# Semilocal convergence by using recurrence relations for a fifth-order method in Banach spaces ☆

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**Abstract**

In this paper, a semilocal convergence result in Banach spaces of an efficient fifth-order method is analyzed. Recurrence relations are used in order to prove this convergence, and some priori error bounds are found. This scheme is finally used to estimate the solution of an integral equation and so, the theoretical results are numerically checked. We use this example to show the better efficiency of the current method compared with other existing ones, including Newton's scheme.

*Keywords:* Nonlinear systems, iterative methods, semilocal convergence, recurrence relations, convergence domain, efficiency index

## 1. Introduction

Newton's method and its variants are used to solve nonlinear equations of the form $F(x) = 0$. This equation can represent differential equations, integral equations or a system of nonlinear equations. The convergence of Newton's method in Banach spaces was established by Kantorovich in [11]. The convergence of the sequence obtained by the iterative expression is derived from the convergence of majorizing sequences. This technique has been used by many authors in order to establish the order of convergence of the variants of Newton's methods (see, for example, [8] and [17]).

Rall in [14] suggested a different approach for the convergence of these methods, based on recurrence relations. Amat, Hernández and Romero ([1], [9]), Ezquerro and Hernández ([5]), Gutiérrez and Hernández ([6] and [7]), Parida and Gupta in [13] and Candela and Marquina ([3] and [4]) used this idea to prove the semilocal convergence for several methods of different orders.

☆This research was supported by Ministerio de Ciencia y Tecnología MTM2011-28636-C02-{01,02}

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In this paper, we analyze the semilocal convergence of a fifth-order method M5 considered in [2] for solving systems of nonlinear equations. In order to get this aim, we use the technique of recurrence relations, that consists of generate a sequence of positive real numbers that guarantees the convergence of the iterative scheme in Banach spaces, providing a suitable convergence domain. This technique allows us to establish weak semilocal convergence conditions for an iterative method with fifth-order of convergence. Even more, we get a result of semilocal convergence under the same conditions of Kantorovich Theorem for Newton's method, which has quadratic convergence. This allows us to apply the fifth-order convergence method for solving nonlinear equations $F(x) = 0$ under the same conditions that assures us the convergence of Newton's method.

Another important aspect of this work is the comparative study of the efficiency of the proposed scheme with the one of other known high-order methods, such as Jarratt's method (see [10]) and the one introduced by Wang et al. in [16], by using the classical efficiency index defined by Ostrowski in [12] and the computational efficiency index described by Traub in [15]. In addition, we include in our comparative study of the efficiency the most used iterative process, the Newton method. Noting that the proposed iterative process M5 is also more efficient than Newton's method when trying to approximate a solution of a system with more than two equations.

Finally, we make some test on integral equations in order to check the theoretical results. Noting that the proposed method M5 is more efficient for the approximation of a solution.

The rest of the paper is organized as follows: in Section 2 we describe the recurrence relations and the properties needed to prove the semilocal convergence of method M5 in Section 3. In Section 4 the comparative analysis of the efficiency is made. Finally, in Section 5 an application on integral equation of mixed Hammerstein type is illustrated.

## 2. Recurrence relations

Let $X$, $Y$ be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain $\Omega$. The fifth-order method M5, which semilocal convergence we are going to study can be found in [2] and its iterative expression is:

$$\left\{ \begin{array}{lcl} y_n & = & x_n - \Gamma_n F(x_n), \\ z_n & = & y_n - 5\Gamma_n F(y_n), \\ x_{n+1} & = & z_n - \frac{1}{5}\Gamma_n \left(-16F(y_n) + F(z_n)\right), \end{array} \right. \tag{1}$$

where $\Gamma_n = [F'(x_n)]^{-1}$, for $n \in \mathbb{N}$.

Let us assume that the inverse of $F'$ at $x_0$, $F'(x_0)^{-1} = \Gamma_0 \in \mathcal{L}(Y, X)$ exists at some $x_0 \in \Omega$, where $\mathcal{L}(Y, X)$ is the set of bounded linear operators from $Y$ into $X$.

In the following we will assume that $y_0, z_0 \in \Omega$ and

(i) $\|\Gamma_0\| \le \beta$,

(ii) $\|\Gamma_0 F(x_0)\| \le \eta$,

(iii) $\|F'(x) - F'(y)\| \leq K\|x - y\|, \qquad x, y \in \Omega,$

in order to obtain the recurrence relations that satisfy the steps that appear in the iterative process (1).

Notice that these are the classical Kantorovich's conditions [11] for the semilocal convergence of Newton's method.

