Geometric Properties and Continuity of the Pre-duality Mapping in Banach Space

Z.H. Zhang[*], V. Montesinos[†], C.Y. Liu[‡], and W.Z. Gong[§]

Abstract

We use the preduality mapping in proving characterizations of some geometric properties of Banach spaces. In particular, those include nearly strongly convexity, nearly uniform convexity —a property introduced by K. Goebel and T. Sekowski—, and nearly very convexity.

1 Introduction

Let $(X, \|\cdot\|)$ be a Banach space. The *duality mapping* $D : S_X \to \mathcal{P}(X^*)$, where S_{X^*} is the unit sphere and $\mathcal{P}(X^*)$ is the family of all subsets of X^*, is a multivalued mapping defined by $D(x) := \{f \in S_{X^*} : \ f(x) = 1\}$ for $x \in S_X$. Geometric properties of the norm have been characterized in terms of the continuity of this mapping or of its selectors. For example, the norm $\|\cdot\|$ is Gâteaux (Fréchet) differentiable at $x_0 \in S_X$ if and only if every selector of the duality mapping is norm-weak*-continuous (respectively, norm-norm-continuous) at x_0 (see, e.g., [**?**, Theorems II§1.1 and II§2.1]; see also [**?**]).

The purpose of this note is to complete some previous results by characterizing some other geometric properties of Banach spaces, this time in terms of the so-called *pre-duality mapping*. This multivalued mapping, denoted by D^{-1} for obvious reasons, sends $f \in S_0(X^*)$ (where $S_0(X^*)$ denotes the subset of S_{X^*} consisting of all functionals that attain their supremum on B_X) to the set $\{x \in S_X : \ f(x) = 1\}$. Predecessors of the results here, formulated in terms of the duality mapping are, e.g., in [**?**], and of the pre-duality mapping $D^{-1} : S_0(X^*) \to S_X$, in [**?**].

[*]College of Fundamental Studies, Shanghai University of Engineering Science, 201600 Shanghai, PR China. E-mail: zhz@sues.edu.cn

[†]Instituto de Matemática Pura y Aplicada, Universitat Politècnica de València, C/Vera, s/n, 46022 València, Spain. E-mail: vmontesinos@mat.upv.es.

[‡]College of Fundamental Studies, Shanghai University of Engineering Science, 201600 Shanghai, PR China. E-mail: cyl@sues.edu.cn

[§]Department of Mathematics, Anhui Normal University, 241003 Wuhu, PR China. E-mail gongwanzhong@shu.edu.cn

2010 *Mathematics Subject Classification*: Primary 46B20; Secondary 46E30.

Key words and phrases: Duality mapping; Pre-duality mapping; α-upper semi-continuity; Usco mapping; Nearly strongly convex space; Nearly uniformly convex space; Nearly very convex space.

The geometric concepts we deal with are presented below. They have been used in fixed point theory, approximation theory, and in other branches of the general and applied theory of Banach spaces. References will be provided in due term.

The techniques used here range from the proof of the Brøndsted-Rockafellar theorem to James' reflexivity theorem, among some others related to compactness and to convexity. The terminology and notation follow standard texts in Banach space theory, e.g., [**?**]. Concepts and symbols not defined there will be introduced along the text.

2 Definitions, notation, and an earlier result

We collect here some definitions that will be used along this note.

As usual, B_X (S_X) denotes the closed unit ball (the unit sphere, respectively) of a Banach space $(X, \|\cdot\|)$.

In [**?**] (see also [**?**]), S. Rolewicz introduced a property of the norm of a Banach space related to the so-called drop property (see op. cit. for definitions). He called this property α, since it was formulated in terms of the Kuratowski index α of non-compactness. This last concept fits naturally in the metric space context: If M is a metric space and B is a bounded subset of M, the *Kuratowski index of non-compactness* $\alpha(B)$ *of* B is

$$\alpha(B) := \inf\{r > 0 : B \text{ can be covered by a finite family of sets of diameter less than } r\}.$$

If $f \in S_{X^*}$ and $0 < \delta < 1$, the set $S(B_X, f, \delta) := \{x \in B_X : f(x) > 1 - \delta\}$ is called a *slice of* B_X *determined by* f. Rolewicz says that a Banach space X has *property* α *at* $f \in S_{X^*}$ whenever

$$\lim_{\delta \to 0^+} \alpha(S(B_X, f, \delta)) = 0,$$

and that X has *property* α whenever it has property α at every $f \in S_{X^*}$. It was proved in [**?**] that *every space with the drop property has property* α, and in [**?**] that *both properties are indeed equivalent*. A uniform version of property α follows: If for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\alpha(S(B_X, f, \delta)) < \varepsilon$ for all $f \in S_{X^*}$, then X is said to have *property* U_α. It is known [**?**] that *a Banach space X is NUC if and only if it has property* U_α. For some other references on *NUC* spaces see also [**?**], [**?**], [**?**], and references therein.

