Two weighted eight-order classes of iterative root-finding methods [*]

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Abstract

In this paper we design, by using the weight function technique, two families of iterative schemes with order of convergence eight. These weight functions depend on one, two and three variables and they are used in the second and third step of the iterative expression. Dynamics on polynomial and non-polynomial functions is analyzed and they are applied on the problem of preliminary orbit determination by using a modified Gauss method. Finally, some standard test functions are to check the reliability of the proposed schemes and allow us to compare them with other known methods.

Key Words: Nonlinear equation, iterative schemes, optimal methods, weight functions technique, efficiency index.

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1 Introduction

Nonlinearity appears frequently in real problems, which are modeled by different branches of science and engineering. These models try to simplify the original problems holding their main properties. So, they usually remain nonlinear. The use of iterative techniques to estimate the solutions of these kind of problems has revealed itself as a fruitful area of research.

In general, we try to find a simple real solution $\xi \in I$ of a nonlinear equation $f(x) = 0$. The most of the computationally useful techniques to determine roots can be classified as: (a) iterative methods that require only functional evaluations of f, and (b) methods whose iterative formula require evaluations of the function and its derivatives. There are two known simple and effective methods that represent these classes: Steffensen' and Newton's methods, respectively, both with order of convergence two. The search of variants of these methods with an accelerated convergence and a reduced number of operations and functional evaluations has resulted in the last decades in multistep methods. These schemes belong to the class of most powerful methods that overcome the limitations of the methods of a point with respect to the order of convergence and computational efficiency. This type of iterative methods were extensively studied in Traub's text, [1] and more recently, in [2].

The most of multistep iterative methods modify Newton's scheme to solve nonlinear equations with a high-order of convergence (see [3, 4, 5, 6, 7]). However, they frequently use derivatives, which is a serious disadvantage. Sometimes, the applications of iterative algorithms that depend on derivatives are restricted in engineering and science. For these cases, different authors have developed derivative-free iterative methods in numerous papers, that only need the additional evaluations of the function (see [8, 9, 10, 11]).

In this paper (following the ideas presented in [12] and [13]) we design, in Section 2, two families of iterative methods belonging to classes (a) and (b), respectively, both of eighth-order of convergence. Their stability properties are tested on different functions in Section 3. In Section 4, an application is made on the problem of preliminary orbit determination. This problem is used to test the robustness and efficiency of the proposed methods, compared with other partners of the same order and with the original scheme used by Gauss and introduced in [14]. To get this aim, variable precision arithmetics is used to work with a high number of digits and a very restrictive stopping criterium which are not needed in real cases. Finally, in Section 5 some test

functions are used to compare the numerical results of our methods with the obtained by other eighth-order schemes.

2 Design of the families

Let us remember that the iterative expression of Steffensen's scheme is

$$x_{k+1} = x_k - \frac{f(x_k)}{f[z_k, x_k]},$$

where $z_k = x_k + \beta f(x_k)$, $f[.,.]$ denotes the usual first-order divided differences and Newton's one is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

If we compose any of these methods with itself, schemes of order 4 with 4 functional evaluations are obtained in both cases. That makes them non-optimal, according to the conjecture of Kung-Traub [15].

In double Newton scheme, let us consider a frozen derivative in the second step, obtaining the non-optimal Traub's method of order 3 (see [1]). The authors in [13], using the technique of weight functions, got to increase to four the order of convergence of the Traub's scheme without adding new functional evaluations. The iterative expression of the obtained method is

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\
x_{k+1} &= y_k - H(u_k)\frac{f(y_k)}{f'(x_k)},
\end{aligned}
\tag{1}
$$

where $H(u_k)$ represents a real-valued function and $u_k = \dfrac{f(y_k)}{b_1 f(x_k) + b_2 f(y_k)}$, being b_1 and b_2 real parameters.

Now, let us consider the double Steffensen scheme, holding the divided difference in the denominator in the second step. By using the technique of weight functions, Petković et al. in [12] increase to four the order of convergence of double Steffensen method without adding new functional evaluations. The iterative scheme of the resulting method is

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f[z_k, x_k]}, \\
x_{k+1} &= y_k - H(u_k, v_k)\frac{f(y_k)}{f[z_k, x_k]},
\end{aligned}
\tag{2}
$$

where $z_k = x_k + \beta f(x_k)$, $u_k = \dfrac{f(y_k)}{f(x_k)}$ and $v_k = \dfrac{f(y_k)}{f(z_k)}$. Under some conditions on H and for nonzero arbitrary values of β, (2) is an optimal method of order four.

Following these ideas, we design two classes of three-steps methods, that satisfy the following results. Let us notice that a particular subclass of the family presented in Theorem 1 was designed by Džunić et al. in [16].

Theorem 1 *Let $\xi \in I$ be a simple zero of a sufficiently differentiable function $f : I \subset \mathbb{R} \to \mathbb{R}$ on an open interval I. Let H and G be sufficiently differentiable real functions and x_0 be an initial approximation close enough to ξ. If $\beta = 1$ and H and G satisfy $H(0) = 1$, $H'(0) = 2b_1$, $H''(0) = 2b_1(2b_1 + b_2)$, $G(0,0) = G_v(0,0) = 1$, $G_u(0,0) = 2b_1$, $G_{uv}(0,0) = 4b_1$, $G_{uu}(0,0) = 2b_1(3b_1 + b_2)$ and $|G_{vv}(0,0)| < +\infty$. Then, the method*

$$
\begin{aligned}
y_k &= x_k - \beta\frac{f(x_k)}{f'(x_k)}, \\
z_k &= y_k - H(u_k)\frac{f(y_k)}{f'(x_k)}, \\
x_{k+1} &= z_k - G(u_k, v_k)\frac{f(z_k)}{f'(x_k)},
\end{aligned}
\tag{3}
$$

where $u_k = \dfrac{f(y_k)}{b_1 f(x_k) + b_2 f(y_k)}$, *being* b_1 *and* b_2 *arbitrary real parameters and* $v_k = \dfrac{f(z_k)}{f(y_k)}$, *has order of conver-gence eight for any value* b_1 *different from zero.*

Proof. Let us introduce the Taylor's series expansion of f and f' around ξ

$$f(x_k) = f'(\xi) \sum_{j=1}^{8} c_j e_k^j + O(e_k^9) \quad \text{and} \quad f'(x_k) = f'(\xi) \sum_{j=1}^{8} (j+1) c_{j+1} e_k^j + O(e_k^9),$$

where $c_j = \dfrac{f^{(j)}(\xi)}{j! f'(\xi)}$ and $e_k = x_k - \xi$. Applying these developments on the damped Newton's method

$$y_k = x_k - \beta \frac{f(x_k)}{f'(x_k)} = \xi + \sum_{j=1}^{8} A_j e_k^j + O(e_k^9),$$

where $A_1 = 1 - \beta$, $A_2 = -2\beta c_2$, $A_3 = 2\beta(c_3 - c_2^2)$ and $A_4 = \beta(4c_2^3 - 7c_2 c_3 + 3c_4)$. Using again Taylor's series expansion, we get

$$f(y_k) = f'(\xi) \sum_{j=1}^{8} B_j e_k^j + O(e_k^9),$$

where $B_1 = A_1$, $B_2 = (\beta^2 - \beta + 1)c_2$, $B_3 = -2\beta^2 c_2^2 + (1 - \beta + 3\beta^2 - \beta^3)c_3$ and $B_4 = (1 - \beta + 6\beta^2 - 4\beta^3 + \beta^4)c_4 + 5\beta^2 c_2^3 + \beta^2(3\beta - 10)c_3 c_2$.

