uniquely expressed as

$$B_0 = \sum_{i<j} \rho_{ij} B_{ij}, \tag{8}$$

where the matrices B_{ij} are defined in (4). Therefore, if the skew-Hermitian matrix B_0 is decomposed as in (8) and $\mathbf{y} \in \mathbb{R}^n$, then by applying Lemma 1 and Lemma 2 we obtain

$$\operatorname{tr}(B_0^T B_{rs}) = 2\rho_{rs} \qquad \text{and} \qquad \operatorname{tr}(B_0^T \phi_n(\mathbf{y})) = 2 \sum_{i<j} \rho_{ij} \mathbf{d}_{ij}^T \mathbf{y}. \tag{9}$$

The following theorem is the main result of the article. Observe that if $A \in M_{n,n}^+$ is a reciprocal incomplete matrix, as we showed in the example after Problem 3, then $L(A)$ can be written as $L(A) = B_0 + \sum_{r=1}^k \lambda_r B_{i_r j_r}$. Observe furthermore that the (i_r, j_r) entry of B_0 is zero for $r = 1, \cdots, k$.

Theorem 4. *Let $B_0 \in M_{n,n}$ be a skew-Hermitian matrix represented as in (8) and $1 \leq i_1, j_1, \ldots, i_k, j_k \leq n$ with $i_r < j_r$ (for $r = 1, \ldots, k$). Assume that the (i_r, j_r) entry of B_0 is zero for $r = 1, \cdots, k$. The solution of Problem 4 satisfies*

$$\boldsymbol{\lambda} = S\boldsymbol{\mu}, \qquad \left(D - \frac{1}{n} S^T S\right) \boldsymbol{\mu} = \mathbf{b},$$

where $\boldsymbol{\lambda} = [\lambda_1 \ \cdots \ \lambda_k]^T$, $\boldsymbol{\mu} = [\mu_1 \ \cdots \ \mu_{n-1}]^T$, S is the $k \times (n-1)$ matrix whose (r, s) entry is $\mathbf{d}_{i_r j_r}^T \mathbf{y}_s$, D is the diagonal $(n-1) \times (n-1)$ matrix whose (s, s) entry is $\|\mathbf{y}_s\|^2$, and $\mathbf{b} = [\mathbf{w}^T \mathbf{y}_1 \ \cdots \ \mathbf{w}^T \mathbf{y}_{n-1}]^T$, being $\mathbf{w} = \frac{1}{n} \sum_{i<j} \rho_{ij} \mathbf{d}_{ij}$.

Proof. By Theorem 3 we know that a solution $\lambda_1, \ldots, \lambda_k,\ \mu_1, \ldots, \mu_{n-1}$ of Problem 4 exists, and the matrix

$$B_0 + \sum_{r=1}^{k} \lambda_r B_{i_r j_r} - \sum_{s=1}^{n-1} \mu_s \phi_n(\mathbf{y}_s)$$

is orthogonal to $B_{i_1 j_1}, \ldots, B_{i_k j_k}$ and $\phi_n(\mathbf{y}_1), \ldots, \phi_n(\mathbf{y}_{n-1})$. Let $r \in \{1, \ldots, k\}$ be fixed. Since the (i_r, j_r) entry of B_0 is zero, then $\langle B_0, B_{i_r j_r} \rangle = 0$. We obtain from (9), Lemma 1, and Lemma 2

$$0 = \left\langle B_0 + \sum_{r=1}^{k} \lambda_r B_{i_r j_r} - \sum_{s=1}^{n-1} \mu_s \phi_n(\mathbf{y}_s), B_{i_r j_r} \right\rangle = 2\lambda_r - 2 \sum_{s=1}^{n-1} \mu_s \mathbf{d}_{i_r j_r}^T \mathbf{y}_s. \quad (10)$$

Now, fix any $s \in \{1, \ldots, n-1\}$. We obtain from (7), (9), Lemma 2, and Theorem 1

$$0 = \left\langle B_0 + \sum_{r=1}^{k} \lambda_r B_{i_r j_r} - \sum_{s=1}^{n-1} \mu_s \phi_n(\mathbf{y}_s), \phi_n(\mathbf{y}_s) \right\rangle = 2 \sum_{i<j} \rho_{ij} \mathbf{d}_{ij}^T \mathbf{y}_s + 2 \sum_{r=1}^{k} \lambda_r \mathbf{d}_{i_r j_r}^T \mathbf{y}_s - 2n\mu_s \|\mathbf{y}_s\|^2. \quad (11)$$