Let us also denote by $a_0 = K\beta\eta$ and define the sequence $a_{n+1} = a_n f(a_n)^2 g(a_n)$, where

$$f(x) = \frac{1}{1 - x(h(x) + 1)}, \tag{2}$$

$$g(x) = \frac{1}{2}x + (x + 1)h(x) + \frac{1}{2}xh(x)^2 \tag{3}$$

and

$$h(x) = \frac{1}{2}x + \frac{1}{2}x^2 + \frac{5}{8}x^3. \tag{4}$$

To study the convergence of $\{x_n\}$ defined by (1) to a solution of $F(x) = 0$ in a Banach space, we have to prove that $\{x_n\}$ is a Cauchy sequence. To do this, we need to analyze some properties of sequence $\{a_n\}$ and, previously, of the real functions described in (2), (3) and (4), respectively.

**Lemma 1.** *Let $f(x)$, $g(x)$ and $h(x)$ be the real functions described in (2), (3) and (4). Then,*

*(i) $f$ is increasing and $f(x) > 1$ for $x \in (0, 0.6)$.*

*(ii) $h$ and $g$ are increasing for $x \in (0, 0.6)$.*

**Lemma 2.** *Let $f(x)$ and $g(x)$ as before and $a_0 \in (0, 0.2931...)$. Then,*

*(i) $f(a_0)^2 g(a_0) < 1$,*

*(ii) $f(a_0)g(a_0) < 1$,*

*(iii) the sequence $\{a_n\}$ is decreasing and $a_n < 0.2931...$, for $n \geq 0$.*

**Proof:** From the definition of functions $f$ and $g$ (i) follows trivially. From (i) and $f(a_0) > 1$, we obtain (ii). We are going to prove (iii) by induction on $n \geq 0$. Firstly, from (i) and the definition of $a_1$, we have that $a_1 < a_0$. Now, it is supposed that $a_k < a_{k-1}$, for $k \leq n$. Then,

$$a_{n+1} = a_n f(a_n)^2 g(a_n) < a_{n-1} f(a_n)^2 g(a_n) < a_{n-1} f(a_{n-1})^2 g(a_{n-1}) = a_n,$$

as $f$ and $g$ are increasing and $f(x) > 1$.

Finally, for all $n \geq 0$, $a_n < 0.2931...$, since $\{a_n\}$ is a decreasing sequence and $a_0 < 0.2931...$. □

Let us note that $a_0 = 0.2931...$ is the value of the solution of equation $f(a_0)^2 g(a_0) - 1 = 0$.

Using Taylor's expansion of $F(y_0)$ around $x_0$,

$$z_0-x_0 = y_0-x_0-5\Gamma_0 F(y_0) = y_0-x_0-5\Gamma_0 \int_0^1 (F'(x_0+t(y_0-x_0))-F'(x_0))(y_0-x_0)dt,$$

so,

$$\|z_0 - x_0\| \leq \|y_0 - x_0\| + \frac{5}{2}K\beta\|y_0 - x_0\|^2.$$

In a similar way,

$$\|z_0 - y_0\| \leq \frac{5}{2}a_0\|y_0 - x_0\|.$$

Now, by using Taylor's expansion of $F(z_0)$ and (1), we have

$$\begin{aligned}
\|x_1 - x_0\| &= \left\|-\Gamma_0\left(F(x_0) + \frac{9}{5}F(y_0) + \frac{1}{5}F(z_0)\right)\right\| \\
&= \left\|y_0 - x_0 - \frac{9}{5}\Gamma_0\int_{x_0}^{y_0}(F'(x) - F'(x_0))dx \right. \qquad (5)\\
&\quad \left. +\Gamma_0\int_{x_0}^{y_0}(F'(x) - F'(x_0))dx - \frac{1}{5}\Gamma_0\int_{x_0}^{z_0}(F'(x) - F'(x_0))dx\right\| \\
&\leq \|y_0 - x_0\| + \frac{4}{10}K\beta\|y_0 - x_0\|^2 + \frac{1}{10}K\beta\|z_0 - x_0\|^2 \\
&\leq (1 + \frac{1}{2}a_0 + \frac{1}{2}a_0^2 + \frac{5}{8}a_0^3)\eta = (1 + h(a_0))\eta.
\end{aligned}$$

Now, assuming that $a_0 < 0.6$ and applying assumptions (i) to (iii), we have

$$\|I - \Gamma_0 F'(x_1)\| \leq \|\Gamma_0\|\|F'(x_1) - F'(x_0)\| \leq \beta K\|x_1 - x_0\| \leq a_0(1 + h(a_0)) < 1,$$

and, by the Banach Lemma, $\Gamma_1$ exists and

$$\|\Gamma_1\| \leq \frac{1}{1 - a_0(1 + h(a_0))}\|\Gamma_0\| = f(a_0)\|\Gamma_0\|.$$

