Convergence, efficiency and dynamics of new fourth and sixth order families of iterative methods for nonlinear systems

José L. Hueso[†,*], Eulalia Martínez[‡] and Carles Teruel[†]
[†] *Instituto Universitario de Matemática Multidisciplinar*
[‡] *Instituto Universitario de Matemática Pura y Aplicada*
[†‡] *Universitat Politécnica de Valéncia*

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Abstract

In this work we present a new family of iterative methods for solving nonlinear systems that are optimal in the sense of Kung and Traub's conjecture for the unidimensional case. We generalize this family by performing a new step in the iterative method, getting a new family with order of convergence six. We study the efficiency of these families for the multidimensional case by introducing a new term in the computational cost defined by Grau-Sánchez et al. A comparison with already known methods is done by studying the dynamics of these methods in an example system.

keywords: Nonlinear systems, iterative methods, convergence order, optimal methods, computational cost, efficiency, dynamics.

1 Introduction

Finding iterative methods with high order of convergence in order to approximate the solution of a nonlinear system $F(x) = 0$ is an active field in numerical analysis. Nowadays, the range of applications where it is required to use a high level of numerical precision is increasing.

In the scalar case, a recent publication, [1], makes an interesting compilation of multipoint iterative methods and analyzes their efficiency, accuracy and optimality.

Focusing on higher order iterative methods for the multidimensional case, we can mention, among others, some recently published works: [2]-[7], where,

*Corresponding author: José L. Hueso (jlhueso@mat.upv.es)

as we can see, different techniques can be applied in order to improve the computational cost and so the effectiveness of the procedures for approximating solutions of nonlinear systems.

In this work we generalize the technique used in [8], obtaining a new family of iterative methods with fourth order of convergence. The procedures used in [9, 6] for increasing the convergence order of an iterative method, that is, to perform another Newton's step avoiding the evaluation of the jacobian matrix in order to get the maximum efficiency, do not work for the optimal method introduced in [8], so we propose a new procedure to increase the order with a reasonable efficiency.

Obviously, performing a new step in an iterative method carries more function evaluations and so one has to check if the gain in convergence order justifies the increase of the computational cost. A thorough study of the cost and efficiency of iterative methods for nonlinear systems can be found in [5, 10]. Nevertheless, we introduce a new term in the cost expression to take into account matrix-vector operations that occur in some iterative methods such as the considered here.

The paper is organized as follows. New families of iterative methods, Jarratt's two point and three point methods, are obtained in section 2. In section 3, we analyze the computational efficiency for the new methods and in section 4 the new methods are applied in order to approximate the solutions of some nonlinear systems. Finally, section 5 studies the dynamics of these methods for a particular nonlinear system and section 5 is devoted to the conclusions.

2 New families of iterative methods

Our aim is to develop high order methods for nonlinear systems, motivated by the techniques exposed in section 2.6 of chapter 3 of [1] for obtaining multipoint iterative methods of Jarratt's type in the unidimensional case. We try to apply some of the ideas of [11] and [12] to the fourth order method recently published by Sharma et al. [8]. First of all, we generalize this technique by introducing a new term in their proposal, obtaining a new family of fourth order iterative methods.

That is, we consider the family of methods given by:

$$y_n = x_n - \theta\, \Gamma_{x_n} F(x_n) \tag{1}$$
$$H(x_n, y_n) = \Gamma_{x_n} F'(y_n) \tag{2}$$
$$G_s(x_n, y_n) = s_1 I + s_2 H(y_n, x_n) + s_3 H(x_n, y_n) + s_4 H(y_n, x_n)^2 \tag{3}$$
$$z_n = x_n - G_s(x_n, y_n)\Gamma_{x_n} F(x_n) \tag{4}$$
$$x_{n+1} = z_n \tag{5}$$

where $\Gamma_{x_n} = F'(x_n)^{-1}$, and θ, s_1, s_2, s_3, s_4 are constants that we determine in order to get a new family of fourth order optimal methods. Notice that, in the unidimensional case, we evaluate just three functions, $F(x_n)$, $F'(x_n)$

and $F'(y_n)$, so that the family is optimal in the sense of Kung and Traub's conjecture, [13].

By adequately using Taylor's expansion we prove the following result about the convergence order.

Theorem 1 *Let $F:\mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently Fréchet differentiable function in a convex neighborhood of D, containing α, that is a solution of the system $F(x)=0$, whose jacobian matrix is continuous and nonsingular in D. Then, for an initial approximation sufficiently close to α, the family of methods defined by (1-5) has local order of convergence 4 for the following relations among the parameters: $s_1=\frac{5-8s_2}{8}$, $s_3=\frac{s_2}{3}$, $s_4=\frac{9-8s_2}{24}$; $\forall s_2\in\mathbb{R}$ and for $\theta=\frac{2}{3}$.*

The error equation obtained is as follows:

$$e_{n+1}=\frac{(64s_2+117)c_2^3-81c_1c_3c_2+9c_1^2c_4}{81c_1^3}\,e_n^4+O\left(e_n^5\right)$$

where $e_n=x_n-\alpha$ and $c_k=\frac{F^{(k)}(\alpha)}{k!}$, $k\geq 1$.

