On a matrix group constructed from an 
\(\{R, s + 1, k\}\)-potent matrix

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June 12, 2016

Abstract

For a \(\{k\}\)-involutory matrix \(R \in \mathbb{C}^{n \times n}\) (that is, \(R^k = I_n\)) and \(s \in \{0, 1, 2, 3, \ldots\}\), a matrix \(A \in \mathbb{C}^{n \times n}\) is called \(\{R, s + 1, k\}\)-potent if \(A\) satisfies \(RA = A^{s+1}R\). In this paper, a matrix group corresponding to a fixed \(\{R, s + 1, k\}\)-potent matrix is explicitly constructed and properties of this group are derived and investigated. This constructed group is then reconciled with the classical matrix group \(G_A\) that is associated with a generalized group invertible matrix \(A\).

Keywords: \(\{R, s + 1, k\}\)-potent matrix; group inverse; matrix group.

AMS subject classification: Primary: 15A09; Secondary: 15A21

1 Introduction

For a matrix \(A \in \mathbb{C}^{n \times n}\), the group inverse, if it exists, is the unique matrix \(A^\#\) satisfying the matrix equations

\[
AA^\#A = A, \quad A^\#AA^\# = A^\#, \quad AA^\# = A^\#A.
\]

It is well known that \(A^\#\) exists if and only if rank \(A^2 = \text{rank } A\). Further information on group inverses and their applications can be found in [4], and a collection of results on the importance of group inverses of certain classes of singular matrices in several application areas can be found in the recent book [5]. Theorem 7.2.5 in [4] pp. 124] states that a

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square matrix $A$ of rank $r > 0$ belongs to a (multiplicative) matrix group $G_A$ if and only if rank $A^2 = \text{rank } A$. In this case, $A \in \mathbb{C}^{n \times n}$ has the canonical form

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1},$$

(2)

where $P \in \mathbb{C}^{n \times n}$ and $C \in \mathbb{C}^{r \times r}$ are nonsingular matrices. The matrix group $G_A$ corresponding to $A$ is then given by

$$G_A = \left\{ P \begin{bmatrix} X & O \\ O & O \end{bmatrix} P^{-1} : X \in \mathbb{C}^{r \times r}, \text{rank}(X) = r \right\}.$$  

(3)

The identity element in $G_A$ is

$$E = P \begin{bmatrix} I_r & O \\ O & O \end{bmatrix} P^{-1},$$

where $I_r \in \mathbb{C}^{r \times r}$ is the identity matrix, and the inverse of $A$ in this group is

$$A^g = P \begin{bmatrix} C^{-1} & O \\ O & O \end{bmatrix} P^{-1}.$$

Some results related to matrix groups on nonnegative matrices can be found in [1].

Note that the inverse $A^g$ of $A$ in $G_A$ satisfies the matrix equations in (1), and by uniqueness, $A^g = A^\#; the identity element $E$ in $G_A$ satisfies $E = AA^\# = A^\# A$.

For $p \in \{2, 3, \ldots\}$, a matrix $A$ is called $\{p\}$-group involutory if the group inverse of $A$ exists and satisfies $A^\# = A^{p-1}$; in such a case, an equivalent condition is that $A^{p+1} = A$ (see [2, 3]).

Throughout this paper we will use matrices $R \in \mathbb{C}^{n \times n}$ such that $R^k = I_n$ where $k \in \{2, 3, 4, \ldots\}$. These matrices $R$ are called $\{k\}$-involutory [11, 12, 14], and they generalize the well-studied involutory matrices ($k = 2$). Note that the definition given in [11, 12] differs from that in [14]; in this paper we adopt the definition given in [14], namely that $R$ is $\{k\}$-involutory does not require that $k$ be minimal with respect to $R^k = I$.