Let us remark that we need $a_0 < 0.6$ in order to guaranty $a_0(1 + h(a_0)) < 1$. Let us also note that $K\|\Gamma_0\|\|y_0 - x_0\| \leq a_0$, so it can be deduced that $x_1$ is well defined and

$$\|x_1 - x_0\| \leq \|\Gamma_0\| \left\|F(x_0) + \frac{9}{5}F(y_0) + \frac{1}{5}F(z_0)\right\| \leq (h(a_0) + 1)\|\Gamma_0 F(x_0)\|. \qquad (6)$$

Then, assuming that $x_n, y_n, z_n \in \Omega$ and $a_n < 0.6$, for $n \geq 1$, the following estimations can be proved by induction on $n \geq 1$:

$(I_n)$ $\|\Gamma_n\| \leq f(a_{n-1})\|\Gamma_{n-1}\|$,

$(II_n)$ $\|y_n - x_n\| = \|\Gamma_n F(x_n)\| \leq f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\|$,

$(III_n)$ $\|z_n - y_n\| \leq \frac{5}{2}\beta K f(a_0)^n\|y_n - x_n\|^2$,

$(IV_n)$ $K\|\Gamma_n\|\|y_n - x_n\| \leq a_n$,

$(V_n)$ $\|x_n - x_{n-1}\| \leq (1 + h(a_{n-1}))\|y_{n-1} - x_{n-1}\|$.

Let us consider $n = 1$. So, $(I_1)$ has been proved before.
$(II_1)$: Using Taylor's formula,

$$\begin{aligned} F(x_1) &= F(y_0) + F'(y_0)(x_1 - y_0) + \int_{y_0}^{x_1} (F'(x) - F'(y_0))dx \\ &= \int_0^1 (F'(x_0 + t(y_0 - x_0)) - F'(x_0))\,(y_0 - x_0)dt \\ &\quad -(F'(y_0) - F'(x_0) + F'(x_0))\Gamma_0(\frac{9}{5}F(y_0) + \frac{1}{5}F(z_0)) \\ &\quad -\Gamma_0(\frac{9}{5}F(y_0) + \frac{1}{5}F(z_0)) \int_0^1 (F'(y_0 + t(x_1 - y_0)) - F'(y_0))\,dt. \end{aligned}$$

On the other hand,

$$\left\| \frac{9}{5}F(y_0) + \frac{1}{5}F(z_0) \right\| \leq \eta \frac{h(a_0)}{\beta}.$$

Then,

$$\begin{aligned} \|F(x_1)\| &\leq \frac{1}{2}K\eta^2 + K\eta^2(\frac{1}{2}a_0 + \frac{1}{2}a_0^2 + \frac{5}{8}a_0^3) + \eta(\frac{1}{2}\frac{a_0}{\beta} + \frac{1}{2}\frac{a_0^2}{\beta} + \frac{5}{8}\frac{a_0^3}{\beta}) \\ &\quad + \frac{1}{2}K\eta^2(\frac{1}{2}a_0 + \frac{1}{2}a_0^2 + \frac{5}{8}a_0^3)^2. \end{aligned}$$

Therefore,

$$\|y_1 - x_1\| \leq \|\Gamma_1\|\|F(x_1)\| \leq f(a_0)\|\Gamma_0\|\|F(x_1)\| \leq f(a_0)g(a_0)\|y_0 - x_0\|.$$

$(III_1)$: It is clear that

$$\begin{aligned} \|z_1 - y_1\| &\leq 5\|\Gamma_1\|\|F(y_1)\| \leq 5\beta f(a_0) \left\| \int_0^1 (F'(x_1 + t(y_1 - x_1)) - F'(x_1))\,(y_1 - x_1)dt \right\| \\ &\leq \frac{5}{2}\beta K f(a_0)\|y_1 - x_1\|^2. \end{aligned}$$

$(IV_1)$: By using $(I_1)$ and $(II_1)$,

$$K\|\Gamma_1\|\|y_1 - x_1\| \leq Kf(a_0)\|\Gamma_0\|f(a_0)g(a_0)\|y_0 - x_0\| = a_1.$$

$(V_1)$: Has been shown in (6) that $\|x_1 - x_0\| \leq (1 + h(a_0))\|y_0 - x_0\|$.

By considering that the induction hypothesis of items $(I_n)$ to $(V_n)$ are true for a fixed $n \geq 1$, it can be proved $(I_{n+1})$ to $(V_{n+1})$ in a similar way and the induction is complete. □

Let us note that condition $a_n < 0.6$, for $n \geq 1$, is necessary for the existence of operators $\Gamma_n$, $n \geq 1$.

The above recurrence relations for method M5 given in (1) allow us to establish a new semilocal convergence result for this method under mild conditions.