Now, the expansion of $u_k = \dfrac{f(x_k)}{b_1 f(x_k) + b_2 f(y_k)}$ can be calculated and then, a Taylor's series expression is obtained for the weight function $H(u_k) \approx H(0) + H'(0)u_k + (1/2)H''(0)u_k^2$. So,

$$z_k = y_k - H(u_k) \frac{f(y_k)}{f'(x_k)} = \xi + \sum_{j=1}^{8} Z_j e_k^j + O(e_k^9)$$

and the error equation of the second step is $e_{z_k} = \displaystyle\sum_{j=1}^{8} Z_j e_k^j + O(e_k^9)$ whose first term depend on the factor

$$Z_1 = (1 - \beta) \left[1 - H(0) + \frac{1 - \beta}{2(b_1 + b_2(1 - \beta))^2} (-2b_1 H'(0) + (\beta - 1)(2b_2 H'(0) + H''(0))) \right].$$

The condition for obtaining at least convergence order 2 is $Z_1 = 0$, so we need $\beta = 1$. Introducing this value in the error equation of this step we obtain $e_{z_k} = \displaystyle\sum_{j=2}^{8} \bar{Z}_j e_k^j + O(e_k^9)$, where

$$\bar{Z}_2 = (1 - H(0))c_2, \quad \text{and} \quad \bar{Z}_3 = 2(1 - H(0))c_3 - (2 - 4H(0) + H'(0)/b_1)c_2^2.$$

Solving the system of simultaneous equations $\bar{Z}_2 = 0$ and $\bar{Z}_3 = 0$ we get $H(0) = 1$ and $H'(0) = 2b_1$. Substituting these conditions we obtain the error expression

$$e_{z_k} = \frac{1}{2b_1^2} \left((4b_1 b_2 + 10b_1^2 - H''(0))c_2^2 - 2b_1^2 c_3 \right) c_2 e_k^4 + \sum_{j=5}^{8} \bar{\bar{Z}}_j e_k^j + O(e_k^9).$$

From this expression we conclude that the second step of the iterative expression (3) can have convergence order no more than 4 as $b_1 \neq 0$. By using again Taylor's expansion $f(z) = f'(\xi) \displaystyle\sum_{j=4}^{8} D_j e_k^j + O(e_k^9)$, where $D_4 = \frac{1}{2b_1^2} \left((4b_1 b_2 + 10b_1^2 - H''(0))c_2^2 - 2b_1^2 c_3 \right) c_2$.

Now, we define the new variable $v_k = \dfrac{f(z_k)}{f(y_k)}$ and the weight function $G(u_k, v_k) \approx G(0,0) + G_u(0,0)u_k + G_v(0,0)v_k + \frac{1}{2}(G_{uu}(0,0)u_k^2 + G_{vv}(0,0)v_k^2) + G_{uv}(0,0)u_k v_k$.

Then, as $x_{k+1} = z_k - G(u_k, v_k)\dfrac{f(z_k)}{f'(x_k)}$ we obtain

$$e_{k+1} = \frac{1}{2b_1^2}(1 - G(0,0))[(4b_1 b_2 + 10b_1^2 - H''(0))c_2^2 - 2b_1^2 c_3]c_2 e_k^4 + \sum_{j=5}^{8} L_j e_k^j + O(e_k^9).$$

If we choose $G(0,0) = 1$, the order of convergence will be at least 5. Substituting these condition in the error equation we obtain the new expression of it

$$e_{k+1} = \frac{1}{2b_1^3}(2b_1 - G_u(0,0))[(4b_1 b_2 + 10b_1^2 - H''(0))c_2^2 - 2b_1^2 c_3]c_2 e_k^5 + \sum_{j=6}^{8} \bar{L}_j e_k^j + O(e_k^9).$$

Let us observe that if $G_u(0,0) = 2b_1$, the term of order 5 vanishes. Again, we substitute this condition in the error equation and simplifying we obtain

$$e_{k+1} = \sum_{j=6}^{8} \bar{\bar{L}}_j e_k^j + O(e_k^9),$$

where $\bar{\bar{L}}_6 = \dfrac{1}{4b_1^4}((4b_1 b_2 + 10b_1^2 - H''(0))c_2^2 - 2b_1^2 c_3)[(r_1 c_2^2 + 2b_1^2(1 - G_v(0,0))c_3]c_2$ and $r_1 = 4b_1 b_2(G_v(0,0) - 1) + 2b_1^2(5G_v(0,0) - 6) + G_{uu}(0,0) - G_v(0,0)H''(0)$. If $G_v(0,0) = 1$ and $G_{uu}(0,0) = 2b_1^2 + H''(0)$ the order of convergence is at least 7 and the error expression obtained $e_{k+1} = \bar{\bar{\bar{L}}}_7 e_k^7 + \bar{\bar{\bar{L}}}_8 e_k^8 + O(e_k^9)$, where

$$\bar{\bar{\bar{L}}}_7 = \frac{1}{4b_1^5}(nc_2^2 - 2b_1^2 c_3)[r_2 c_2^2 - (8b_1^3 + 2b_1^2 G_{uv}(0,0))c_3]c_2^2.$$

In order to reach eighth-order of convergence we need $\bar{\bar{\bar{L}}}_7 = 0$, that is, $G_{uv}(0,0) = 4b_1$ and $H''(0) = 2(2b_1^2 + b_1 b_2)$.

Finally,

$$e_{k+1} = \frac{1}{2b_1^3}[(3b_1 + b_2)c_2^2 - b_1 c_3][r_3 c_3^2 + r_4 c_2^4 + r_5 c_2^2 c_3 - 2b_1^2 c_2 c_4]e_k^8 + O(e_k^9),$$

where $r_3 = b_1^2(G_{vv} - 2)$, $r_4 = 9b_1^2(G_{vv} - 6) + b_2^2(G_{vv} - 2) + 2b_1 b_2(3G_{vv} - 13)$ and $r_5 = b_1^2(34 - G_{vv}) - 2b_1 b_2(G_{vv} - 5)$ and the proof of the result is finished. ∎

By choosing different weight functions verifying the conditions imposed in Theorem 1, different classes of methods can be found. Specifically, in the next sections we will use

$$\begin{aligned} H(u) &= 1 + 2b_1 u + b_1(2b_1 + b_2)u^2, \\ G(u,v) &= 1 + 2b_1 u + v + b_1(3b_1 + b_2)u^2 + 4b_1 uv. \end{aligned} \tag{4}$$

Let us note that if $b_1 = 1$ and $b_2 = -2$ then $H(u) = 1 + 2u$ and the first two steps of (3) correspond to the Ostrowski's method. In this case function G is $G(u,v) = 1 + 2u + v + u^2 + 4uv$.