Let $R \in \mathbb{C}^{n \times n}$ be a $\{k\}$-involutory matrix and $s \in \{0, 1, 2, 3, \ldots\}$. A matrix $A \in \mathbb{C}^{n \times n}$ is called $\{R, s + 1, k\}$-potent if it satisfies

$$RA = A^{s+1} R.$$  

(4)

These matrices generalize centrosymmetric matrices (that is, matrices $A \in \mathbb{C}^{n \times n}$ such that $AJ = JA$ where $J$ is the $n \times n$ antidiagonal matrix; see [13]), the matrices $A \in \mathbb{C}^{n \times n}$ such that $AP = PA$ where $P$ is an $n \times n$ permutation matrix (see [10]), and $\{K, s + 1\}$-potent matrices (that is, matrices $A \in \mathbb{C}^{n \times n}$ for which $KAK = A^{s+1}$ where $K^2 = I_n$; see [7, 8]). For a study of $\{R, s + 1, k\}$-potent matrices we refer the reader to [6] where, in particular, the following characterization was given.

**Theorem 1.** [6, Theorem 1] Let $R \in \mathbb{C}^{n \times n}$ be a $\{k\}$-involutory matrix, $s \in \{1, 2, 3, \ldots\}$, $n_{s,k} = (s + 1)^k - 1$, and $A \in \mathbb{C}^{n \times n}$. Then the following conditions are equivalent:
(a) $A$ is $\{R, s+1, k\}$-potent.

(b) $A$ is an $\{n, k\}$-group involutory matrix and there exist disjoint projectors $P_0, P_1, \ldots, P_{n, k}$ with

$$
A = \sum_{j=1}^{n, k} \omega^j P_j \quad \text{and} \quad \sum_{j=0}^{n, k} P_j = I_n,
$$

where $\omega = e^{2\pi i n, k}$, and $P_j = O$ when $\omega^j \notin \sigma(A)$ and $P_0 = O$ when $0 \notin \sigma(A)$, and such that the projectors $P_0, P_1, \ldots, P_{n, k}$ satisfy

(i) For each $i \in \{1, \ldots, n, k - 1\}$, there exists a unique $j \in \{1, \ldots, n, k - 1\}$ such that $RP_i R^{-1} = P_j$,

(ii) $RP_{n, k} R^{-1} = P_{n, k}$, and

(iii) $RP_0 R^{-1} = P_0$.

(c) $A$ is diagonalizable and there exist disjoint projectors $P_0, P_1, \ldots, P_{n, k}$ satisfying conditions (i), (ii), and (iii) given in (b).

In [9], a matrix group constructed from a given $\{K, s+1\}$-potent matrix was presented and studied. The goal of this paper is to construct a matrix group corresponding to a given $\{R, s+1, k\}$-potent matrix. We then reconcile this constructed group with the matrix group $G_A$ given in [8].

2 First results

In this section we assume $s \geq 1$. We now establish properties of $\{R, s+1, k\}$-potent matrices.

Lemma 1. Suppose that $A \in \mathbb{C}^{n \times n}$ is an $\{R, s+1, k\}$-potent matrix. Then the following properties hold.

(a) $A^{(s+1)^k} = A$.

(b) $A^\# = A^{(s+1)^k-2}$ and the group projector $AA^\#$ satisfies $AA^\# = A^{(s+1)^k-1}$.

(c) $(A^{(s+1)^k-1})^j = A^{(s+1)^k-1}$ for every $j \in \{1, 2, 3, \ldots\}$.

(d) $R^p A^j = A^{j(s+1)^p} R^p$ for every $p \in \{1, 2, \ldots, k\}, j \in \{1, 2, \ldots, (s+1)^k - 1\}$. In particular, $R^p$ and $A^{(s+1)^k-1}$ commute, the matrices $A^j$ are $\{R, s+1, k\}$-potent and $A$ is $\{R^p, (s+1)^p - 1, k\}$-potent.