## 3. Semilocal convergence analysis

From the technical Lemmas 1 and 2 and recurrence relations proved in the previous section, we are able to prove the semilocal convergence result for method (1) under mild conditions. In the previous results we have used different conditions for parameter $a_0$. In the following, we will consider the most restrictive one in order to prove the semilocal convergence.

**Theorem 1.** *Let $X$ and $Y$ be Banach spaces and $F : \Omega \subseteq X \to Y$ be a nonlinear twice Fréchet differentiable operator in an open convex domain $\Omega$. Let us assume that $\Gamma_0 = [F'(x_0)]^{-1} \in \mathcal{L}(Y, X)$ exists at some $x_0 \in \Omega$ and assumptions*

*(i)* $\|\Gamma_0\| \leq \beta$,

*(ii)* $\|\Gamma_0 F(x_0)\| \leq \eta$,

*(iii)* $\|F'(x) - F'(y)\| \leq K\|x - y\|, \qquad x, y \in \Omega$,

*are satisfied. Let us denote $a_0 = K\beta\eta$ and suppose that $a_0 < 0.2931....$ Then, if $B(x_0, R\eta) = \{x \in X : \|x - x_0\| < R\eta\} \subset \Omega$, where $R = \frac{5}{2}a_0 + \frac{h(a_0)+1}{1-f(a_0)g(a_0)}$, the sequence $\{x_n\}$ defined in (1) and starting at $x_0$ converges to a solution $x^*$ of the equation $F(x) = 0$. In that case, the solution $x^*$ and the iterates $x_n$, $y_n$ and $z_n$ belong to $\overline{B(x_0, R\eta)}$, and $x^*$ is the only solution of $F(x) = 0$ in $B\left(x_0, \frac{2}{K\beta} - R\eta\right) \cap \Omega$.*

**Proof:** Firstly, let us recall that $\Gamma_n$ exists for $n \geq 1$, since $a_0 < 0.2931....$ Moreover, we are going to prove that $y_n$ and $z_n$ belong to $B(x_0, R\eta) \subset \Omega$. By recurrence relation $(V_n)$, it is easy to observe that

$$\begin{aligned} \|x_n - x_0\| &\leq \|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\| + \cdots + \|x_1 - x_0\| \\ &\leq (1 + h(a_0))\|y_0 - x_0\| \sum_{k=0}^{n-1} (f(a_0)g(a_0))^k. \end{aligned}$$

So,

$$\begin{aligned} \|y_n - x_0\| &\leq \|y_n - x_n\| + \|x_n - x_0\| \\ &\leq (1 + h(a_0))(f(a_0)g(a_0))^n\|y_0 - x_0\| + (1 + h(a_0))\|y_0 - x_0\| \sum_{k=0}^{n-1} (f(a_0)g(a_0))^k \\ &< (1 + h(a_0))\frac{1 - (f(a_0)g(a_0))^{n+1}}{1 - f(a_0)g(a_0)}\eta < R\eta. \end{aligned}$$

Now, by applying recurrence relations $(I_n)$ and $(II_n)$,

$$\begin{aligned} \|z_n - y_n\| &\leq 5\|\Gamma_n\|\|F(y_n)\| \\ &\leq \frac{5}{2}K\beta f(a_0)^n\|y_n - x_n\|^2 \\ &\leq \frac{5}{2}a_0(f(a_0)^3 g(a_0)^2)^n\|y_0 - x_0\|. \end{aligned}$$

Therefore,

$$\begin{aligned}\|z_n - x_0\| &\leq \|z_n - y_n\| + \|y_n - x_0\| \\ &\leq \frac{5}{2}a_0(f(a_0)^3 g(a_0)^2)^n \|y_0 - x_0\| + (1 + h(a_0))\frac{1 - (f(a_0)g(a_0))^{n+1}}{1 - f(a_0)g(a_0)}\|y_0 - x_0\| \\ &< \left(\frac{5}{2}a_0 + (1 + h(a_0))\frac{1 - (f(a_0)g(a_0))^{n+1}}{1 - f(a_0)g(a_0)}\right)\eta < R\eta.\end{aligned}$$

In order to prove the convergence of the sequence $\{x_n\}$, let us state that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq (1 + h(a_n))\|y_n - x_n\| \\ &\leq (1 + h(a_n))f(a_{n-1})g(a_{n-1})\|y_{n-1} - x_{n-1}\| \\ &\leq \cdots \leq (1 + h(a_n))\left[\prod_{j=0}^{n-1} f(a_j)g(a_j)\right]\|y_0 - x_0\|,\end{aligned} \tag{7}$$

by $(V_n)$ and $(II_n)$.