Respect to the class of derivative-free methods, mentioned in the Introduction, and taking into account Petković 's idea showed in (2), we establish the following result.

Theorem 2 *Let $\xi \in I$ be a simple zero of a sufficiently differentiable function $f : I \subset \mathbb{R} \to \mathbb{R}$ on an open interval I and be x_0 an initial approximation close enough to ξ. Let H and G be sufficiently differentiable real functions which satisfy: $H(0,0) = H_u(0,0) = H_v(0,0) = 1$, $H_{uu}(0,0) = H_{vv}(0,0) = 2$, $H_{uv}(0,0) = 0$, $G(0,0,0) = G_u(0,0,0) = G_v(0,0,0) = G_w(0,0,0) = 1$, $G_{uu}(0,0,0) = G_{vv}(0,0,0) = G_{uw}(0,0,0) = G_{vw}(0,0,0) = 2$, $G_{uv}(0,0,0) = 1$ and $|G_{ww}(0,0,0)| < \infty$. Then, the method*

$$\begin{aligned} y_k &= x_k - \frac{f(x_k)}{f[z_k, x_k]}, \\ t_k &= y_k - H(u_k, v_k)\frac{f(y_k)}{f[z_k, x_k]}, \\ x_{k+1} &= t_k - G(u_k, v_k, w_k)\frac{f(t_k)}{f[z_k, x_k]}, \end{aligned} \tag{5}$$

4

where $z_k = x_k + \beta f(x_k)$, $u_k = \dfrac{f(y_k)}{f(x_k)}$, $v_k = \dfrac{f(y_k)}{f(z_k)}$ and $w_k = \dfrac{f(t_k)}{f(y_k)}$, has order of convergence eight for any value β different from zero and the error equation is

$$
\begin{aligned}
e_{k+1} &= -\frac{1}{2}\rho^2 c_2 (3c_2^2 - c_3)((9G_{ww}(0,0,0)\rho^2 + 2(-27 - 54\gamma - 28\gamma^2 + \gamma^3) + \gamma^4))c_2^4 + 2\rho(17 + 17\gamma + 2\gamma^2 \\
&- 3G_{ww}(0,0,0)\rho)c_2^2 c_3 + (-2 + G_{ww}(0,0,0))\rho^2 c_3^2 - 2\rho^2 c_2 c_4))e_k^8 + O(e_k^9),
\end{aligned}
$$

where $\rho = 1 + \gamma$ and $\gamma = f'(\xi)\beta$.

Proof. By using the Taylor's series expansion of $f(x_k)$ introduced in the previous theorem, we have

$$
z_k = \rho e_k + \gamma \sum_{j=2}^{8} c_j e_k^j + O(e_k^9).
$$

So,

$$
f(z_k) = f'(\xi) \sum_{j=1}^{8} Z_j e_k^j + O(e_k^9),
$$

where $Z_1 = \rho$, $Z_2 = (\gamma + \rho^2)c_2$, $Z_3 = 2\gamma\rho c_2^2 + (\gamma + \rho^3)c_3$ and $Z_4 = 3\gamma\rho^2 c_2 c_3 + \gamma c_2(\gamma c_2^2 + 2\rho c_3) + (\gamma + \rho^4)c_4$; and,

$$
f[x_k, z_k] = \frac{f(z_k) - f(x_k)}{z_k - x_k} = f'(\xi)\left[1 + \sum_{j=1}^{8} D_j e_k^j + \right] + O(e_k^9),
$$

where $D_1 = (1 + \rho)c_2$, $D_2 = \gamma c_2^2 + (3\rho + \gamma^2)c_3$ and $D_3 = f'(\xi)(1 + \rho)(2\gamma c_2 c_3 + (2\rho + \gamma^2)c_4)$.

Therefore, the error at the first step of the method is

$$
e_{y_k} = y_k - \xi = \sum_{j=2}^{8} Y_j e_k^j + O(e_k^9), \tag{6}
$$

where $Y_2 = \rho c_2$, $Y_3 = -(2\rho + \gamma^2)c_2^2 + (2 + 3\gamma + \gamma^2)c_3$ and $Y_4 = (4 + 5\gamma + 3\gamma^2 + \gamma^3)c_2^3 - (7 + 10\gamma + 7\gamma^2 + 2\gamma^3)c_2 c_3 + (3 + 6\gamma + 4\gamma^2 + \gamma^3)c_4$.

Let us calculate the Taylor's expressions for the variables $u_k = \dfrac{f(y_k)}{f(x_k)}$ and $v_k = \dfrac{f(y_k)}{f(z_k)}$ and the weight function H around $(0,0)$

$$
H(u_k, v_k) \approx H(0,0) + H_u(0,0)u_k + H_v(0,0)v_k + \frac{1}{2}H_{uu}(0,0)u_k^2 + \frac{1}{2}H_{vv}(0,0)v_k^2 + H_{uv}(0,0)u_k v_k. \tag{7}
$$

Then, the error equation of the second step is

$$
e_{t_k} = t_k - \xi = \sum_{j=2}^{8} R_j e_k^j + O(e_k^9), \tag{8}
$$

where $R_2 = -(-1 + H(0,0))\rho c_2$ and $R_3 = -(2\rho + H_v(0,0) + \gamma H_v(0,0) + \gamma^2 + \rho^2 H_u(0,0) - H(0,0)(4 + 5\gamma + 2\gamma^2))c_2^2 - (-1 + H(0,0))(2 + 3\gamma + \gamma^2)c_3$. In order to t_k reaches fourth-order of convergence, we need to assume that $H(0,0) = H_u(0,0) = H_v(0,0) = 1$. With these values,

$$
e_{t_k} = \sum_{j=4}^{8} \bar{R}_j e_k^j + O(e_k^9), \tag{9}
$$

where $\bar{R}_4 = -\frac{1}{2}\rho(r_{4,1}c_2^3 + r_{4,2}c_2 c_3)$, $r_{4,1} = -2(5\rho + \gamma^2) + h_{vv}(0,0) + 2h_{uv}(0,0)\rho + h_{uu}(0,0)\rho^2$ and $r_{4,2} = 2\rho$. Then,

$$
f(t_k) = f'(\xi)\sum_{j=4}^{8} \bar{R}_j e_k^j + O(e_k^9). \tag{10}
$$

5

Now, we calculate the expression of $w_k = \frac{f(t_k)}{f(y_k)}$ and the expression of the weight function $G(u_k, v_k, w_k)$ around $(0, 0, 0)$

$$
\begin{aligned}
G(u_k, v_k, w_k) &\approx G(0,0,0) + G_u(0,0,0)u_k + G_v(0,0,0)v_k + G_w(0,0,0)w_k + \frac{1}{2}G_{uu}(0,0,0)u_k^2 + \frac{1}{2}G_{vv}(0,0,0)v_k^2 \\
&+ \frac{1}{2}G_{ww}(0,0,0)w_k^2 + G_{uv}(0,0,0)u_k v_k + G_{uw}(0,0,0)u_k w_k + G_{vw}(0,0,0)v_k w_k.
\end{aligned}
$$