(e) $(A^j R^p)^m = A^{j[(s+1)^mp-1]/[(s+1)^p-1]} R^{mp}$, for every $j \in \{1, 2, \ldots, (s+1)^k - 1\}$, $p \in \{1, 2, \ldots, k\}$, $m \in \{1, 2, \ldots, k\}$. In particular,

\[ (A^j R)^m = A^{(s+1)m-1} R^m \text{ for every } m \in \{1, 2, \ldots, k\}. \]
For every \( j, \ell \in \{1, 2, \ldots, (s + 1)^k - 1\} \), \( p, m \in \{1, 2, \ldots, k\} \), \( (A^j R_p)(A^i R^m) = A^{j+i} R^{p+m} \), where \( \ell \equiv \ell (s + 1)^j + j \mod ((s + 1)^k - 1) \) and \( p \equiv p + m \mod (k) \).

**Proof.** Statements (a) and (b) were proved in [6]. Using (a),

\[
(A^{(s+1)^k-1})^2 = A^{(s+1)^k} A^{(s+1)^{k-2}} = AA^{(s+1)^{k-2}} = A^{(s+1)^{k-1}},
\]

and now (c) follows by induction.

We next prove (d). First note that

\[
RAR^{-1} = A^{s+1}
\]

implies \( RA^j R^{-1} = A^{j(s+1)} \), for all \( j \geq 1 \). Thus, if \( A \) is \( \{R, s + 1, k\}\)-potent then so is \( A^j \) for all \( j \geq 1 \). In particular, let \( j = s + 1 \). Then

\[
RA^{s+1} R^{-1} = A^{(s+1)^2},
\]

and (5) and (6) gives \( R^2 A R^{-2} = A^{(s+1)^2} \). By induction, \( R^p A R^{-p} = A^{j(s+1)^p} \) for all \( p \geq 1 \). Since for all \( j > 1 \), \( A^j \) is also \( \{R, s + 1, k\}\)-potent, it follows that \( R^p A^j R^{-p} = A^{j(s+1)^p} \) for all \( j \geq 1 \) and all \( p \geq 1 \). This proves (d).

For (e), the equality is clear for \( m = 1 \). For \( m = 2 \), we have

\[
(A^j R^p)^2 = A^j R^p A^j R^p = A^j A^{j(s+1)^p} R^{2p}, \text{ by (d)}
\]

\[= A^j A^{j(s+1)^{p+1}} R^{2p}.\]

The general case \( (A^j R^p)^m = A^j[1+(s+1)^p+(s+1)2^p+\ldots+(s+1)^{m-1}p]^m R^{mp} \) follows by induction. The identity \([((s + 1)^p - 1)((s + 1)^{(m-1)p} + \ldots + (s + 1)^{p} + 1)] = (s + 1)^{mp} - 1 \) yields the result.

For the proof of (e'), it is enough to set \( j = s \) and \( p = 1 \) in (e).

Statement (f) follows easily from (d). Next, by using (c) and (d),

\[
(A^j R^p) A^{(s+1)^k-1} = A^j A^{(s+1)^k-1} R^p = A^{j-1} A^{(s+1)^k} R^p = A^{j-1} A R^p = A^j R^p
\]

for every \( j \in \{1, 2, \ldots, (s + 1)^k - 1\} \) and \( p \in \{1, 2, \ldots, k\} \). This proves one equality in (g). The other equality can be directly shown as

\[
A^{(s+1)^k-1}(A^j R^p) = A^{(s+1)^k} A^{-1} R^p = A^j R^p.
\]
For the proof of (h), let \( j \in \{1, 2, \ldots, (s + 1)^k - 1\} \). By (d), there exists \( \ell \) such that \((A^\ell R^k)^p(A^\ell R^p) = A^{(s + 1)^k - 1}\) if and only if \( A^{\ell + j(s + 1)^k - p} = A^{(s + 1)^k - 1}\). This last equality holds if and only if \( \ell \equiv -j(s + 1)^k - p \mod ((s + 1)^k - 1)\). Using this value of \( \ell \) we can get
\[
\ell(s + 1)^p \equiv -j(s + 1)^k \mod ((s + 1)^k - 1).
\]