We list below some definitions concerning continuity of set-valued mappings.

Definition 2.1. Let Φ be a set-valued mapping from a topological space X into a topological space Y.

(i) Φ is said to be *upper semicontinuous at* $x \in X$ if for any open set N in Y containing $\Phi(x)$, there exists an open neighborhood U of x in X such that $\Phi(U) \subset N$. Further, Φ is said to be *usco at* $x \in X$ if Φ is upper semicontinuous at $x \in X$ and the set $\Phi(x)$ is nonempty and compact. The mapping Φ is said to be *usco* if it is usco at every point $x \in X$.

(ii) If Y is a metric space, the mapping Φ is said to be *α-upper semicontinuous at* $x \in X$ if for any $\varepsilon > 0$, there exists an open neighborhood U of x such that $\alpha(\Phi(U)) < \varepsilon$.

Assume now that X and Y are normed spaces. The following are some uniform geometric concepts used in this note.

(iii) Φ is said to be *nearly uniformly upper semicontinuous on* S_X if Φ is usco and for any $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $\{x_n\}$ is a sequence in S_X and $\operatorname{diam}(\{x_n\}) < \delta$, then $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which satisfies $\Phi(x_{n_i}) \subset \Phi(x_{n_j}) + \varepsilon B_Y$ for all $i, j \geq 1$.

(iv) Φ is said to be *uniformly α-upper semicontinuous on* S_X if for any $\varepsilon > 0$, there exists $\delta\ (= \delta(\varepsilon)) \in (0,1)$ such that $\alpha(\Phi(B(x,\delta))) < \varepsilon$ for all $x \in S_X$.

The following definition collects several geometric properties of a Banach space that will be used below. They have been studied, e.g., in [**?**, **?**, **?**, **?**, **?**, **?**]. In particular, [**?**, **?**, **?**, **?**] contain applications to approximation theory.

Definition 2.2 ([**?**]). A Banach space X is said to be *strongly convex* (*very convex* / *nearly strongly convex* / *nearly very convex*) if for any $x \in S_X$ and any sequence $\{x_n\}$ in B_X such that for some $x^* \in D(x)$ we have $x^*(x_n) \to 1$ as $n \to \infty$, then $x_n \to x$ as $n \to \infty$ (respectively $x_n \xrightarrow{w} x$ as $n \to \infty$ / the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact / the set $\{x_n : n \in \mathbb{N}\}$ is weakly relatively compact).

By [**?**, **?**] it is known that if a Banach space X is strongly convex, then X is midpoint locally uniformly rotund and has the Kadec–Klee property, and also that if a Banach space X is very convex, then X is weakly midpoint locally uniformly rotund (for definitions, see op. cit.). It is clear that strongly convex $\Rightarrow$ nearly strongly convex $\Rightarrow$ nearly very convex, and that strongly convex $\Rightarrow$ very convex $\Rightarrow$ nearly very convex. None of the implications can be reversed (see [**?**, Examples 2.4, 2.5, and 2.7]).

The following result characterizes those geometric properties in terms of continuity properties of the duality mapping.

Theorem 2.3 ([**?**]). *Let X be a Banach space. Then*

(i) *X is nearly strongly convex if and only if D^{-1} is norm-norm-upper semicontinuous on $S_0(X^*)$ with norm-compact images.*

(ii) *X is nearly very convex if and only if D^{-1} is norm-weak-upper semicontinuous on $S_0(X^*)$ with weakly compact images.*

(iii) *X is strongly convex if and only if D^{-1} is norm-norm-continuous on $S_0(X^*)$ and single-valued.*

(iv) *X is very convex if and only if D^{-1} is norm-weak-continuous on $S_0(X^*)$ and single-valued.*

3 The Results

3.1 Nearly strong convexity

The following lemma is a straightforward consequence of the Brøndsted-Rockafellar Theorem (see, e.g., [?, Theorem 3.17]). It will be used in the proof of Theorem **??** below.

Lemma 3.1. *Suppose $\varepsilon > 0$, $x_0 \in S_X$, $f_0 \in S_{X^*}$, and $f_0(x_0) > 1 - \varepsilon$. Then there are $x_\varepsilon \in S_X$ and $f_\varepsilon \in D(x_\varepsilon)$ such that*

$$\|x_\varepsilon - x_0\| < 2\sqrt{\varepsilon}, \quad \|f_\varepsilon - f_0\| < 2\sqrt{\varepsilon}.$$

Theorem **??** below is an extension, in a localized setting, of characterization (i) of nearly strong convexity in Theorem **??**. Observe that a multi-valued mapping from a normed space into another normed space that at some point is norm-upper semicontinuous and has a compact value, is norm-α-upper semicontinuous at that point.