If we define $\mathbf{w} = \frac{1}{n} \sum_{i<j} \rho_{ij} \mathbf{d}_{ij}$, then (11) reduces to

$$\mu_s \|\mathbf{y}_s\|^2 = \mathbf{w}^T \mathbf{y}_s + \frac{1}{n} \sum_{r=1}^{k} \lambda_r \mathbf{d}_{i_r j_r}^T \mathbf{y}_s.$$

Set $\alpha_{rs} = \mathbf{d}_{i_r j_r}^T \mathbf{y}_s$ for $r = 1, \ldots, k$ and $s = 1, \ldots, n-1$. Equalities (10) and (11) can be written as

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -\alpha_{11} & -\alpha_{12} & \cdots & -\alpha_{1n-1} \\ 0 & 1 & \cdots & 0 & -\alpha_{21} & -\alpha_{22} & \cdots & -\alpha_{2n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{k1} & -\alpha_{k2} & \cdots & -\alpha_{kn-1} \\ -\frac{\alpha_{11}}{n} & -\frac{\alpha_{21}}{n} & \cdots & -\frac{\alpha_{k1}}{n} & \|\mathbf{y}_1\|^2 & 0 & \cdots & 0 \\ -\frac{\alpha_{12}}{n} & -\frac{\alpha_{22}}{n} & \cdots & -\frac{\alpha_{k2}}{n} & 0 & \|\mathbf{y}_2\|^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\alpha_{1n-1}}{n} & -\frac{\alpha_{2n-1}}{n} & \cdots & -\frac{\alpha_{kn-1}}{n} & 0 & 0 & \cdots & \|\mathbf{y}_{n-1}\|^2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \\ \mu_1 \\ \vdots \\ \mu_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \mathbf{w}^T \mathbf{y}_1 \\ \vdots \\ \mathbf{w}^T \mathbf{y}_{n-1} \end{bmatrix}.$$

By following the notation of the Theorem, the above matrix equality can be written as

$$\begin{bmatrix} I_k & -S \\ -\frac{1}{n} S^T & D \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix}.$$

By a simple Gaussian block elimination,

$$\left[\begin{array}{cc|c} I_k & -S & \mathbf{0} \\ -\frac{1}{n} S^T & D & \mathbf{b} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} I_k & -S & \mathbf{0} \\ 0 & D - \frac{1}{n} S^T S & \mathbf{b} \end{array}\right].$$

It should be evident that $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ satisfy the thesis of the theorem. ∎

Next we present an m-file that can be executed in Matlab or Octave. This file computes the most consistent completion of an incomplete reciprocal matrix in the sense of Theorem 4 when this completion is unique. We maintain a similar notation with Theorem 4, with the only difference that matrix S is stored in `alpha`.

```
function [lambda mu] = th4(A, P)

% A is the matrix to be completed (if we do not know a {ij}, then A(i,j)=1.
% P is a (0, 1)-matrix such that if we do not know a {ij}, then P(i,j)=1.
% if we know a {ij}, then P(i, j)=0.
% Use: [lambda mu] = th4(A, P)

[n, m] = size(A);
B = log(A);
auxP = triu(P);
[noi noj] = find(auxP==1);
auxP = triu(P+ones(n,n),1);
[sii sij] = find(auxP==1);
kn = length(noi); ks = length(sii);
Y = y(n);
D = diag(ones(1,n)*Y.^2);
I = eye(n);
w = zeros(n,1);
alpha = zeros(kn,n-1);
for r = 1:kn
    for s = 1:n-1
        i = noi(r); j = noj(r);
        alpha(r,s) = (I(j,:)-I(i,:))*Y(:,s);
    end
end
for index = 1:ks
    i = sii(index); j = sij(index);
    w = w+B(i,j)*(I(:,j)-I(:,i));
end
w = w/n;
b = Y'*w;
mu = (D - alpha'*alpha/n)\b;
lambda = alpha*mu;
```

This file uses another m-file, `y.m` which computes the vectors $\mathbf{y}_1, \cdots, \mathbf{y}_{n-1}$ (see [4]). We include this here for the sake of the completeness.

```
function y = y (n)
y = zeros(n,n-1);
for k = 1:n-1
    y (1:k,k) = ones (k,1);
    y (k + 1,k) = -k;
end
```

We will now compare our procedure with two previous procedures that also complete an incomplete reciprocal matrix.