Then, from (7),

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\leq (1 + h(a_{n+m-1}))\eta \prod_{j=0}^{n+m-2} f(a_j)g(a_j) + (1 + h(a_{n+m-2}))\eta \prod_{j=0}^{n+m-3} f(a_j)g(a_j) \\ &\quad + \cdots + (1 + h(a_n))\eta \prod_{j=0}^{n-1} f(a_j)g(a_j),\end{aligned}$$

and, as $h$ is increasing and $\{a_n\}$ is decreasing by Lemmas 1 and 2,

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq (1 + h(a_0))\eta \sum_{l=0}^{m-1}\left[\prod_{j=0}^{n+l-1} f(a_j)g(a_j)\right] \\ &\leq (1 + h(a_0))\eta \sum_{l=0}^{m-1} (f(a_0)g(a_0))^{l+n},\end{aligned}$$

since $f$ and $g$ are also increasing. So, by applying the partial sum of a geometric sequence,

$$\|x_{n+m} - x_n\| \leq (1 + h(a_0))\frac{1 - (f(a_0)g(a_0))^m}{1 - f(a_0)g(a_0)}(f(a_0)g(a_0))^n \eta. \tag{8}$$

Then, we conclude that $\{x_n\}$ is a Cauchy sequence if $f(a_0)g(a_0) < 1$.

In order to prove that $x^*$ is a solution of $F(x) = 0$, we start with the bound of $\|F'(x_n)\|$,

$$\begin{aligned}\|F'(x_n)\| &\leq \|F'(x_0)\| + \|F'(x_n) - F'(x_0)\| \\ &\leq \|F'(x_0)\| + K\|x_n - x_0\| \\ &\leq \|F'(x_0)\| + KR\eta,\end{aligned} \tag{9}$$

by applying hypothesis $(iii)$ and Lemmas 1 and 2.

Then, by (7),

$$\begin{aligned}\|F(x_n)\| &\leq \|F'(x_n)\|\|y_n - x_n\| \\ &\leq \|F'(x_n)\| \left[\prod_{j=0}^{n-1} f(a_j)g(a_j)\eta\right],\end{aligned}$$

and, as $f$ and $g$ are increasing and $\{a_n\}$ is decreasing,

$$\|F(x_n)\| \leq \|F'(x_n)\|(f(a_0)g(a_0))^n\eta.$$

Since $\|F'(x_n)\|$ is bounded (see (9)), and $(f(a_0)g(a_0))^n$ tends to zero when $n \to \infty$, we conclude that $\|F(x_n)\| \to 0$. By continuity of $F$ in $\Omega$, $F(x^*) = 0$.

Let us observe that, if $a_0 \in (0, 0.2931...)$, $\frac{2}{K\beta} - R\eta > 0$. So, we are going to prove the uniqueness of $x^*$ in $B\left(x_0, \frac{2}{K\beta} - R\eta\right) \cap \Omega$. Let us assume that $y^*$ is a solution of $F(x) = 0$ in $B\left(x_0, \frac{2}{K\beta} - R\eta\right) \cap \Omega$. Then, in order to prove that $y^* = x^*$, and taking into account the Taylor expansion

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*),$$

it is necessary to show that the operator $\int_0^1 F'(x^* + t(y^* - x^*))dt$ is invertible. So, by applying hypothesis (iii),

$$\begin{aligned}\|\Gamma_0\| \int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\|dt &\leq K\beta \int_0^1 \|x^* + t(y^* - x^*) - x_0\|dt \\ &\leq K\beta \int_0^1 \left((1-t)\|x^* - x_0\| + t\|y^* - x_0\|\right) dt \\ &< \frac{K\beta}{2}\left(R\eta + \frac{2}{K\beta} - R\eta\right) = 1,\end{aligned}$$

by the Banach Lemma, the integral operator is invertible and hence $y^* = x^*$. □

**Remark.** Let us remark that if $0.2794... < a_0 < 0.2931...$, the radius of the existence ball is greater than the one of the uniqueness. However, if we choose as an strict upper bound of $a_0$ the root of the equation $R(a_0) - \frac{1}{a_0} = 0$, it is, $a_0 < 0.2794\ldots$, then we can establish an analogous result to Theorem 1, but in this case $R\eta < \frac{2}{K\beta} - R\eta$.

## 4. A study of the efficiency of iterative method M5

In this section, we study the efficiency of iterative method M5, such that we consider this situation $X = Y = \mathbb{R}^m$.