Now,
\[
(A^\ell R^p)(A^\ell R^{k-p}) = A^\ell A^{(s+1)^k} R^p R^{k-p} = A^{(s+1)^k} A^{\ell(s+1)^k} = A^{(s+1)^k + \ell(s+1)^p} = A^{(s+1)^k - 1},
\]
which leads to (h). Observe that \( \ell \equiv -j(s + 1)^k - p \mod ((s + 1)^k - 1)\) is equivalent to \( j(s + 1)^k \equiv -\ell(s + 1)^p \mod ((s + 1)^k - 1)\).

Finally, by setting \( j = p = 1 \) and \( m = k \) in (e), we obtain
\[
(AR)^{(s+1)^k+1} = [(AR)^k]^s\ AR = \left[A^{\frac{(s+1)^k-1}{s}}\right]^s\ AR = A^{(s+1)^k - 1}\ AR = AR,
\]
where the last equality follows from (a). This proves statement (i), and completes the proof of Lemma \( \square \)

3 Construction of the matrix group

Using Lemma \( \square \) we construct, from a given \( \{R, s + 1, k\}\)-potent matrix, a matrix group containing a cyclic subgroup of \( \{R, s + 1, k\}\)-potent matrices. Throughout this section we assume \( s \geq 1 \).

**Theorem 2.** Suppose \( A \in \mathbb{C}^{n \times n} \) is an \( \{R, s + 1, k\}\)-potent matrix, and assume that \( A^i \neq A^j \) for all distinct \( i, j \in \{1, 2, \ldots, (s + 1)^k - 1\} \). Then the set
\[
G = \{A^j R^p : j \in \{1, 2, \ldots, (s + 1)^k - 1\}, p \in \{1, 2, \ldots, k\}\}
\]
is a group under matrix multiplication, and the following statements hold.

(a) \( A \) is an element of order \( (s + 1)^k - 1 \), and the set
\[
S_A = \{A^j, j \in \{1, 2, \ldots, (s + 1)^k - 1\}\}
\]
is a cyclic subgroup of \( G \). Moreover, \( S_A \) is the smallest (in the inclusion sense) subgroup of \( G \) that contains \( A, A^\#, \) and \( AA^\# \).

(b) \( A^s R \) and \( A^{(s + 1)^k - 1} R^{k-1} \) are elements of order \( k \) of \( G \).

(c) \( (A^s R)A(A^s R)^{k-1} = A^{s+1} \).

(d) The set \( S_A \) is a normal subgroup of \( G \) and all its elements are \( \{R, s + 1, k\}\)-potent matrices.

(e) The order of \( G \) is \( k((s + 1)^k - 1) \) and \( G \) is not commutative.
Proof. Properties (f) − (h) in Lemma 1 show that $G$ is a group under multiplication with identity element $A^{(s+1)^k-1}$.

Statement (a) follows from properties (a) − (c) in Lemma 1 and the assumption that the powers $A^i$ are distinct for $i \in \{1, 2, \ldots, (s+1)^k - 1\}$.

By setting $m = k$ in property (e)' in Lemma 1 we obtain $(A^sR)^k = A^{(s+1)^k-1}$. On the other hand, since $A^{(s+1)^k-1}$ and $R^{k-1}$ commute by property (d) in Lemma 1

$$(A^{(s+1)^k-1}R^{k-1})^k = (A^{(s+1)^k-1})^k(R^{k-1}) = A^{(s+1)^k-1},$$

proving statement (b).

By setting $m = k - 1$ in property (e)' in Lemma 1 we obtain

$$(A^sR)A(A^sR)^{k-1} = A^sRA^{(s+1)^k-1}R^{k-1} = A^sA^{(s+1)^k-1(s+1)}RR^{k-1} = A^{s+1}.$$ 

proving statement (c).