In [17], instead of managing with multiplicative properties (Saaty's treatment), the emphasis is on additive properties of the given matrices. A matrix $R = (r_{ij})$ is said to be *reciprocal in the additive sense* if $(r_{ik} - 0.5) + (r_{kj} - 0.5) = r_{ij} - 0.5$ for any i, j, k. The purpose of [17] is to minimize the *global inconsistency index* defined by $\rho = \sum_{ijk}(r_{ik} + r_{kj} - r_{ij} - 0.5)^2$ considering the missing entries as variables. This solution involves solving a linear system. But, in several cases, the solution obtained must be computed by solving a quadratic programming problem. Also, notice that the condition $k \leq n - 2$ must be satisfied to ensure that the aforementioned linear system is nonsingular (here, $2k$ is the number of unknown entries and n is the size of the incomplete matrix).

In [7], the *logarithmic least squares method for incomplete matrices* is solved. Explicitly, given A an incomplete reciprocal matrix, if $E = \{(i, j) : a_{ij}$ and a_{ji} are given $\}$ the problem consists in finding $\mathbf{w} = (w_1, \ldots, w_n)$ such that $\mathbf{w}$ is the solution of

$$\min \sum_{(i,j)\in E} [\log a_{ij} - \log(w_i/w_j)]^2 + [\log a_{ji} - \log(w_j/w_i)]^2, \quad w_1 + \cdots + w_n = 1, w_i > 0.$$

This vector $\mathbf{w}$ gives the priority vector. The authors showed that the solution is unique if, and only if, some graph (related only with the positions $(i, j) \in E$) is connected. If this graph is connected, the solution $\mathbf{w}$ can be obtained by solving a linear system whose matrix has a strong connection with the Laplacian matrix of the aforementioned graph.

Observe that our method, although based on the minimisation of a natural distance in $M^+_{n,n}$, boils down to just the solution of a system of linear equations, so that no minimisation process is needed. The method uses the multiplicative condition, as proposed by Saaty, instead of the additive condition as in [17]; and last but not least, it produces an entire consistent comparison matrix, instead of just the priority vector as in [7]. Obtaining the entire consistent comparison matrix is crucial, for example if the aggregation of individual judgments, as opposed to aggregation of individual priorities, is necessary within a given participative process.

5 Application in the context of leak management

In this section we discuss how the above methodology is used in a decision process in which the actor consulted made no comparative judgments between two sets of criteria with respect to the adoption of a certain policy on leakage control. The main objective is to minimize water loss through an appropriate leakage control policy. In a simplified framework, we consider two alternatives: ALC (active leakage control) and PLC (passive leakage control). The first consists of taking a priori actions in the supply system for prevention; while the second involves repairing reported and/or obvious leaks [16]. Various criteria, including tangible and intangible factors and qualitative factors, can be used to decide on the alternatives. To illustrate the application of the methodology developed, we consider a set of five criteria:

C_1: Cost of development planning and implementation;

C_2: budget and credit;

C_3: payback;

C_4: social costs;

C_5: environmental costs.

It is noted that the first three criteria are directly related to economic and financial aspects. These criteria are intended to assess the economic cost of the project, the availability of financial resources to meet the investment, and the recovery of investments. The last two criteria relate to social and environmental impacts brought about by either the development of a leakage control project with a proactive vision (as is the active control of leakage) or conversely, the externalities resulting from only reacting to evident or reported leaks, thus opting for a passive management of leakage control.