Notice that method M5, as has been proved in the previous theorem, has a high-order of convergence. Nevertheless, it is not the only advantage of the scheme: the number of evaluations of the nonlinear function $F$ and its associated Jacobian matrix is much lower than the respective one of known methods. The most used tool to compare the efficiency of different iterative schemes is the efficiency index, defined by Ostrowski as $EI = p^{\frac{1}{d}}$, where $p$ is the order of convergence and $d$ is the total number of functional evaluations per iteration. The efficiency index of method (1) is $EI_{M5} = 5^{\frac{1}{m^2+3m}}$. We compare it (see Figure 1) with the index of classical Newton's method, $EI_N = 2^{\frac{1}{m^2+m}}$, but also with some high-order procedures, as fourth-order Jarratt's scheme, $EI_J = 4^{\frac{1}{2m^2+m}}$, whose iterative expression is

$$\left\{ \begin{array}{lcl} y_n & = & x_n - \Gamma_n F(x_n), \\ z_n & = & x_n + \frac{2}{3}(y_n - x_n), \\ x_{n+1} & = & x_n - [6F'(z_n) - 2F'(x_n)]^{-1}[3F'(z_n) + F'(x_n)]\Gamma_n F(x_n), \end{array} \right. \quad (10)$$

and the recent Wang-Kou-Gu's method (WKG) of order of convergence six, $EI_{WKG} = 6^{\frac{1}{2m^2+2m}}$, given by

$$\left\{ \begin{array}{lcl} y_n & = & x_n - \Gamma_n F(x_n), \\ z_n & = & x_n + \frac{2}{3}(y_n - x_n), \\ w_n & = & x_n - [6F'(z_n) - 2F'(x_n)]^{-1}[3F'(z_n) + F'(x_n)]\Gamma_n F(x_n), \\ x_{n+1} & = & w_n - \left[\frac{3}{2}F'(z_n)^{-1} - \frac{1}{2}\Gamma_n\right] F(w_n), \end{array} \right. \quad (11)$$

for different sizes of the system.

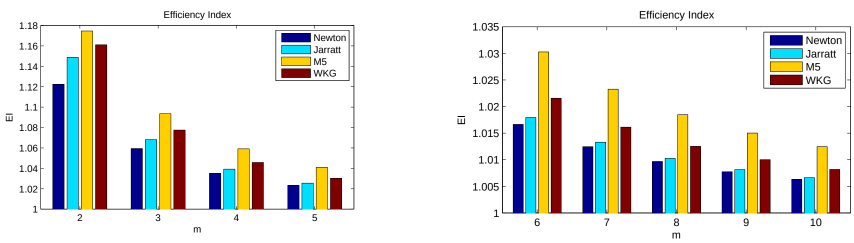


Figure 1: Efficiency indices for Newton, Jarratt, M5 and Wang's methods

In Figure 1 the efficiency indices for systems of size $m \leq 10$ can be observed. Let us remark that the best efficiency index is the one of M5. In a similar way, the same conclusion can be reached for higher sizes of the system.

However, it is interesting to note that in the case of non-linear systems of equations, the computational cost of evaluating operators $F$ and $F'$ is not similar, as it happens in the case of scalar equations. Therefore, this efficiency index is not a good efficiency measurement for an iterative process in the multivariate case. In any way, the efficiency index is not the only way to compare iterative schemes: the computational efficiency index introduced by Traub in [15], is also a useful tool. It is defined by $CE = p^{\frac{1}{op}}$, where *op* is the number of products and quotients per iteration. In the multidimensional case, it is very important to take into account the number of operations performed, since for each iteration a number of linear systems must be solved. We recall that the number of products/quotients that we need for solving a linear system of size $m \times m$, by using $LU$ factorization, is $\frac{m^3+3m^2-m}{3}$, that is, $\frac{m^3-m}{3}$ operations for $LU$ factorization and $m^2$ operations for solving the two triangular systems. In addition, the computational cost for solving $k$ systems with the same matrix of coefficient is only $\frac{m^3+3km^2-m}{3}$.

Therefore, the computational efficiency index of Newton's method:

$$\left\{ \begin{array}{rcl} F'(x_n)c_n & = & -F(x_n), \\ x_{n+1} & = & x_n + c_n \end{array} \right.$$

is $CE_N = 2^{\frac{3}{m^3+3m^2-m}}$, as one linear system must be solved.

Let us note that the iterative expression of Jarratt's scheme,

$$\left\{ \begin{array}{ccl} F'(x_n)c_n & = & -F(x_n), \\ y_n & = & x_n + c_n, \\ z_n & = & x_n + \frac{2}{3}c_n, \\ u_n & = & -\left[3F'(z_n) + F'(x_n)\right]c_n, \\ \left[6F'(z_n) - 2F'(x_n)\right]d_n & = & u_n, \\ x_{n+1} & = & x_n + d_n, \end{array} \right.$$

involves the solution of two different linear systems and one product matrix-vector (whose number of products/quotients is $m^2$). So its computational efficiency index is $CE_J = 4^{\frac{3}{2m^3+9m^2-2m}}$.