Proof: By applying the Taylor's expansion of $F(x_n)$ about α and taking into account that $F(\alpha)=0$, we have

$$F(x_n)=c_1e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+O(e_n^5) \tag{6}$$

where $c_k=\frac{F^{(k)}(\alpha)}{k!}\in\mathcal{L}_k(\mathbb{R}^n,\mathbb{R}^n)$, $k\geq 1$. By differentiating, one has

$$F'(x_n)=c_1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+O(e_n^5) \tag{7}$$

and then the following Taylor's development:

$$\Gamma_{x_n}F(x_n)=e_n-\frac{c_2}{c_1}\,e_n^2+\frac{2\left(c_2^2-c_1c_3\right)}{c_1^2}\,e_n^3+\frac{-4c_2^3+7c_1c_2c_3-3c_1^2c_4}{c_1^3}\,e_n^4+O(e_n^5)$$

By substituting in the first step (1), we have:

$$y_n-\alpha=(1-\theta)\,e_n+\frac{c_2\,\theta}{c_1}\,e_n^2+\frac{2\,\theta\left(c_1c_3-c_2^2\right)}{c_1^2}\,e_n^3+\frac{\theta\left(4c_2^3-7c_1c_2c_3+3c_1^2c_4\right)}{c_1^3}\,e_n^4+O(e_n^5)$$

And, then

$$\begin{aligned} F'(y_n)&=c_1-2\left(c_2(\theta-1)\right)e_n+\left(3c_3(\theta-1)^2+\frac{2c_2^2\theta}{c_1}\right)e_n^2-\frac{2}{c_1^2}\big(2c_1^2c_4(\theta-1)^3\\ &+2c_2^3\theta+c_1c_2c_3\theta(3\theta-5)\big)e_n^3+\frac{1}{c_1^3}\Big(6c_1^2c_4c_2\theta\left(2\theta^2-4\theta+3\right)+8c_2^4\theta\\ &+c_1c_3c_2^2\theta(15\theta-26)+c_1^2(\theta-1)\left(5c_1c_5(\theta-1)^3-12c_3^2\theta\right)\Big)e_n^4+O(e_n^5)\end{aligned} \tag{8}$$

and by substituting now in (2) we get:

$$
\begin{aligned}
H(x_n,y_n) = \Gamma_{x_n}F'(y_n) = 1 - \frac{2c_2\theta}{c_1}\, e_n + \frac{3\theta\left(c_1c_3(\theta-2)+2c_2^2\right)}{c_1^2}\, e_n^2 \qquad (9)\\
- \frac{4\left(\theta\left(c_1^2c_4\left(\theta^2-3\theta+3\right)+c_1c_3c_2(3\theta-7)+4c_2^3\right)\right)}{c_1^3}\, e_n^3\\
+ \frac{1}{c_1^4}\theta\big(2c_1^2c_4c_2\left(10\theta^2-24\theta+25\right)+c_1^2\big(5c_1c_5\left(\theta^3-4\theta^2+6\theta-4\right)\\
+ c_3^2(30-21\theta)\big)+c_1c_3c_2^2(39\theta-100)+40c_2^4\big)\, e_n^4 + O(e_n^5)
\end{aligned}
$$

and

$$
\begin{aligned}
H(y_n,x_n) = \Gamma_{y_n}F'(x_n) = 1 + \frac{2\theta c_2}{c_1}\, e_n + \frac{\theta\left((4\theta-6)c_2^2-3(\theta-2)c_1c_3\right)}{c_1^2}\, e_n^2 \qquad (10)\\
+ \frac{4\theta\left(2\left(\theta^2-3\theta+2\right)c_2^3+\left(-3\theta^2+9\theta-7\right)c_1c_3c_2+\left(\theta^2-3\theta+3\right)c_1^2c_4\right)}{c_1^3}\, e_n^3\\
+ \frac{1}{c_1^4}\theta\big(4\left(4\theta^3-18\theta^2+25\theta-10\right)c_2^4+\left(-36\theta^3+156\theta^2-223\theta+100\right)c_1c_3c_2^2\\
+ 2\left(8\theta^3-34\theta^2+48\theta-25\right)c_1^2c_4c_2+c_1^2\big(\left(9\theta^3-36\theta^2+57\theta-30\right)c_3^2\\
- 5\left(\theta^3-4\theta^2+6\theta-4\right)c_1c_5\big)\big)\, e_n^4 + O(e_n^5)
\end{aligned}
$$