In its proof we shall need the following simple lemma:

Lemma 3.2. *Given a non-empty subset A of a Banach space X, $n \in \mathbb{N}$, $I_n := \{1, 2, \dots, n\}$, and a subset $\{f_i : i \in I_n\}$ of X^*, and real numbers $\alpha_1, \alpha_2, \dots, \alpha_m$,*

$$\begin{aligned} &\{x^{**} \in \overline{A}^{w^*} : f_i(x^{**}) > \alpha_i : i \in I_n\} \\ &\quad \subset \overline{\{x \in A : f_i(x) > \alpha_i : i \in I_n\}}^{w^*} &(3.1)\\ &\quad \subset \{x^{**} \in \overline{A}^{w^*} : f_i(x^{**}) \geq \alpha_i : i \in I_n\}, &(3.2)\end{aligned}$$

*and the second inclusion is an equality if A is convex and the intermediate set in (**??**) is non-empty.*

Proof. The two inclusions are almost obvious. We shall prove then the last statement. Assume that A is convex and that there exists $a_0 \in A$ such that $f_i(a_0) > \alpha_i$ for $i \in I_n$. Find $\varepsilon > 0$ such that $f_i(a_0) - \varepsilon > \alpha_i$ for $i \in I_n$. Put $H := \{a \in A :$ there exists $i \in I_n$ such that $f_i(a) = \alpha_i\}$ (hence $A \setminus H = \{a \in A : f_i(a) \neq \alpha_i,$ for all $i \in I_n\}$. We claim that $\overline{A}^{w^*} = \overline{A \setminus H}^{w^*}$. Indeed, if $a \in H$, we can find a sequence $\{a_m\}$ in the segment $[a_0, a)$ such that $a_m \in A \setminus H$ for every $m \in \mathbb{N}$, and $a_m \to a$ (in the norm and in the weak topology). It follows that $A \subset \overline{A \setminus H}^{w^*}$, so $\overline{A}^{w^*} \subset \overline{A \setminus H}^{w^*}$, and the claim is proved.

Fix now $x^{**} \in \overline{A}^{w^*}$ such that $f_i(x^{**}) \geq \alpha_i$ for all $i \in I_n$. Put $P := \{i \in I_n : f_i(x^{**}) = \alpha_i\}$. The previous claim allows us to build a net $\{x_j : j \in J, \leq\}$ in $A \setminus H$ (hence $f_i(x_j) \neq \alpha_i$ for $i \in I_n$ and $j \in J$) that weak*-converges to x^{**}. In particular, $f_i(x_j) \to_j f_i(x^{**})$ for $i \in I_n$. Thus, without loss of generality we may assume that, for all $j \in J$, we have $f_i(x_j) > \alpha_i$ for all $i \in I_n \setminus P$ and $f_i(a_0) - f_i(x_j) > \varepsilon$ for all $i \in P$. Note that $f_i(x_j) \neq \alpha_i$ $(= f_i(x^{**}))$ for each $i \in P$ and $j \in J$. Choose

$$\lambda_j \in \left(\max\left\{ \frac{|\alpha_i - f_i(x_j)|}{f_i(a_0) - f_i(x_j)},\ i \in P \right\}, \min\left\{ \frac{2|\alpha_i - f_i(x_j)|}{f_i(a_0) - f_i(x_j)},\ i \in P \right\} \right), \quad j \in J,$$

and put $c_j := \lambda_j a_0 + (1-\lambda_j)x_j$ for all $j \in J$. It is simple to see that the following properties hold:

(i) $c_j \in A$ for all $j \in J$.

(ii) The net $\{\lambda_j : j \in J, \leq\}$ converges to 0.

(iii) $f_i(c_j) > \alpha_i$ for all $i \in I_n$ and $j \in J$.

Thus, the net $\{c_j : j \in J, \leq\}$ is in $\{a \in A : f_i(a) > \alpha_i, i \in I_n\}$ and weak*-converges to x^{**}. This proves the statement. □

Theorem 3.3. *Let X be a Banach space, and let $f_0 \in S_0(X^*)$. Then the following are equivalent.*

(i) *For any sequence $\{x_n\}$ in S_X with $f_0(x_n) \to 1$, the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact (i.e., X is nearly strongly convex at f_0).*

(ii) *D^{-1} is norm-upper semi continuous at f_0, and $D^{-1}(f_0)$ is norm-compact.*

(iii) *X has property α at f_0.*

(iv) *D^{-1} is norm-α-upper semicontinuous at f_0.*

(v) *For any sequence $\{F_n\}$ in $S_{X^{**}}$ such that $F_n(f_0) \to 1$ as $n \to \infty$, the set $\{F_n : n \in \mathbb{N}\}$ is norm-relatively compact.*

Proof We shall proceed in the following way:

$$\text{(i)} \Leftarrow \text{(v)} \Leftarrow \text{(iii)} \Rightarrow \text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Leftrightarrow \text{(iv)}$$