In order to apply the WKG method, four linear systems must to be solved, two of them with the same matrix of coefficients, and one product matrix-vector must be done.

$$\left\{ \begin{array}{ccl} F'(x_n)c_n & = & -F(x_n), \\ y_n & = & x_n + c_n, \\ z_n & = & x_n + \frac{2}{3}c_n, \\ u_n & = & -\left[3F'(z_n) + F'(x_n)\right]c_n, \\ \left[6F'(z_n) - 2F'(x_n)\right]d_n & = & u_n, \\ w_n & = & x_n + d_n, \\ F'(x_n)e_n & = & -F(w_n), \\ F'(z_n)f_n & = & -F(w_n), \\ x_{n+1} & = & w_n + \frac{3}{2}f_n - \frac{1}{2}e_n. \end{array} \right.$$

Then, its computational efficiency index is $CE_{WKG} = 6^{\frac{1}{m^3+5m^2-m}}$.

Finally, the computational efficiency index of M5

$$\left\{ \begin{array}{rcl} F'(x_n)c_n & = & -F(x_n), \\ y_n & = & x_n + c_n, \\ F'(x_n)d_n & = & -F(y_n), \\ z_n & = & y_n + 5d_n, \\ F'(x_n)e_n & = & -F(z_n), \\ x_{n+1} & = & z_n - \frac{16}{5}d_n + \frac{1}{5}e_n, \end{array} \right.$$

is $CE_{M5} = 5^{\frac{3}{m^3+9m^2-m}}$, as the three linear systems to be solved per iteration have the same matrix of coefficients.

All of them can be visually compared in Figure 2. It can be observed that classical Newton's method has the best computational efficiency index for $m = 2$, but for higher systems ($2 < m \leq 10$), M5 has better behavior than the rest of the analyzed methods. This fact holds for higher size of the system.

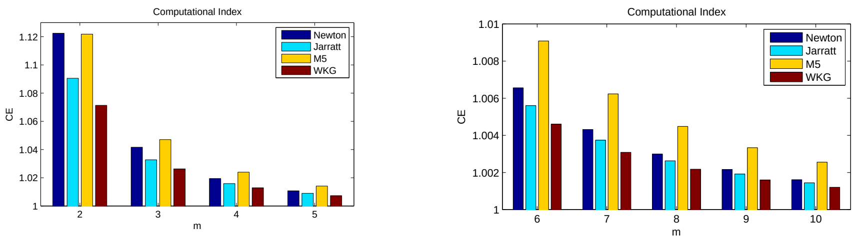


Figure 2: Computational efficiency indices for Newton, Jarratt, M5 and Wang's methods

## 5. Numerical results

In order to check the performance of the iterative methods presented above, we performe tests on nonlinear integral equation of mixed Hammerstein type. In particular, we consider the following nonlinear integral equation of mixed Hammerstein type

$$x(s) = 1 + \int_0^1 G(s,t)\, x(t)^2\, dt, \quad s \in [0,1], \tag{12}$$

where $x \in C[0,1]$, $t \in [0,1]$ and the kernel $G$ is $G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases}$

To solve (12), we transform it into a finite dimensional problem by using a process of discretization. For this, we approximate the integral that appears in (12) by the Gauss-Legendre formula

$$\int_0^1 h(t)\, dt \simeq \sum_{i=1}^{m} w_i h(t_i),$$

where the nodes $t_i$ and the weights $w_i$ are known.

If we denote the approximation of $x(t_i)$ by $x_i$ ($i = 1, 2, \ldots, m$), then (12) is equivalent to the following nonlinear system of equations

$$x_i = 1 + \sum_{j=1}^{m} a_{ij}\, x_j^2, \quad j = 1, 2, \ldots, m, \tag{13}$$

where

$$a_{ij} = \begin{cases} w_j t_j (1 - t_i) & \text{if } j \leq i, \\ w_j t_i (1 - t_j) & \text{if } j > i. \end{cases}$$

System (13) is now written as

$$F(\mathbf{x}) \equiv \mathbf{x} - \mathbf{1} - A\mathbf{v}_{\mathbf{x}} = 0, \qquad F : \mathbb{R}^m \longrightarrow \mathbb{R}^m, \tag{14}$$

where

$\mathbf{x} = (x_1, x_2, \ldots, x_m)^T,\ \mathbf{1} = (1, 1, \ldots, 1)^T,\ A = (a_{ij})_{i,j=1}^m,$ and $\mathbf{v}_{\mathbf{x}} = (x_1^2, x_2^2, \ldots, x_m^2)^T$.

Moreover, $F'(\mathbf{x}) = I - 2AD(\mathbf{x})$, where $D(\mathbf{x}) = \text{diag}\{x_1, x_2, \ldots, x_m\}$.

Usually these systems are solved by Newton's method, however we show that the application of the proposed method M5 is more favorable to apply the classical Newton method. To do this, we consider a combination of indexes considered previously, the efficiency index and the computational efficiency index. So, we consider another measure of the efficiency of an iterative process which takes into account both the operational cost of the evaluations that are required and the operational cost of doing an step of the algorithm. Notice that when the operator $F$ is known both operational costs can be computed.