Therefore, the error equation of the method is

$$
e_{k+1} = \sum_{j=4}^{8} L_j e_k^j + O(e_k^9), \tag{11}
$$

where $L_4 = \frac{1}{2}(-1 + G(0,0,0))\rho(r_{4,1}c_2^3 + r_{4,2}c_2 c_3)$. If we assume $G(0,0,0) = 1$, then the fourth-order term vanishes and $\bar{L}_5 = \frac{1}{2}(-2 + G_u(0,0,0) + G_v(0,0,0) - \gamma(1 - G_u(0,0,0)))\rho(r_{4,1}c_2^4 + r_{4,2}c_2^2 c_3)$.

Now, if $G_u(0,0,0) = G_v(0,0,0) = 1$, we reach the sixth-order and $\bar{\bar{L}}_6 = \frac{1}{4}\rho(r_{4,1}c_2^3 + r_{4,2}c_2 c_3)((-2(6\rho + \gamma^2) + 2G_{uv}(0,0,0)\rho + G_{vv}(0,0,0) + G_{uu}(0,0,0)\rho^2 + G_w(0,0,0)(2(5\rho^2 + \gamma^2) - H_{uu}(0,0)\rho^2 - 2H_{uv}(0,0)\rho - H_{vv}(0,0))c_2^3 - 2(G_w(0,0,0) - 1)\rho c_2 c_3)$.

Again, by imposing conditions $G_{uv}(0,0,0) = 1 + H_{uv}(0,0)$, $G_w(0,0,0) = 1$ and $H_{uu}(0,0) = G_{uu}(0,0,0)$, we obtain order 7 with

$$
\begin{aligned}
\bar{\bar{\bar{L}}}_7 = &-\frac{1}{4}\left(\rho c_2^2\left((-10 + 2H_{uv}(0,0) + H_{vv}(0,0) - 10\gamma + 2\gamma H_{uv}(0,0) - 2\gamma^2 + G_{uu}(0,0,0)\rho^2\right)c_2^2 + 2\rho c_3\right) \\
&((32 - 10G_{vw}(0,0,0) - 4H_{uv}(0,0) + 2G_{vw}(0,0,0)H_{uv}(0,0) - 2H_{vv}(0,0) + G_{vw}(0,0,0)H_{vv}(0,0) \\
&+48\gamma - 10\gamma G_{vw}(0,0,0) - 6\gamma H_{uv}(0,0) + 2\gamma G_{vw}(0,0,0)H_{uv}(0,0) - \gamma H_{vv}(0,0) + 20\gamma^2 \\
&-2\gamma^2 G_{vw}(0,0,0) - 2\gamma^2 H_{uv}(0,0) + 2\gamma^3 + G_{uu}(0,0,0)\rho^2(-2 + G_{uw}(0,0,0) + G_{vw}(0,0,0) \\
&-\gamma + \gamma G_{uw}(0,0,0)) + G_{uw}(0,0,0)\rho\left(H_{vv}(0,0) + 2H_{uv}(0,0)\rho - 2\left(5 + 5\gamma + \gamma^2\right)\right))c_2^2 \\
&+2\rho(-4 + G_{uw}(0,0,0) + G_{vw}(0,0,0) - 2\gamma + \gamma G_{uw}(0,0,0))c_3)).
\end{aligned}
$$

Finally, conditions $G_{uu}(0,0,0) = G_{uw}(0,0,0) = G_{vw}(0,0,0) = 2$, $H_{uv}(0,0) = 0$ and $H_{vv}(0,0) = 2$, allows us to gain eighth-order with the following error equation

$$
\begin{aligned}
e_{k+1} = &-\frac{1}{2}\rho^2 c_2(3c_2^2 - c_3)((9G_{ww}(0,0,0)\rho^2 + 2(-27 - 54\gamma - 28\gamma^2 + \gamma^3) + \gamma^4))c_2^4 + 2\rho(17 + 17\gamma + 2\gamma^2 \\
&- 3G_{ww}(0,0,0)\rho)c_2^2 c_3 + (-2 + G_{ww}(0,0,0))\rho^2 c_3^2 - 2\rho^2 c_2 c_4))e_k^8 + O(e_k^9)
\end{aligned}
$$

and the proof is finished. ∎

Remark 1 *The families of three-point methods (3) and (5) require four functional evaluations and have order of convergence eight. Therefore, these families are optimal in the sense of the Kung-Traub conjecture and have computational efficiency index $8^{1/4} \approx 1.6818$.*

In this case, some of the functions H and G satisfying conditions of Theorem 2 are

$$
\begin{aligned}
H(u, v) &= 1 + u + v + u^2 + v^2, \\
G(u, v, w) &= 1 + u + v + w + u^2 + v^2 + uv + 2(vw + uw). \tag{12}
\end{aligned}
$$

In the following sections we are going to use the elements of the families (3), for $b_1 = 1$ and $b_2 = 0$, and (5) obtained by choosing the weight functions (4) and (12). These elements of the class will be denoted by $M_1 8$ and $M_2 8$, respectively.

3 Dynamical aspects

The dynamical properties of the rational function associated to an iterative method acting on a polynomial give us important information about numerical features of the method as its stability and reliability (see [17]).

The dynamical behavior of the orbit of a point on the complex plane can be classified depending on its asymptotic behavior. In this way, a point in the Riemann sphere $z_0 \in \hat{\mathbb{C}}$ is a fixed point of R if $R(z_0) = z_0$. A fixed point is attracting, repelling or parabolic if $|R'(z_0)|$ is less than, greater than or equal to 1, respectively. Moreover, if $|R'(z_0)| = 0$, the fixed point is superattracting.

If z^* is an attracting fixed point of the rational function R, its basin of attraction $\mathcal{A}(z^*)$ is defined as the set of pre-images of any order such that

$$\mathcal{A}(z_f^*) = \left\{ z_0 \in \hat{\mathbb{C}} : R^n(z_0) \to z^*, n \to \infty \right\}.$$

The set of points whose orbits tends to an attracting fixed point z_f^* is defined as the Fatou set, $\mathcal{F}(R)$. The complementary set in the Riemann sphere, the Julia set $\mathcal{J}(R)$, is the closure of the set consisting of its repelling fixed points, and establishes the borders between the basins of attraction.