These expressions (9) and (10) allow us going to (3) obtain:

$$
\begin{aligned}
G_s(x_n,y_n) &= s_1I + s_2H(x_n,y_n) + s_3H(y_n,x_n) + s_4H(x_n,y_n)^2\\
&= (s_1+s_2+s_3+s_4) + \frac{2(s_2-s_3+2s_4)\,\theta\, c_2}{c_1}\, e_n + \frac{2\theta}{c_1^2}\big((3(s_3+2s_4(\theta-1))\\
&+ s_2(2\theta-3))c_2^2 - 3(s_2-s_3+2s_4)(\theta-2)c_1c_3\big)\, e_n^2 + \frac{4\theta}{c_1^3}\big(2\big(-2s_3+s_2(\theta^2\\
&- 3\theta+2)+s_4\left(4\theta^2-9\theta+4\right)\big)c_2^3 - \big(s_3(3\theta-7)+s_2\left(3\theta^2-9\theta+7\right)\\
&+ s_4\left(9\theta^2-24\theta+14\right)\big)c_1c_3c_2 + (s_2-s_3+2s_4)\left(\theta^2-3\theta+3\right)c_1^2c_4\big)\, e_n^3\\
&+ \frac{1}{c_1^4}\theta\Big(4\big(10s_3+s_2\left(4\theta^3-18\theta^2+25\theta-10\right)+s_4(20\theta^3-72\theta^2+75\theta\\
&- 20)\big)c_2^4 + \big(s_3(39\theta-100)+s_2\left(-36\theta^3+156\theta^2-223\theta+100\right)\\
&+ 2s_4\left(-72\theta^3+270\theta^2-315\theta+100\right)\big)c_1c_3c_2^2 + 2\big(s_3\left(10\theta^2-24\theta+25\right)\\
&+ s_2\left(8\theta^3-34\theta^2+48\theta-25\right)+2s_4\left(12\theta^3-46\theta^2+60\theta-25\right)\big)c_1^2c_4c_2\\
&+ c_1^2\big(3\big(s_3(10-7\theta)+s_2\left(3\theta^3-12\theta^2+19\theta-10\right)+s_4(9\theta^3-36\theta^2+50\theta+\\
&- 20)\big)c_3^2 - 5(s_2-s_3+2s_4)\left(\theta^3-4\theta^2+6\theta-4\right)c_1c_5\big)\Big)\, e_n^4 + O(e_n^5)
\end{aligned}
$$

By substituting in (4) and (5), we can obtain the error equation:

$$
\begin{aligned}
e_{n+1} &= (-s_1 - s_2 - s_3 - s_4 + 1)\, e_n + \frac{(s_1 + s_2 + s_3 + s_4 - 2s_2\theta + 2s_3\theta - 4s_4\theta)c_2}{c_1}\, e_n^2 \\
&+ \frac{1}{c_1^2}\Big(\big(-3s_3\theta^2 + 6s_4\theta^2 + 6s_3\theta - 12s_4\theta + 2s_1 + 2s_3 + 2s_4 + s_2\left(3\theta^2 - 6\theta + 2\right)\big)\, c_1c_3 \\
&- 2\left(2s_2\theta^2 + 6s_4\theta^2 - 4s_2\theta + 4s_3\theta - 8s_4\theta + s_1 + s_2 + s_3 + s_4\right) c_2^2\Big) e_n^3 \\
&+ \frac{1}{c_1^3}\Big(\big(- 32s_4\theta^3 + 84s_4\theta^2 + 26s_3\theta - 52s_4\theta + 4s_1 + 4s_3 + 4s_4 + s_2\big(- 8\theta^3 + 28\theta^2 \\
&- 26\theta + 4\big)\big)c_2^3 - \big(- 36s_4\theta^3 - 15s_3\theta^2 + 102s_4\theta^2 + 38s_3\theta - 76s_4\theta + 7s_1 + 7s_3 \\
&+ 7s_4 + s_2\left(-12\theta^3 + 39\theta^2 - 38\theta + 7\right)\big)c_1c_3c_2 + \big(4s_3\theta^3 - 8s_4\theta^3 - 12s_3\theta^2 + 24s_4\theta^2 \\
&+ 12s_3\theta - 24s_4\theta + 3s_1 + 3s_3 + 3s_4 + s_2\left(-4\theta^3 + 12\theta^2 - 12\theta + 3\right)\big)c_1^2c_4\Big)\, e_n^4 + O(e_n^5)
\end{aligned}
$$

Finally, to obtain a fourth order iterative method, we will find a solution of the system given by conditions C_1, C_2, C_3 and C_4. Furthermore, to get a higher order of convergence we should also solve the system given C_5,C_6 and C_7.