Thus, we define the measure of the efficiency of an iterative process applied to an operator $F$ given as follows

$$E(method, F) = p^{1/(\mu+\sigma)},$$

where the operational cost of the evaluations that are required and the operational cost of doing an step of the algorithm are denoted by $\mu$ and $\sigma$, respectively. In these case the number of operations related to evaluate $F(x_n)$ and $F'(x_n)$ are $m^2 + m$ and $m^2$, respectively.

So, for solving nonlinear system (14), the particular cases of the Newton, Jarratt, M5 and WKG methods require $(m^3 + 9m^2 + 2m)/3$, $(2m^3 + 18m^2 + m)/3$, $(m^3 + 21m^2 + 8m)/3$ and $m^3 + 9m^2 + m$ operations per step, respectively.

Using this efficiency measure we can see in Figure 3, if $m > 2$, that method (1), with $E(M5, F) = 5^{\frac{3}{m^3+21m^2+8m}}$, is more efficient than the Newton, Jarratt and WKG methods, with $E(Newton, F) = 2^{\frac{3}{m^3+9m^2+2m}}$, $E(Jarrat, F) = 4^{\frac{3}{2m^3+18m^2+m}}$ and $E(WKG, F) = 6^{\frac{1}{m^3+9m^2+m}}$, respectively.

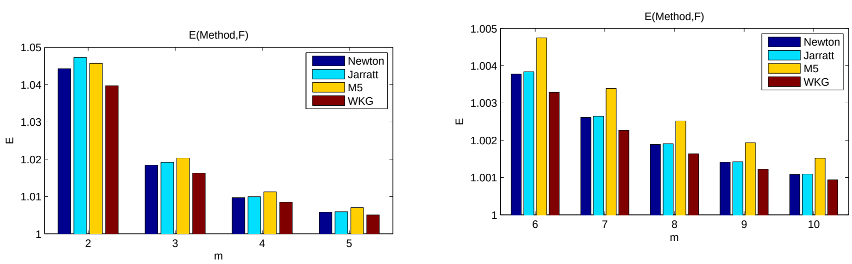


Figure 3: Efficiency of Newton, Jarratt, M5 and Wang's methods applied to $F$

We denote the $n$th iteration of (1) by $\mathbf{x}_n = (x_1^{(n)}, x_2^{(n)}, \ldots, x_m^{(n)})^T$. Choosing as starting point $\mathbf{x_0} = (1.6, 1.6, \ldots, 1.6)^T$, $m = 8$ and the max-norm, we obtain $K =$

$0.2471\ldots$, $\beta = 1.5969\ldots$, $\eta = 0.5911\ldots$, $a_0 = K\beta\eta = 0.2333\ldots < 0.2793\ldots$. In consequence, we can apply method (1) to solve system (13), since condition for $a_0$ given in Lemma 2 is satisfied. Moreover, by Theorem 1, the existence of the solution is guaranteed in $\overline{B(\mathbf{x_0}, 1.5165\ldots)}$ and the uniqueness in $B(\mathbf{x_0}, 3.5515\ldots)$.

By Theorem 1, method (1) is convergent and, after four iterations and using the stopping criterion $\|\mathbf{x_n}-\mathbf{x_{n-1}}\|_\infty < 10^{-180}$ or $\|\mathbf{f}(\mathbf{x_n})\|_\infty < 10^{-180}$, we obtain the numerical approximation $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_8^*)^T$ to the solution of (12) which is shown in Table 1.

| $i$ | $x_i^*$ | $i$ | $x_i^*$ | $i$ | $x_i^*$ | $i$ | $x_i^*$ |
|---|---|---|---|---|---|---|---|
| 1 | $1.01222\ldots$ | 3 | $1.11807\ldots$ | 5 | $1.15980\ldots$ | 7 | $1.05842\ldots$ |
| 2 | $1.05842\ldots$ | 4 | $1.15980\ldots$ | 6 | $1.11807\ldots$ | 8 | $1.01222\ldots$ |

Table 1: Numerical solution $\mathbf{x}^*$ of (13)

In Table 2, we show some computational aspects of the different mentioned methods applied to this problem with the same numerical settings. It can be observed that the best efficiency of M5 is translated as the lowest total number of operations and the highest index E.

| Method | Iter | Total $\sigma + \mu$ | Index E |
|---|---|---|---|
| Newton | 8 | 2944 | 1.000235 |
| Jarratt | 4 | 2912 | 1.000476 |
| Wang et al. | 3 | 3288 | 1.000545 |
| M5 | 4 | 2560 | 1.000629 |

Table 2: Numerical results for the different methods used

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