For the representation of the convergence basins of every iterative procedure we have used the software described in [18]. In order to draw the dynamical planes, each point of the complex plane is considered as a starting point of the iterative scheme and it is painted in different colors depending on the point which it has converged to (the roots of the polynomial are marked in the figures by white stars). The figures has been generated for values of z_0 in $[-2, 2] \times [-2, 2]$, with a mesh of 800×800 points and 80 iterations per point. Depending on the number of iterations needed to converge, the color of the starting point will be brighter (less iterations) or darker (more iterations). We will represent the dynamical behavior of the mentioned elements of the two suggested classes, (3) and (5), for low-degree polynomials and non-polynomial functions, showing their stability and the amplitude of the convergence regions in these cases.
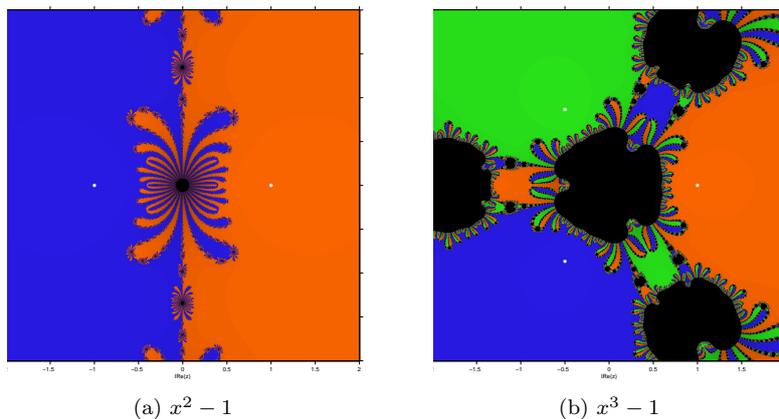

(a) $x^2 - 1$　　　　　　　(b) $x^3 - 1$

Figure 1: Dynamical planes for the scheme $M_1 8$

When $M_1 8$ is considered, we observe in Figure 1 the basin of attraction of the different roots of low-degree polynomials. It is possible to observe some regions of slow convergence, in form of "flowers", whose petals make narrower while the convergence is slower. The black point in the center of the flowers correspond to regions of no convergence. The observed dynamical planes define wide amplitudes for the different basins of the roots.

If we consider now the rational function associated to the element of family (5) and analyze its dynamical behavior, we find that there exist black regions in the dynamical planes (see Figure 2) whose orbits do not tend to any of the roots, but to the infinity. So, the are wide regions of no convergence. Moreover, it can be observed that the amplitude of the convergence regions is narrower in some cases. Nevertheless, the convergence improves when β is closer to zero than when it is near one.

In Figure 3, we show the dynamical planes corresponding to function $f(x) = 10xe^{-x^2} - 1$ in $[-2, 2] \times [-2, 2]$, where two simple roots appear. These planes have been obtained by using Newton, $M_1 8$ and $M_2 8$ methods. In all cases, it can be observed that wide regions of non convergence appear. Some of them are due to poles of the function f. We observe that the behavior of the proposed methods is not very different from the one of Newton's scheme.
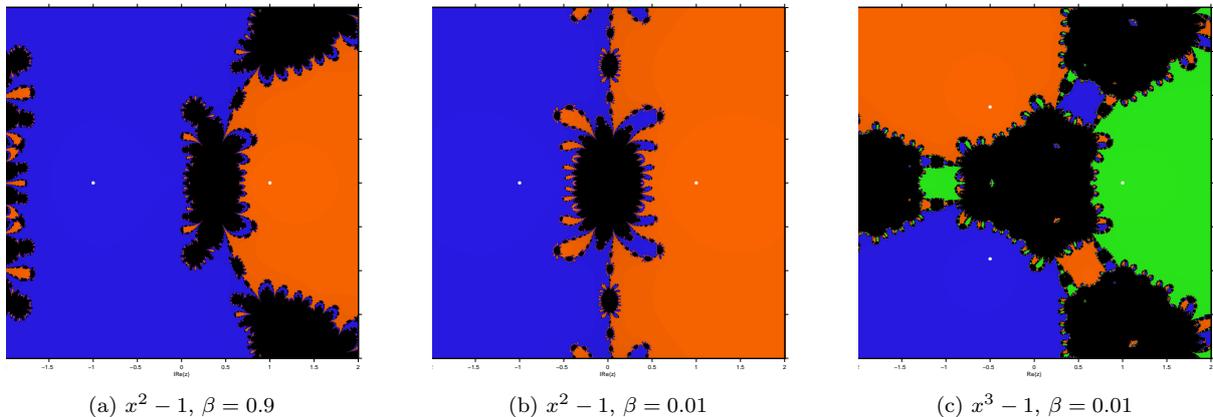
(a) $x^2 - 1$, $\beta = 0.9$ (b) $x^2 - 1$, $\beta = 0.01$ (c) $x^3 - 1$, $\beta = 0.01$

Figure 2: Dynamical planes for the method $M_2 8$

(a) Newton   (b) $M_1 8$ (c) $M_2 8$, $\beta = 0.9$ (d) $M_2 8$, $\beta = 0.01$
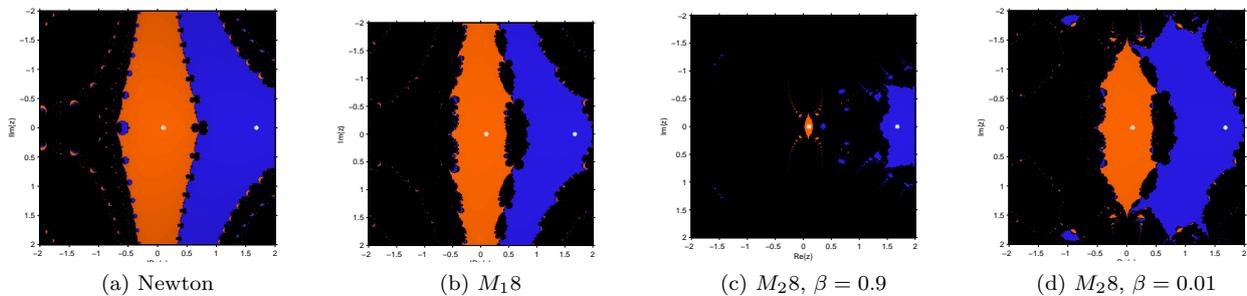
Figure 3: Dynamical planes for function $10xe^{-x^2} - 1$

4 Preliminary orbit determination

A classical reference in preliminary orbit determination is F. Gauss (1777-1855), who deduced the orbit of the minor planet Ceres, discovered in 1801 and afterwards lost. The calculation of its trajectory by means of the procedure designed by Gauss and the re-discovering in following Gauss' calculations, marked the international recognition of Gauss and his work.