(iii)⇒(v). Assume that X has property α at f_0. Then, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\alpha(S(B_X, f_0, \delta)) < \varepsilon$. This implies that $S(B_X, f_0, \delta)$ can be covered by a finite number of closed balls $B(x_i, \varepsilon)$, $i = 1, 2, \cdots, n$. Use inclusion (**??**) in Lemma **??** above to get

$$S(B_{X^{**}}, f_0, \delta) \subset \bigcup_{i=1}^{n} \overline{B(x_i, \varepsilon)}^{w^*}. \tag{3.3}$$

Since the weak*-closure of a ball in X is a ball in X^{**} with the same center and the same radius, from (**??**) we get that $\{F_n : n \in \mathbb{N}\}$ is relatively compact if $\{F_n\}$ is a sequence as in (v).

(v)⇒(i) is trivial.

(iii)⇒(i) Assume that X has property α at f_0. Let $\{x_n\}$ be a sequence as in (i). Since $\alpha(S(B_X, f_0, \delta)) \to 0$ as $\delta \to 0$, we get that the set $\{x_n : n \in \mathbb{N}\}$ is relatively compact.

(i)⇒(ii) Assume that D^{-1} is not norm-upper semicontinuous at f_0. Then there exists a norm-open subset U of X such that $D^{-1}(f_0) \subset U$ and for every norm-open (relatively to $S_0(X^*)$) subset V and containing f_0, we have $D^{-1}(V) \not\subset U$. This shows in particular the existence, for each $n \in \mathbb{N}$, of $f_n \in S_0(X^*)$ and $x_n \in D^{-1}(f_n)$ such that $\|f_n - f_0\| < 1/n$ and $x_n \notin U$. Since $f_0(x_n) = 1 + (f_0 - f_n)(x_n)$ for all $n \in \mathbb{N}$, we get $f_0(x_n) \to 1$ as $n \to \infty$, hence $\{x_n : n \in \mathbb{N}\}$ is a norm-relatively compact subset of X by (i). The sequence $\{x_n\}$

has a norm-convergent subsequence, say $\{x_{n_k}\}_{k=1}^{\infty}$. Let x_0 ($\in D^{-1}(f_0)$) be its limit. This contradicts that $x_n \notin U$ for each $n \in \mathbb{N}$.

That $D^{-1}(f_0)$ is norm-relatively compact follows from the very definition of the near strong convexity at f_0.

(ii)⇒(iii) This is true for any multivalued mapping between normed spaces, as we mentioned above. We spell out the details: Assume that D^{-1} is norm-upper semicontinuous at f_0 and $D^{-1}(f_0)$ is norm-compact. Fix $\varepsilon > 0$. Find a norm-open (relatively to $S_0(X^*)$) subset V of X^* such that $f_0 \in V$ and $D^{-1}(V) \subset D^{-1}(f_0) + (\varepsilon/2)B_X$. Since $D^{-1}(f_0)$ is norm-compact, we get that $\alpha(D^{-1}(V)) \le \varepsilon$.

(iii)⇒(iv) Assume X has property α at f_0. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $\alpha(S(B_X, f_0, \delta)) < \varepsilon$. Let $g \in S_0(X^*)$ be such that $\|f_0 - g\| < \delta$. Then, if $x \in D^{-1}(g)$ we have $f_0(x) = g(x) + f_0(x) - g(x) = 1 + (f_0(x) - g(x)) > 1 - \delta$, hence $D^{-1}(B(f_0, \delta) \cap S_0(X^*)) \subset S(B_X, f_0, \delta)$. Since $\alpha(S(B_X, f_0, \delta) < \varepsilon$, this proves the α-upper semi-continuity of D^{-1} at f_0.

(iv)⇒(iii) If D^{-1} is α-upper semicontinuous at f_0 then for all $\varepsilon > 0$ there exists $\delta > 0$ (and we may assume, without loss of generality, that $\delta < \varepsilon/2$) such that $\alpha(D^{-1}(B(f_0, \delta) \cap S_0(X^*))) < \varepsilon/2$. Take $\eta = \delta^2/16$ and $x_0 \in S(B_X, f_0, \eta)$. Then, by Lemma **??**, there exist $g \in B(f_0, 2\sqrt{\eta}) \cap S_0(X^*)$ and $x \in D^{-1}(g)$ such that $\|x_0 - x\| < 2\sqrt{\eta}$, hence $x \in D^{-1}(B(f_0, 2\sqrt{\eta}) \cap S_0(X^*))$. It follows that

$$S(B_X, f_0, \eta) \subset D^{-1}(B(f_0, 2\sqrt{\eta}) \cap S_0(X^*)) + 2\sqrt{\eta}B_X,$$

hence $\alpha(S(B_X, f_0, \eta)) < \varepsilon/2 + 4\sqrt{\eta} < \varepsilon$. □

Remark 3.4. If $f \in S(X^*)$ satisfies statement (i) in Theorem **??**, then f is called a *strong support functional* in [**?**]. It was proved there that (i) implies (ii) in the aforesaid theorem. Theorem **??** above shows that in fact (i) and (ii) are equivalent.