For the proof of statement (d), let $j, t \in \{1, 2, \ldots, (s+1)^k - 1\}$, $p \in \{1, 2, \ldots, k\}$, and $\ell \in \{1, 2, \ldots, (s+1)^k - 1\}$ such that $j(s+1)^k \equiv -\ell(s+1)^p$ [mod $((s+1)^k - 1]$). Using property (d) of Lemma 1 we obtain

$$(A^\ell R^p)A^\ell A^{(s+1)^p}R^pA^\ell R^{k-p} = A^\ell A^{(s+1)^p}A^{(s+1)^p}R^pR^{k-p} = A^{(s+1)^p}.$$ 

Hence, $S_A$ is a normal subgroup of $G$, and by setting $p = 1$ in property (d) in Lemma 1 we find that the elements of $S_A$ are \{R, s + 1, k\}-potent matrices.

For the proof of statement (e), we show that the elements $A^jR^p$, $j \in \{1, \ldots, (s+1)^k - 1\}$ and $p \in \{1, \ldots, k\}$, are pairwise distinct.

First we show that for fixed $p \in \{1, \ldots, k - 1\}$, $AR^p \neq A^j$ for any $j \in \{1, \ldots, (s+1)^k - 1\}$. Otherwise, $AR^pA = A^{j+1}$, and using property (d) in Lemma 1 $A(R^pA) = A(A^{(s+1)^p}R^p) = A^{(s+1)^p+j}$. But then, $A^{j+1} = A^{(s+1)^p+j}$, contradicting the assumption that the powers $A^i$ are pairwise distinct for $i \in \{1, \ldots, (s+1)^k - 1\}$. Next, since for $p \in \{1, \ldots, k - 1\}$, $AR^p \neq A^j$ for any $j \in \{1, \ldots, (s+1)^k - 1\}$, it follows that for any $\ell \in \{1, 2, \ldots, (s+1)^k - 1\}$ and $p \in \{1, \ldots, k - 1\}$, $A^\ell R^p \neq A^j$ for any $j \in \{1, 2, \ldots, (s+1)^k - 1\}$. Finally, if $A^\ell R^p = A^\ell R^m$ for some $j, \ell \in \{1, 2, \ldots, (s+1)^k - 1\}$ and $p, m \in \{1, \ldots, k\}$ with $(j, p) \neq (\ell, m)$, then $A^\ell R^{p-m} = A^\ell$, contradicting the previous assertion. Thus, the elements $A^jR^p$, $j \in \{1, \ldots, (s+1)^k - 1\}$ and $p \in \{1, \ldots, k\}$, are pairwise distinct, and the order of $G$ is $k[(s+1)^k - 1]$. In order to show that $G$ is not commutative, it is enough to see that $(AR)(A^{s+1}R^{k-1}) = (A^{s+1}R^{k-1})(AR)$ gives $A^{(s+1)^2+1} = A^{(s+1)^{k-1}+s+1}$ which leads to a contradiction.

Theorem 3.1 (e) in [9] states that for a \{K, s + 1\}-potent matrix, the associated matrix group $G$ either has order $(s + 1)^2 - 1$ and is commutative, or has order $2((s + 1)^2 - 1)$ and is not commutative; Theorem 2 (e) now asserts that the former case does not occur.