The so-called Gauss' method is based on the rate y between the triangle and the ellipse sector defined by two position vectors, from astronomical observations. This proportion is related with the geometry of the orbit and the observed position by

$$y = 1 + X(l + x), \tag{13}$$

where

$$l = \frac{r_1 + r_2}{4\sqrt{r_1 r_2} \cos\left(\frac{\nu_2 - \nu_1}{2}\right)} - \frac{1}{2}, \quad x = \sin^2\left(\frac{E_2 - E_1}{4}\right),$$

and

$$X = \frac{E_2 - E_1 - \sin\left(E_2 - E_1\right)}{\sin^3\left(\frac{E_2 - E_1}{2}\right)}.$$

The angles E_i, ν_i, $i = 1, 2$, are the eccentric and true anomalies, respectively, associated to the observed positions $\overrightarrow{r_1}$ and $\overrightarrow{r_2}$ (let us denote by r_i the modulus of vector $\overrightarrow{r_i}$, $i = 1, 2$).

Equation (13) is, actually, the composition of the First Gauss Equation

$$y^2 = \frac{m}{l + x}$$

and the Second Gauss Equation

$$y^2(y - 1) = mX,$$

where $m = \dfrac{\mu \tau^2}{[2\sqrt{r_1 r_2}\cos(\frac{\nu_2 - \nu_1}{2})]^3}$, μ is the gravitational parameter of the motion and τ is a modified time variable.

The original iterative procedure used to solve the nonlinear Gauss equation (13) is the Fixed Point method (see, for example, [14]) and is described in the following scheme:

(i) From the initial estimation $y_0 = 1$, $x_0 = \frac{m}{y_0^2} - l$ is obtained (it is possible to calculate m and l from the observed positions $\vec{r_1}$ and $\vec{r_2}$ and the time τ.

(ii) From x_0 and

$$\cos\left(\frac{E_2 - E_1}{2}\right) = 1 - 2x_0, \quad \sin\left(\frac{E_2 - E_1}{2}\right) = +\sqrt{4x_0(1 - x_0)},$$

we calculate $E_2 - E_1$. Then, we obtain $X_0 = \dfrac{E_2 - E_1 - \sin(E_2 - E_1)}{\sin^3\left(\frac{E_2 - E_1}{2}\right)}$.

(iii) By using the combined Gauss equation (13)

$$y_1 = 1 + X_0(l + x_0),$$

a new iteration is calculated and the process start again.

The iterative process follows as described above, getting new estimations of the ratio, until it does not vary within a given tolerance. Once the method has converged, the semi-major axis a, can be calculated by means of equation

$$y = \frac{\sqrt{\mu p} \cdot \tau}{r_2 r_1 \sin(\nu_2 - \nu_1)} = \frac{\sqrt{\mu} \cdot \tau}{2\sqrt{a}\sqrt{r_2 r_1}\sin\left(\frac{E_2 - E_1}{2}\right)\cos\left(\frac{\nu_2 - \nu_1}{2}\right)},$$

from the last estimations of ratio and difference of eccentric anomalies, and the last phase is then initiated, to determine velocity and orbital elements.

Let us note that the initial estimation used is good only if the observations are close in time, so it is a weakness implicit in Gauss method. Indeed, the original fixed point scheme, with linear convergence, has been improved by the recent optimal high-order schemes. In particular, we are going to apply the proposed optimal eight-order methods, joint with some existing schemes of the same rate of convergence, in order to get an efficient modified-Gauss procedure of preliminary orbit determination, by solving the combined Gauss equation (13), with $y = 1$ as an initial estimation.

All the iterative schemes introduced in the following are optimal in the sense of Kung-Traub's conjecture and have been designed with the weight-function technique, so they are fully comparable with the new ones designed in this paper. Let us refer now to the procedure that Kim present in [4]: a three-step eighth-order method, whose iterative expression is

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\
z_k &= y_k - \frac{1 + u_k + 2/3u_k^2}{1 - u_k - 2u_k^2}\frac{f(y_k)}{f'(x_k)}, \\
x_{k+1} &= z_k - \frac{1 - 2u_k + v_k}{1 - 3u_k - 2v_k}\frac{f(z_k)}{f'(x_k) + f[y_k, x_k, z_k](z_k - x_k)},
\end{aligned}
\tag{14}
$$

where $u_k = \frac{f(y_k)}{f(x_k)}$, $v_k = \frac{f(z_k)}{f(x_k)}$ and $f[\cdot, \cdot, \cdot]$ denotes the divided difference of order two. We will denote this scheme by K8.

Another recent eighth-order method has been developed by Khan et. al. in [5]. Its iterative expression is:

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\
z_k &= y_k - \frac{f(x_k)^2}{f(x_k)^2 - 2f(x_k)f(y_k) + f(y_k)^2}\frac{f(y_k)}{f'(x_k)}, \\
x_{k+1} &= z_k - \frac{1}{1 + v_k^2}\frac{f(z_k)}{K - C(y_k - z_k) - D(y_k - z_k)^2},
\end{aligned}
\tag{15}
$$

where $v_k = \frac{f(z_k)}{f(x_k)}$, $K = f[y_k, z_k]$, $H = f[x_k, y_k]$, $C = \frac{H-K}{(x_k-y_k)(x_k-z_k)} - D(x_k + y_k - 2z_k)$, $D = \frac{f'(x_k)-H}{(x_k-y_k)(x_k-z_k)} - \frac{H-K}{(x_k-z_k)^2}$ and $f[\cdot, \cdot]$ denotes the divided difference of order 1. In the following, we will denote this method by F8.

The third scheme that we will use to compare with, denoted by D8, was obtained by Džunić and Petković in [19]. It is based on Ostrowski's fourth-order scheme (see [20]) and its iterative expression is

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\
z_k &= y_k - \frac{f(x_k)}{f(x_k) - 2f(y_k)} \frac{f(y_k)}{f'(x_k)}, \\
x_{k+1} &= z_k - \frac{(1+w_k)(1+2v_k)}{1 - 2u_k - u_k^2} \frac{f(z_k)}{f'(x_k)},
\end{aligned}
\tag{16}
$$

where $u_k = \frac{f(y_k)}{f(x_k)}$, $v_k = \frac{f(z_k)}{f(x_k)}$ and $w_k = \frac{f(z_k)}{f(y_k)}$.

Finally, we will also compare our new schemes with the method designed by Soleymani et al. in [21] (denoted by S8), also initialized with Ostrowski's procedure,

$$
\begin{aligned}
y_k &= x_k - \frac{f(x_k)}{f'(x_k)}, \\
z_k &= y_k - \frac{f(x_k)}{f(x_k) - 2f(y_k)} \frac{f(y_k)}{f'(x_k)}, \\
x_{k+1} &= z_k - \frac{f(z_k)}{2f[y_k, x_k] - f'(x_k) + f[z_k, x_k, x_k](z_k - y_k)} \left(1 + w_k + 2v_k - 2u_k^3 + \frac{2}{5}\frac{f(z_k)}{f'(x_k)}\right),
\end{aligned}
\tag{17}
$$

where $u_k = \frac{f(y_k)}{f(x_k)}$, $v_k = \frac{f(z_k)}{f(x_k)}$ and $w_k = \frac{f(z_k)}{f(y_k)}$.