For this problem, Table 1 presents the views of the actor involved using the Saaty scale. For this specific matrix, the pairwise comparison process was undertaken by an employee who is part of the team overseeing the renovation of the drinking water network in the city of Valencia (Spain). This task is within a project for assessing the best leakage control alternative [14] that was specifically developed by the authors with Valencia's main water utility. Many of the matrices obtained in the course of this project have already been used by the authors in previous works [3, 4, 5, 6, 13]. The matrix in Table 1 was not considered in the main decision-making process as it was incomplete. However, despite its lack of completeness, the matrix is worth considering precisely in the context of dealing with incomplete information (an objective clearly within the scope of this paper). The incompleteness of this matrix is justified as follows. The employee has the appropriate background to be considered an expert on the problem being treated as he supervises works to improve the water supply infrastructure. He is usually present when works are being carried out and so has some knowledge of the social impact. However, he considered his familiarity with the environmental costs as weak; and consequently, he limited his judgments to those elements in which he was fully acquainted.

The asterisks indicate judgments not provided.

	C_1	C_2	C_3	C_4	C_5
C_1	1	1	2	9	$*$
C_2	1	1	3	9	7
C_3	1/2	1/3	1	7	5
C_4	1/9	1/9	1/7	1	$*$
C_5	$*$	1/7	1/5	$*$	1

Table 1: Incomplete matrix of comparison of criteria

After applying the described process, the following values are obtained

$$\lambda_1 = 2.07, \qquad \lambda_2 = -0.24.$$

With these values the entire matrix in Table 2 is built. The priority vector for this consistently

	C_1	C_2	C_3	C_4	C_5
C_1	1	1	2	9	7.92
C_2	1	1	3	9	7
C_3	1/2	1/3	1	7	5
C_4	1/9	1/9	1/7	1	0.79
C_5	0.13	1/7	1/5	1.27	1

Table 2: Consistently completed matrix of comparison of criteria

completed matrix (with CR = 1.60%) is

$$Z = [0.351\ 0.380\ 0.189\ 0.035\ 0.045]^T,$$

showing a clear dominance of the economic criteria. The actor in question avoided comparing environmental costs with those of planning and implementation because he dared not estimate how much more importance to give planning and implementation over environmental costs. Moreover, although he had no clear ideas on the subject, he considered environmental costs to be less important than budgeting and investment recovery. Finally, he chose not to issue a comparative judgment between social and environmental costs.

The developed 'consistent completion mechanism' reveals, based on the comparisons issued, that environmental costs, together with social costs, are not among the top priorities for the actor involved.

We omit in this document, the final aggregation process, which would normally take place and make use of a comparison of alternatives for each criterion, since the main objective of this article has already been shown, namely, consistent completion.

It should be noted that this mechanism of consistent completion is clearly explicit and involves only a few simple matrix calculations. As a result, its application does not involve any special calculation burden. In fact, it can be simply and directly implemented in any computing environment that includes matrix functions.

6 Conclusions

In [5] we consider that AHP methodology can be thought as a dynamic model, following the idea that input in decision-making depends on whether actors indicate their preferences once or at several times. We presented mechanisms for adding and deleting criteria efficiently to preserve the work previously done on consistency with the criteria used so far. In this contribution, we have considered the scenario in which the actors consulted are not familiar with the effects of some elements. As it can be unreasonable to wait for complete data collection and quality information on the preferences of decision makers, we think that data on user preferences may be completed several times at the user's convenience. As a result, the static input mode can be easily modified so that partial results are considered (based on partial preference data obtained from data collected at different times). The final decision will be produced when the information is complete, using, for example, the mechanisms described in this paper and [5]. Enabling a dynamic approach in traditional techniques will undoubtedly open the door to a variety of topics for future research.

The method we propose is computationally efficient and is based on a linearization process previously introduced by the authors [4]. In order to illustrate the operation of the algorithm, we have presented an experiment that considers a problem of decision making in a water supply utility in relation to the adoption of a leakage control policy -active or passive control being the alternatives under consideration.

Part of the future work should focus on the formalization of methods of consensus decision-making in which it is desirable or necessary to include the views of various actors -as is the trend in many current processes of governance.

Acknowledgments

This work has been funded as part of project IDAWAS, DPI2009-11591 by the Dirección General de Investigación del Ministerio de Ciencia e Innovación (Spain); with supplementary support from ACOMP/2011/188 from the Conselleria d'Educació of the Generalitat Valenciana. The first author also received support from Spanish project PAID-06-12. The use of English in this paper was revised by John Rawlins.

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