We have shown that $A$, $A^\#$, and $AA^\#$ belong to $S_A$. Is $I_n - AA^\#$ also an element of the group $G$?
Proposition 1. If \( A \in \mathbb{C}^{n \times n} \) is a nonzero \( \{R, s + 1, k\}\)-potent matrix then the eigenprojection at zero does not belong to \( G \), that is, 
\[
I_n - AA^# \notin G.
\]

Proof. If we suppose that \( I_n - AA^# \in G \) then there exist \( j \in \{1, 2, \ldots, (s + 1)^k - 1\} \), \( p \in \{1, 2, \ldots, k\} \) such that \( I_n - AA^# = A^j R^p \). Pre-multiplying by \( A \) we get \( A^{j+1} = O \), that is, \( A \) is nilpotent. Since \( A \) is diagonalizable, we arrive at \( A = O \), which is a contradiction. \( \square \)

Let \( H \) be the set defined by
\[
H = \{ A^{(s+1)^k-1} R^p : p \in \{1, 2, \ldots, k\} \}.
\]

Then under matrix multiplication, \( H \) is a cyclic subgroup of \( G \) that is not normal because if \( g = A^{(s+1)^k-2} \) and \( h = A^{(s+1)^k-1} R^p \) for \( p \in \{1, 2, \ldots, k - 1\} \) then \( ghg^{-1} \notin H \).

Corollary 1. The group \( G \) is a semidirect product of \( H \) acting on \( S_A \).

Proof. Every element \( A^j R^p \) of \( G \) can be written as a product of an element of \( S_A \) and an element of \( H \) as \( A^j R^p = A^j (A^{(s+1)^k-1} R^p) \) and this representation is unique. This uniqueness follows from the fact that \( G \) has order \( k((s + 1)^k - 1) \). \( \square \)

Observe that \( H \simeq \mathbb{Z}_k \), \( S_A \simeq \mathbb{Z}_{(s+1)^k-1} \), and another way to see that \( G \) is isomorphic to a semidirect product of \( \mathbb{Z}_k \) acting on \( \mathbb{Z}_{(s+1)^k-1} \) is by considering its representation in the form \( \langle a, b | a^k = e, b^r = e, aba = b^m \rangle \) where \( m, r \) are coprime. Here \( r = (s + 1)^k - 1 \), \( a = A^s R \), \( b = A \), \( m = s + 1 \).

Moreover, notice that the result presented in Corollary 1 describes the quotient group \( G/S_A \). In fact, the natural embedding \( \iota : H \hookrightarrow G \), composed with the natural projection \( \pi : G \to G/S_A \), gives an isomorphism between \( G/S_A \) and \( H \), which is represented in (8).

\[
\begin{array}{ccc}
G & \xrightarrow{\pi} & G/S_A \\
\downarrow{\iota} & & \downarrow{g} \\
H & & 
\end{array}
\]

We next reconcile the matrix group \( G \) given in Theorem 2 that is constructed from an \( \{R, s + 1, k\}\)-potent matrix \( A \), and the matrix group \( G_A \) given in (3). We begin with the following lemma.

Lemma 2. Suppose that \( R \in \mathbb{C}^{n \times n} \) is \( \{k\}\)-involutory, \( s \in \{1, 2, 3, \ldots\} \), and \( A \in \mathbb{C}^{n \times n} \) has rank \( r > 0 \). Then \( A \) is \( \{R, s + 1, k\}\)-potent if and only if there exists a nonsingular matrix \( P \in \mathbb{C}^{n \times n} \) such that
\[
A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}, \quad R = P \begin{bmatrix} R_1 & O \\ O & R_2 \end{bmatrix} P^{-1},
\]
where \( R_1 \in \mathbb{C}^{r \times r} \), \( R_2 \in \mathbb{C}^{(n-r) \times (n-r)} \) are \( \{k\}\)-involutory, and \( C \in \mathbb{C}^{r \times r} \) is nonsingular and \( \{R_1, s + 1, k\}\)-potent.
Proof. Suppose that $A$ is $\{R, s+1, k\}$-potent. Then $A$ has index at most 1 and so it has the form

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1},$$

(10)

where $C \in \mathbb{C}^{r \times r}$ is nonsingular. We now partition $R$ conformable to $A$ as follows

$$R = P \begin{bmatrix} R_1 & R_3 \\ R_4 & R_2 \end{bmatrix} P^{-1}.$$  

(11)