In the numerical test made, variable precision arithmetics has been used, with 4000 digits of mantissa in Matlab R2011b. Some reference orbits have been used in the test, that can be found in [14]. As orbital elements of each one of the test orbits are known, the vector position in the instants t_1 and t_2 have been re-calculated with 3998 exact digits. Then, our aim is gain from these positions, the orbital elements, with a precision as high as possible:

- Test Orbit I has the position vectors

$$
\vec{r_1} \approx [2.46080928705339, 2.04052290636432, 0.14381905768815]
$$

and

$$
\vec{r_2} \approx [1.98804155574820, 2.50333354505224, 0.31455350605251],
$$

measured in Earth radius (e.r.) at the instants $t_1 = 0$ and $t_2 = 0.01044412000000$ julian days (J.D.). The orbital elements corresponding to the geometry of the orbit are the semimajor axis $a = 4$ e.r., the eccentricity $e = 0.2$, the epoch of the perigee $T_0 = 0h0m0s$, and the Euler angles which fit the orbit in space are the Right Ascension of the ascending node, $\Omega = 30^o$, the argument of the perigee $\omega = 10^o$ and the inclination of the orbit $i = 15^o$.

- Test Orbit II. Position vectors and times:

$$
\vec{r_1} \approx [-1.75981065999937, 1.68112802634201, 1.16913429510899] \quad \text{e.r.,} \quad t_1 = 0 \quad \text{J.D.,}
$$

$$
\vec{r_2} \approx [-2.23077219993536, 0.77453561301361, 1.34602197883025] \quad \text{e.r.,} \, t_2 = 0.01527809 \quad \text{J.D.,}
$$

Orbital elements: $\Omega = 80^o$, $\omega = 60^o$, $i = 30^o$, $a = 3$ e.r., $e = 0.1$, $T_0 = 0h0m0s$.

- Test Orbit III. Position vectors and times:

$$
\vec{r_1} \approx [0.41136206679761, -1.66250000000000, 0.82272413359522] \quad \text{e.r.,} \quad t_1 = 0 \quad \text{J.D.,}
$$

$$
\vec{r_2} \approx [0.97756752977209, -1.64428006097667, -0.04236299091612] \quad \text{e.r.,} \, t_2 = 0.01316924 \quad \text{J.D.,}
$$

Orbital elements: $\Omega = 120^o$, $\omega = 150^o$, $i = 60^o$, $a = 2$ e.r., $e = 0.05$, $T_0 = 0h0m0s$.

By using these test positions vectors and times, we will consider that the solution provided by Newton's scheme with a required accuracy of 10^{-3998} is the exact solution y_{ex}. Then we will compare the exact error at the first three iterations of the proposed eighth-order method M8 and the known schemes K8, F8, D8 and S8. We also include, in Tables 1 to 3, the computational order of convergence (COC) (see [22]), in order to check the computational efficiency of the schemes related to their theoretical rate of convergence. This index is evaluated by the formula

$$COC \approx \frac{\log|(x_{k+1} - \xi)/(x_k - \xi)|}{\log|(x_k - \xi)/(x_{k-1} - \xi)|}.$$

Scheme	Fixed Point	K8	F8	D8	S8	$M_1 8$
$|y_1 - y_{ex}|$	0.006399	8.572e-25	1.717e-18	1.459e-24	1.2369276e-24	1.244e-24
$|y_2 - y_{ex}|$	8.157e-5	8.642e-200	2.389e-96	1.105e-197	2.503241e-198	2.636e-198
$|y_3 - y_{ex}|$	1.03e-6	9.217e-1600	1.246e-485	1.198e-1582	7.043196e-1588	1.075e-1587
COC	1.0	8.0	5.0	8.0	8.0	8.0

Table 1: Comparison of modified-Gauss schemes for Reference Orbit I

The different test orbits have been chosen with an important property: the increasing angle $\nu_2 - \nu_1$. It measures the spread in the observations and, by the design of Gauss' procedure, it induces instability in the system when it gets higher. The difference between the true anomalies of the observations is, for the test orbits I to III, $\nu_2 - \nu_1 = 12.23^o$, $\nu_2 - \nu_1 = 22.06^o$ and $\nu_2 - \nu_1 = 31.46^o$, respectively. It can be observed in Tables 1 to 3 that, when the spread of the observations increases, the precision obtained in the calculations per step reduces in the same rate for any eighth-order method.

Scheme	Fixed Point	K8	F8	D8	S8	$M_1 8$
$|y_1 - y_{ex}|$	0.02314	4.650e-18	1.327e-13	4.57e-18	3.8834479e-18	2.425e-18
$|y_2 - y_{ex}|$	0.001055	1.053e-143	8.253e-70	1.216e-143	2.8055145e-144	4.214e-146
$|y_3 - y_{ex}|$	4.652e-5	7.296-1149	7.694e-351	3.061e-1148	2.0814831e-1153	3.508e-1168
COC	1.0	8.0	5.0	8.0	8.0	8.0

Table 2: Comparison of modified-Gauss schemes for Reference Orbit II

It is clear that the application of high-order schemes to the problem of preliminary orbit calculation by Gauss procedure gets an important success, as the gain in speed and the precision obtained in the calculations are increased. Nevertheless, the behavior of all the high-order schemes are not the same. The great complexity of the calculations involved in the problem and the own stability of each iterative scheme are important factors to take into account: let us note that Khan's procedure is not appropriated for this particular problem, as it converges more slowly than the rest of scheme, being a good method with theoretical order of convergence eight.

Scheme	Fixed Point	K8	F8	D8	S8	$M_1 8$
$|y_1 - y_{ex}|$	0.05052	6.249e-14	9.195e-11	2.6e-14	2.1653574e-14	4.962e-15
$|y_2 - y_{ex}|$	0.004936	2.263e-109	1.888e-54	1.753e-112	3.3635496e-113	7.299e-119
$|y_3 - y_{ex}|$	0.0004492	6.695e-873	6.897e-273	7.503e-898	1.1401025e-903	1.6e-949
COC	1.0	8.0	5.0	8.0	8.0	8.0

Table 3: Comparison of modified-Gauss schemes for Reference Orbit III

Let us note that the precision of the orbital elements calculated with the third estimation provided by any eighth-order method is total, as all the 4000 decimal digits of the solution considered as exact are reached with only three iterations.

5 Numerical results

In this section we show the convergence behavior of the proposed methods $M_1 8$ and $M_2 8$. In the numerical test made, variable precision arithmetics has been used, with 4000 digits of mantissa in Matlab R2010a. In our numerical experiments we compare our schemes with Newton's method (NM), Steffensen's method (SM), Ostrowski's method (OM), and several iterative procedures, quoted below.