A useful result due to G. Choquet (see, e.g., [**?**, Lemma 3.69]) ensures that *if X is a Banach space and x is an extreme point of a convex weakly compact subset C of X, then every weak-neighborhood of x in B_X contains a slice that contains x.* We shall prove below that a similar result holds for the (extremal) subset $D^{-1}(f_0)$ of B_X. We stress that no compactness conditions are assumed on the closed unit ball of X. Since the result has two similar versions, one for the norm topology and the other for the weak topology, we think it better to unify the statement by using a topology τ, that stands for both. When τ is the norm topology, Theorem **??** completes Theorem **??**, while when τ is the weak topology, it provides a localized version of (ii) in Theorem **??**.

Theorem 3.5. *Let X be a Banach space, let $f_0 \in S_0(X^*)$, and let τ be either the norm or weak topology. Then, the pre-duality mapping D^{-1} is τ-usco at $f_0 \in S_0(X^*)$ if and only if the set $D^{-1}(f_0)$ is τ-compact and, simultaneously, for every τ-neighborhood N of the origin in X, the set $D^{-1}(f_0) + N$ contains a slice of B_X determined by f_0.*

Proof. Necessity. If D^{-1} is τ-upper semicontinuous at $f_0 \in S_0(X^*)$, then given a τ-neighborhood N of the origin, there exists $\delta' > 0$ such that

$$D^{-1}(B(f_0, \delta') \cap S_0(X^*)) \subset D^{-1}(f_0) + \frac{1}{2}N.$$

Choose $\delta \in (0, \delta')$ such that $\delta B_X \subset \frac{1}{2}N$. Given $x \in S(B_X, f_0, \delta^2/4)$, by Lemma **??** there exist $g_\delta \in B(f_0, \delta) \cap S_0(X^*)$ and $y_\delta \in D^{-1}(g_\delta)$, such that $\|x - y_\delta\| < \delta$. It follows that $x \in D^{-1}(B(f_0, \delta') \cap S_0(X^*)) + \delta B_X$, hence $S(B_X, f_0, \delta^2/4) \subset D^{-1}(f_0) + N$. Note that the τ-compactness of $D^{-1}(f_0)$ is guaranteed by the τ-usco character of D^{-1}.

Sufficiency. Assume that W is a τ-neighborhood of $D^{-1}(f_0)$. Consider the family of sets $\{D^{-1}(f_0) + N : \ N \in \mathcal{N}\}$, where $\mathcal{N}$ is the family of all τ-neighborhoods of the origin in X. Then $\bigcap_{N \in \mathcal{N}} D^{-1}(f_0) + N = D^{-1}(f_0) \subset W$. It follows from the τ-compactness of $D^{-1}(f_0)$ that there is $N \in \mathcal{N}$ such that $D^{-1}(f_0) + N \subset W$. We know that $D^{-1}(f_0) + N$ contains the slice $S(B_X, f_0, \delta)$ determined by f_0 for some δ. Then for all $g \in B(f_0, \delta) \cap S_0(X^*)$ and $x \in D^{-1}(g)$,

$$f_0(x) = g(x) + f_0(x) - g(x) \geq 1 - \|f_0 - g\| > 1 - \delta.$$

Therefore,

$$D^{-1}(B(f_0, \delta) \cap S_{X^*}) \subset S(B_X, f_0, \delta) \subset D^{-1}(f_0) + N \subset W.$$

This shows that D^{-1} is τ-upper semicontinuous at $f_0 \in S_0(X^*)$. □

The nearly strong convexity at some $f_0 \in S_0(X^*)$ is defined in terms of sequences contained in slices of B_X determined by f_0. It is not completely evident that in fact it implies something formally stronger, the existence of convergent *subnets* of *nets* in slices. This is the content of the next result. As a consequence, it will follow that nearly strongly convex Banach spaces have Kadec property (see Corollary **??**).

Theorem 3.6. *Let X be a Banach space, and let τ be either the norm or weak topology. Let $f_0 \in S_0(X^*)$. Then the pre-duality mapping D^{-1} is τ-usco at f_0 if and only if for every net $\{x_i : \ i \in I, \ \leq\}$ in S_X such that $f_0(x_i) \to 1$, there exists a τ-convergent subnet.*

Proof. Necessity. Since $f_0(x_i) \to 1$, for any $n \in \mathbb{N}$ we can choose i_n such that $f_0(x_i) > 1 - \frac{1}{n}$ for any $i \geq i_n$. Without loss of generality, we may assume that $i_1 \leq i_2 \leq \cdots \leq i_n \leq \cdots$

If there exists $j_0 \in I$ such that $j_0 \geq i_n$ for all $n \in \mathbb{N}$, then $f_0(x_i) = 1$ for every $i \geq j_0$. Therefore, $\{x_i : \ i \in I, \ i \geq j_0, \ \leq\} \subset D^{-1}(f_0)$. Due to the fact that $D^{-1}(f_0)$ is τ-compact, the subnet $\{x_i : \ i \in I, \ i \geq j_0, \ \leq\}$ has a further subnet that converges.