Using expressions (10) and (11) we have that

$$A^{s+1}R = P \begin{bmatrix} C^{s+1}R_1 & C^{s+1}R_3 \\ O & O \end{bmatrix} P^{-1}$$

and

$$RA = P \begin{bmatrix} R_1C & O \\ R_4C & O \end{bmatrix} P^{-1}.$$  

Equating blocks,

$$C^{s+1}R_1 = R_1C, \quad C^{s+1}R_3 = O, \quad \text{and} \quad R_4C = O.$$  

Since $C$ is nonsingular, $R_3 = O$, $R_4 = O$, and so

$$R = P \begin{bmatrix} R_1 & O \\ O & R_2 \end{bmatrix} P^{-1}.$$  

Using $R^k = I_n$, this last expression implies that $R_1$ and $R_2$ are both $\{k\}$-involutory. Hence, $C$ is $\{R_1, s+1, k\}$-potent.

The converse is trivial. \qed

Recall that the elements of $G_A$ have a canonical form as given in (3).

**Theorem 3.** Suppose $A \in \mathbb{C}^{n \times n}$ is an $\{R, s+1, k\}$-potent matrix, and suppose that $A^i \neq A^j$ for all pairwise distinct $i, j \in \{1, 2, \ldots, (s+1)^k - 1\}$. If $A$ and $R$ are expressed as in (9) then

$$G = \left\{ P \begin{bmatrix} C^jR_1^p & O \\ O & O \end{bmatrix} P^{-1} : j \in \{1, 2, \ldots, (s+1)^k - 1\}, p \in \{1, 2, \ldots, k\} \right\}.$$  

Moreover, $G$ is a subgroup of $G_A$.

Proof. The description of the elements of $G$ follows from Theorem 2 and Lemma 2. It is clear that $G \subseteq G_A$. Since $C$ is $\{R_1, s+1, k\}$-potent, $G$ is closed, hence $G$ is a subgroup of $G_A$. \qed
4 Final remarks: the case \( s = 0 \)

For the case \( s = 0 \) in (1), the matrix \( A \) satisfies \( AR = RA \) where \( R^k = I_n \). Notice that property (ii) in Lemma 1 does not give any information. However, if there exists some positive integer \( t \) such that \( A^{t+1} = A \) and \( t \) is the smallest positive integer satisfying this property, then we can construct the group \( G = \{ A^j R^p, j \in \{1,2,\ldots,t\}, p \in \{1,2,\ldots,k\} \} \) having similar properties as in the case \( s \geq 1 \). If such an integer \( t \) does not exist, it is impossible to construct the corresponding group, as the following example shows.

Example 1. Consider the matrices

\[
A = \begin{bmatrix}
\cos(\alpha) & \sin(\alpha) & 0 \\
-\sin(\alpha) & \cos(\alpha) & 0 \\
0 & 0 & 2
\end{bmatrix}
\text{ and } R = \begin{bmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

for some \( \alpha \in \mathbb{R} \), we have that \( R^4 = I_3 \), \( AR = RA \) and

\[
A^m = \begin{bmatrix}
\cos(m\alpha) & \sin(m\alpha) & 0 \\
-\sin(m\alpha) & \cos(m\alpha) & 0 \\
0 & 0 & 2^m
\end{bmatrix}
\text{ for all } m \geq 2.
\]

In general, when \( s = 0 \) there is no relation between the existence of the group inverse of \( A \) and of \( A \) being \( \{R, 1, k\} \)-potent. In Example 1 we have a \( \{R, 1, 4\} \)-potent matrix that is nonsingular whereas in Example 2 below the given \( \{R, 1, 4\} \)-potent matrix does not have a group inverse.

Example 2. Consider the matrices

\[
A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\text{ and } R = \begin{bmatrix}
i & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

In this case, \( AR = RA \), \( R^4 = I_3 \), but the group inverse of \( A \) does not exist.

References