KWM [3]

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$

$$z_k = y_k - H_1 \frac{f(y_k)}{f'(x_k)},$$

$$x_{k+1} = z_k - \left[(1 + H_1)^2 + (1 + 4H_1) H_2 \right] \frac{f(z_k)}{f'(x_k)},$$

where $H_1 = \frac{f(x_k)}{f(x_k) - 2f(y_k)}$ and $H_2 = \frac{f(z_k)}{f(y_k) - 3f(z_k)}$.

CTVM [7]

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$

$$z_k = y_k - H \frac{f(y_k)}{f'(x_k)},$$

$$x_{k+1} = u_k - 3 \frac{u_k - z_k}{y_k - x_k} \frac{f(z_k)}{f'(x_k)},$$

where $H = \frac{f(x_k)}{f(x_k) - 2f(y_k)}$, $u_k = z_k - (H + T/2)^2 \frac{f(z_k)}{f'(x_k)}$ and $T = \frac{f(z_k)}{f(y_k) - 2f(z_k)}$.

ZLHM [8]

$$y_k = x_k - \frac{f(x_k)}{f[x_k, z_k]}, \quad z_k = x_k + \gamma f(x_k),$$

$$u_k = y_k - \frac{f(y_k)}{f[y_k, x_k] + f[y_k, x_k, z_k](y_k - x_k)},$$

$$x_{k+1} = u_k - \frac{f(u_k)}{f[u_k, y_k] + f[u_k, y_k, x_k](u_k - y_k) + f[u_k, y_k, x_k, z_k](u_k - y_k)(u_k - x_k)},$$

SKM [11]

$$y_k = x_k - \frac{f(x_k)}{f[x_k, z_k]}, \quad z_k = x_k - f(x_k),$$

$$u_k = y_k - H \frac{f(x_k)}{f[x_k, z_k]},$$

$$x_{k+1} = u_k - G \frac{f(u_k)}{f[x_k, z_k]},$$

where $H = 1 + w_1(x_k) + w_2(x_k)$, $w_1(x_k) = \frac{f(y_k)}{f(x_k)}$, $w_2(x_k) = \frac{f(y_k)}{f(z_k)}$, $G = g_1 + g_2 + g_3$, $g_1 = 1 + (2 - f[x_k, z_k])w_1(x_k) + (1 - f[x_k, z_k])w_1(x_k)^2$, $g_2 = (4 - f[x_k, z_k](6 + f[x_k, z_k](-4 + f[x_k, z_k])))w_1(x_k)^3$, $g_3 = w_3(x_k) + w_3(x_k)^2 + (4 - 2f[x_k, z_k])w_4(x_k)$, $w_3(x_k) = \frac{f(u_k)}{f(y_k)}$ and $w_4 = \frac{f(u_k)}{f(z_k)}$.

Table 4 shows the expression of the test functions, the roots with sixteen significant digits and the initial approximation x_0 which is the same for all methods. Displayed in Table 5 is the number of iterations (It), the computational order of convergence (COC), the absolute value of the function $|f(x_k)|$ at the last iteration and root obtained at each method ξ.

These numerical results shown are in accordance with the efficiency analysis the show in Theorem 1 and Theorem 2 presented in Section 2. Both the numerical results, such as the dynamic analysis show that derivative-free methods are more sensitive to the initial approximation comparing them with those that use derivatives.

Table 4: Test functions and their roots

Test functions	Roots	Starting points
$f_1 = \sqrt{x^4 + 8}\ \sin\left(\frac{\pi}{x^2+2}\right) + \frac{x^3}{x^4+1} - \sqrt{6} + \frac{8}{17}$	$\xi_1 = -2$	$x_0 = -1.8$
	$\xi_2 \approx -1.149212674609088$	
$f_2 = x \exp(x^2) - \sin(x^2) + 3\cos(x) + 5$	$\xi \approx -1.201576112092293$	$x_0 = 2$
$f_3 = \sqrt{x^2 + 2x + 5} - 2\sin(x) - x^2 + 3$	$\xi_1 \approx +2.3319676558839640$	$x_0 = 1$
	$\xi_2 \approx -2.573166514902827$	
$f_4 = x^4 + \sin\left(\frac{\pi}{x^2}\right) - 5$	$\xi_1 \approx +1.414213562373095$	$x_0 = 2$
	$\xi_2 \approx -1.414213562373095$	
$f_5 = \left(\sin(x) - \frac{x}{2}\right)^2$	$\xi = 0$ (double root)	$x_0 = 0.5$

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Table 5: Comparison of various derivative-free iterative methods

		NM	SM	OM	KWM	CTVM	$M_1 8$	ZLHM	SKM	$M_2 8$
f_1	It	9	9	5	3	3	4	3	4	3
	COC	2.0000	2.0000	4.0000	8.2176	8.3082	8.0000	8.1662	7.9999	8.0000
	$|f(x_n)|$	1.6e-479	2.3e-574	8.8e-752	9.8e-359	1.4e-329	9.6e-2525	6.6e-431	4.2e-1882	9.6e-2522
	root	ξ_1	ξ	ξ_1	ξ_1	ξ_1	ξ_1	ξ_1	ξ_1	ξ_1
f_2	It	14	n.c.	5	3	3	4	4	n.c.	4
	COC	2.0000		4.0000	7.9547	7.9386	8.0203	8.0038		8.0203
	$|f(x_n)|$	2.6e-553		7.9e-1063	3.5e-517	8.6e-485	5.0e-528	3.7e-1066		5.0e-528
	root	ξ		ξ	ξ	ξ_1	ξ	ξ		ξ
f_3	It	8	8	5	3	3	3	3	14	3
	COC	2.0000	2.0000	3.9998	8.0286	7.9075	7.9753	8.0003	7.6578	7.9753
	$|f(x_n)|$	3.2e-422	1.8e-334	2.7e-1292	5.6e-560	6.7e-622	4.0e-545	9.0e-556	4.4e-080	4.0e-545
	root	ξ_1	ξ_1	ξ_1	ξ_1	ξ_1	ξ_1	ξ_1	ξ_1	ξ_1
f_4	It	9	70	5	4	3	n.c.	4	6	n.c.
	COC	2.0000	2.0000	4.0000	8.0232	7.6465		8.0353	8.0000	
	$|f(x_n)|$	8.2e-425	5.1e-306	5.6e-621	1.3e-1714	2.3e-333		1.3e-092	2.7e-1750	
	root	ξ_1	ξ_2	ξ_2	ξ_1	ξ_1		ξ_1	ξ_1	
f_5	It	536	536	268	n.c.	194	226	179	226	225
	COC	1.0000	1.0000	1.0000		1.0000	1.0000	1.0000	1.0000	1.0000
	$|f(x_n)|$	7.1e-325	7.9e-325	1.0e-326		3.1e-325	3.5e-325	2.5e-325	2.0e-324	3.5e-325
	root	ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ	ξ

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