Further, assume that for every $i \in I$ there always exists some i_n such that $i_n \geq i$, then $\{x_{i_n} : n \in \mathbb{N}\}$ is a subnet of $\{x_i : \ i \in I, \ \leq\}$. Let $\mathcal{V}$ be a τ-open covering of $\{x_{i_n} : n \in \mathbb{N}\} \cup D^{-1}(f_0)$. Since $D^{-1}(f_0)$ is τ-compact, $\mathcal{V}$ has a finite subcovering $\{V_1, V_2, \cdots, V_m\}$ of $D^{-1}(f_0)$. Again by the τ-compactness of $D^{-1}(f_0)$, there exists a

τ-neighborhood N of the origin in X such that $D^{-1}(f_0) + N \subset \bigcup_{k=1}^{m} V_k$. By Theorem **??**, there exists $\delta > 0$ such that $S(B_X, f_0, \delta) \subset D^{-1}(f_0) + N$ $(\subset \bigcup_{k=1}^{m} V_k)$, and so $\{x_{i_n} : n \geq n_0\} \subset S(B_X, f_0, \delta)$ $(\subset \bigcup_{k=1}^{m} V_k)$ for some $n_0 \in \mathbb{N}$. This shows that $\{x_{i_n} : n \in \mathbb{N}\} \cup D^{-1}(f_0)$ is τ-compact. Consequently, $\{x_{i_n} : n \in \mathbb{N}\}$ has a τ-convergent subnet.

Sufficiency. By the assumption, $D^{-1}(f_0)$ is τ-compact. Assume that D^{-1} is not τ-usco on $S_0(X^*)$ at f_0. By Theorem **??**, there exists a τ-open neighborhood N of the origin in X such that no slice determined by f_0 is contained in the τ-open set $D^{-1}(f_0) + N$. We can define then a sequence $\{x_n\}$ in S_X such that $f_0(x_n) \to 1$ and $x_n \notin D^{-1}(f_0) + N$ for $n \in \mathbb{N}$. The sequence $\{x_n\}$ has a subnet that τ-converges to some $x \in X$. It follows that $x \notin D^{-1}(f_0) + N$; however, $f_0(x) = 1$, a contradiction. □

Corollary 3.7. *Let X be a Banach space, and let $f_0 \in S_0(X^*)$. If the pre-duality mapping D^{-1} is norm usco at f_0, then for any $x \in D^{-1}(f_0)$ and any net $\{x_i : i \in I, \leq\}$ in S_X, the condition $x_i \xrightarrow{w} x$ implies $x_i \xrightarrow{\|\cdot\|} x$.*

Recall that a Banach space X has the *Kadec property* whenever the weak and norm topologies coincide on S_X. From the equivalence (i)$\Leftrightarrow$(ii) in Theorem **??** and Corollary **??** we get the following result.

Corollary 3.8. *If X is a nearly strongly convex Banach space, then X has Kadec property.*

3.2 Nearly uniform convexity

Given a nonempty subset M of a Banach space X, $\operatorname{sep} M := \inf\{\|x-y\| : x, y \in M, x \neq y\}$ is called the *separation* of the set M. The following notion was introduced by R. Huff and, independently, by K. Goebel and T. Sekowski.

Definition 3.9 ([**?**], [**?**])**.** A Banach space X is said to be *nearly uniformly convex* (*NUC*, in short) if for any $\varepsilon > 0$ there exists $\delta \in (0,1)$ such that for every countable set C in B_X with $\operatorname{sep}(C) \geq \varepsilon$, we have $\operatorname{conv}(C) \cap B(0,\delta) \neq \emptyset$.

In [**?**], a characterization of the near uniform convexity of a Banach space was given in terms of the duality mapping on the space X^*, i.e., the mapping D from S_{X^*} into $S_{X^{**}}$. Precisely, the following result holds.

Theorem 3.10 ([**?**])**.** *For a Banach space X, the following statements are equivalent:*

(i) *X is NUC.*

(ii) *The duality mapping D from S_{X^*} into $S_{X^{**}}$ is nearly uniformly norm-upper semicontinuous on S_{X^*}.*

(iii) *The duality mapping D from S_{X^*} into $S_{X^{**}}$ is uniformly α-upper semicontinuous on S_{X^*}.*

Every NUC space is reflexive (an argument uses James' reflexivity theorem: Given $\varepsilon > 0$ find $\delta \in (0, 1/2)$ according to the definition of NUC. For $f \in S_{X^*}$, the slice $S(B_X, f, \delta)$ is disjoint from δB_X, hence no ε-separated sequence can be found in $S(B_X, f, \delta)$. Since $\varepsilon > 0$ is arbitrary, this shows that f attains its supremum on B_X). It turns out that the aforesaid characterization really deals with the preduality mapping. However, it is a priori somehow inconvenient to check near uniform convexity by looking at the second dual. We provide here a similar characterization, formulated right away in terms of the pre-duality mapping. It is based on Theorem **??** and the fact, proved below, that each of conditions (ii) or (iii) there already imply reflexivity.

Theorem 3.11. *For a Banach space X, the following statements are equivalent:*

(i) *X is NUC.*

(ii) *The pre-duality mapping D^{-1} is nearly uniformly norm-norm-upper semicontinuous on $S_0(X^*)$.*

(iii) *The pre-duality mapping D^{-1} is uniformly α-upper semicontinuous on $S_0(X^*)$.*

Proof. Since every NUC space is reflexive, $D^{-1}(f) = D(f)$ for every $f \in S_{X^*}$, and Theorem **??** shows (i)⇒(ii) and (i)⇒(iii).

(ii)⇒(i). Let us show that (ii) already implies reflexivity —and so Theorem **??** will finalize the proof of the implication. Let $f \in S_{X^*}$. For $n \in \mathbb{N}$, find $x_n \in S_X$ such that $f(x_n) > 1 - 1/n$. Lemma **??** shows the existence of $y_n \in S_X$ and $f_n \in D(y_n)$ such that $\|x_n - y_n\| < 2/\sqrt{n}$ and $\|f - f_n\| < 2/\sqrt{n}$. Given $\varepsilon > 0$, find $\delta > 0$ according to the nearly uniform norm-norm-semi-continuity of D^{-1}. Let $n_0 \in \mathbb{N}$ be such that $2/\sqrt{n} < \delta/2$ for all $n \geq n_0$. Since $\operatorname{diam}\{f_n : n \geq n_0\} < \delta$, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ that satisfies $D^{-1}(f_{n_k}) \subset D^{-1}(f_{n_j}) + \varepsilon B_X$ for all $k, j \in \mathbb{N}$. Observe that, for all $j \in \mathbb{N}$, the set $D^{-1}(f_{n_j})$ is compact. This works for all $\varepsilon > 0$. A diagonal procedure allows to select a subsequence $\{z_n\}$ of $\{y_n\}$ such that the set $\{z_n : n \in \mathbb{N}\}$ has the property that for every $\varepsilon > 0$ there exists a compact subset K_ε of X such that $\{z_n : n \in \mathbb{N}\} \subset K_\varepsilon + \varepsilon B_X$. This shows that $\{z_n : n \in \mathbb{N}\}$ is relatively compact, hence $\{z_n\}$ has a convergent subsequence. Therefore, $\{x_n\}$ has a convergent subsequence, too, and this shows that f attains it supremum on B_X. James' weak compactness theorem ensures that X is reflexive.

(iii)⇒(i). It is enough to show that (iii) already implies the reflexivity of X, as in the proof of (ii)⇒(i) above. The technique is similar to the one used there. □

3.3 Nearly very convexity

The following result gives a sufficient condition for a Banach space being nearly very convex. Note that D on S_{X^*} has set-values in $S_{X^{**}}$.

Proposition 3.12. *Let X be a Banach space. If $\dim D^{-1}(f) = \dim D(f) < +\infty$ for all $f \in S_0(X^*)$, then X is nearly very convex.*

Proof. Let $\{x_n\}$ be a sequence in B_X such that $f(x_n) \to 1$ for some $f \in S_0(X^*)$. Assume that the set $\{x_n : n \in \mathbb{N}\}$ is not relatively weakly compact, and let $x^{**} \in X^{**} \setminus X$ be a weak* cluster point of the set $\{x_n : n \in \mathbb{N}\}$. Then $x^{**} \in D(f)$, and so $\dim D(f) > \dim D^{-1}(f)$. □

Theorem 3.13. *Let X be a Banach space. Then, the pre-duality mapping D^{-1} is weak usco at $f \in S_0(X^*)$ if and only if $D(f) = D^{-1}(f)$.*

Proof. Necessity. Assume that D^{-1} is weak usco at f. Let $x^{**} \in D(f) \setminus D^{-1}(f)$. Due to the fact that $D^{-1}(f)$ is weakly compact, there exists a weak*-closed weak*-neighborhood N^{**} of $D^{-1}(f)$ such that $x^{**} \notin N^{**}$. The set $N^{**} \cap X$ is a weak neighborhood of $D^{-1}(f)$. An argument similar to the one used in the proof of sufficiency in Theorem **??** shows that there exists a weak neighborhood N of the origin in X such that $D^{-1}(f_0) + N \subset N^{**} \cap X$. By Theorem **??**, $D^{-1}(f_0) + N$ contains a slice $S(B_X, f, \delta)$. It follows from Lemma **??** and the fact that N^{**} is weak*-closed that $(x^{**} \in)\ S(B(X^{**}), f, \delta) \subset N^{**}$, a contradiction.

Conversely, if $D^{-1}(f) = D(f)$ the statement follows trivially, as D is always weak*-usco. □

By (i) in Theorem **??**, we have the following corollary:

Corollary 3.14. *A Banach space X is nearly very convex if and only if $D(f) = D^{-1}(f)$ for all $f \in S_0(X^*)$.*

Acknowledgements

We thank a referee for the careful reading of the manuscript. His/her observations substantially improved the overall aspect of the present work, detected several misprints and made some convenient changes.

This work was supported by

(1) The National Natural Science Foundation of China (grant no. 11271248).

(2) Specific Academic Discipline Project of Shanghai Municipal Education Commission (grant no. B-8932-13-0136).

(3) Project MTM2011-22417, Ministerio de Ciencia e Innovación, Spain (V. Montesinos).

References

[BHBT] P. Bandyopadhyay, D. Huang, B.L. Lin, and S.L. Troyanski, *Some generalizations of local uniform rotundity*, J. Math. Anal. Appl. **252** (2000), 906–916.

[BLLN] P. Bandyopadhyay, Y. Li, B. Lin, D. Narayana, *Proximinality in Banach Spaces*, J. Math. Anal. Appl. **341** (2008), 309-317.

[Dies] J. Diestel, *Geometry of Banach Spaces. Selected Topics*, LNM **485**, Springer, 1975.

[FHHMZ] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis*, CMS Books in Mathematics, Springer-Verlag, 2011.

[GGS] J.R. Giles, D.A. Gregory, and B. Sims, *Geometrical implications of upper semi-continuity of the duality mapping on a Banach space*, Pacific J. Math. **79** (1) (1978), 99–109.

[GoSe] K. Goebel and T. Sekowski, *The modulus of non-compact convexity*, Ann. Univ. M. Curie-Sklodowska, Sect. A, **38** (1984), 41–48.

[GM1] A.J. Guirao and V. Montesinos, *A note in approximative compactness and continuity of metric projections in Banach spaces*, J. Convex Anal. **18** (2011), 397–401.

[H] R. Huff, *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. **10** (4) (1980), 743–749.

[KR] D. Kutzarova and S. Rolewicz, *On nearly uniformly convex sets*, Arch. Math. **57** (1991), 385–394.

[KLZ] D. Kutzarova, B. L. Lin, and W. Zhang, *Some geometrical properties of Banach spaces related to nearly uniform convexity*, Contemp. Math, **144** (1993), 165–171.

[KP] D. Kutzarova and S. Prus, *Operators which factor through nearly uniformly convex spaces*, Boll. Un. Mat. Ital. B (7), **9** (1995), 2, 479–494.

[Mo] V. Montesinos, *Drop property equals reflexivity*, Studia Math. **87** (1987), 93–100.

[Ph] R.R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, 2nd Edition, LNM **1364**, Springer-Verlag, 1993.

[Rol1] S. Rolewicz, *On drop property*, Studia Math. **85** (1986), 27-37.

[Rol2] S. Rolewicz, *On Δ-uniform convexity and drop property.* Studia Math. **87** (1987), 181–191.

[WL] C.X. Wu and Y.J. Li, *Strong convexity in Banach spaces*, J. Math., Wuhan Univ. **13** (1) (1993), 105–108.

[WN] J.H. Wang and C.X. Nan, *The continuity of subdifferential mapping*, J. Math. Anal. Appl. **210** (1997), 206–214.

[WZ] J.H. Wang and Z.H. Zhang, *Characterization of the property (C-K)*, Acta Math. Sci. Ser. A Chin. Ed. **17** (A)(3)(1997), 280–284.

[ZL1] Z.H. Zhang and C.Y. Liu, *Some generalization of locally and weakly locally uniformly convex space*, Nonlinear Anal. **74** (12)(2011), 3896–3902.

[ZL2] Z.H. Zhang and C.Y. Liu, *Convexity and proximinality in Banach spaces*, J. Funct. Spaces Appl., Volume 2012, Article ID 724120, 11 pages, Doi:10.1155/2012/724120.

[ZL3] Z.H. Zhang and C.Y. Liu, *Convexity and existence of the farthest point*, Abstract and Applied Analysis Volume 2011, Article ID 139597, 9 pages Doi:10.1155/2011/139597.

[ZS] Z.H. Zhang and Z.R. Shi, *Convexities and approximative compactness and continuity of the metric projection in Banach spaces*, J. Approx. Theory **161** (2) (2009), 802–812.

[ZZ] Z.H. Zhang and C.J. Zhang, *On very rotund Banach spaces*, Appl. Math. Mech. (English Ed.) **21** (8) (2000), 965–970.