$$
\begin{aligned}
&C_1 : (1 - s_1 - s_2 - s_3 - s_4) = 0 \\
&C_2 : (s_1 + s_2 + s_3 + s_4 - 2s_2\theta + 2s_3\theta - 4s_4\theta) = 0 \\
&C_3 : (s_1 + s_2 + s_3 + s_4 - 4s_2\theta + 4s_3\theta - 8s_4\theta + 2s_2\theta^2 + 6s_4\theta^2) = 0 \\
&C_4 : (2s_1 + 2s_3 + 2s_4 + 6s_3\theta - 12s_4\theta - 3s_3\theta^2 + 6s_4\theta^2 + s_2(2 - 6\theta + 3\theta^2)) = 0 \\
&C_5 : (4s_1 + 4s_3 + 4s_4 + 26s_3\theta - 52s_4\theta + 84s_4\theta^2 - 32s_4\theta^3 + s_2(4 - 26\theta + 28\theta^2 - 8\theta^3)) = 0 \\
&C_6 : (7s_1 + 7s_3 + 7s_4 + 38s_3\theta - 76s_4\theta - 15s_3\theta^2 + 102s_4\theta^2 - 36s_4\theta^3 + s_2(7 - 38\theta + 39\theta^2 - 12\theta^3)) = 0 \\
&C_7 : (3s_1 + 3s_3 + 3s_4 + 12s_3\theta - 24s_4\theta - 12s_3\theta^2 + 24s_4\theta^2 + 4s_3\theta^3 - 8s_4\theta^3 + s_2(3 - 12\theta + 12\theta^2 - 4\theta^3)) = 0
\end{aligned}
\tag{11}
$$

Unfortunately, the seven condition system has no solution. However, the conditions C_1, C_2, C_3 and C_4 provide a parametric solution:

$$
\theta = \frac{2}{3}; s1 = \frac{5 - 8s_2}{8}; s_3 = \frac{s_2}{3}; s_4 = \frac{9 - 8s_2}{24}, \qquad \forall s_2 \in \mathbb{R} \tag{12}
$$

Thus, we have obtained a family of fourth order optimal methods, whose error expression is:

$$
e_{n+1} = \frac{(64s_2 + 117)c_2^3 - 81c_1c_3c_2 + 9c_1^2c_4}{81c_1^3}\, e_n^4 + O(e_n^5) \tag{13}
$$

□

We are now interested in improving the convergence order of this family of methods, and so, we substitute the last (trivial) step of the fourth order iteration (5) by a new step similar to (4)

$$
x_{n+1} = z_n - G_t(x_n, y_n)\Gamma_{y_n} F(z_n) \tag{14}
$$

where

$$
G_t(x_n, y_n) = t_1 I + t_2 H(x_n, y_n) + t_3 H(y_n, x_n) + t_4 H(x_n, y_n)^2 \tag{15}
$$

For each value of s_2, we find relations among the constants t_1, t_2, t_3 and t_4 providing a family of sixth order methods according to the following:

Theorem 2 *Considering the same conditions as in Theorem 1, the biparametric family of three-step methods (14)-(15) has local order of convergence 6 for this relation among the constants:* $t_2 = -\frac{3+8t_1}{8}$, $t_3 = \frac{15-8t_1}{24}$, $t_4 = \frac{9+4t_1}{12}$; $\forall(s_2, t_1) \in \mathbb{R}^2$. *The vectorial error difference equation can be written as:*

$$e_{n+1} = \frac{-(64s_2+117)c_2^3 + 81c_1c_3c_2 - 9c_1^2c_4}{81c_1^4}c_3e_n^6 + O\left(e_n^7\right)$$

where $c_k = \frac{F^{(k)}(\alpha)}{k!}, k \geq 1$

Proof: Rewriting the error expression obtained in the preceding Theorem, we have

$$\begin{aligned} z_n - \alpha = & \frac{\big((64s_2+117)c_2^3 - 81c_1c_3c_2 + 9c_1^2c_4\big)}{81c_1^3}\, e_n^4 + \frac{2}{243c_1^4}\big(-2(352s_2+387)c_2^4 \\ & + 144(4s_2+9)c_1c_3c_2^2 - 270c_1^2c_4c_2 + 9c_1^2\left(4c_1c_5 - 27c_3^2\right)\big)\, e_n^5 + \frac{2}{243c_1^5}\big(7\big(448s_2 \\ & + 297\big)c_2^5 - (5056s_2+5625)c_1c_3c_2^3 + (832s_2+1953)c_1^2c_4c_2^2 - 9c_1^2c_2\big(45c_1c_5 \\ & - (128s_2+315)c_3^2\big) + 9c_1^3\left(7c_1c_6 - 99c_3c_4\right)\big)\, e_n^6 + O(e_n^7) \end{aligned}$$

Reasoning as in the previous theorem, we obtain expansions for the terms of (14) and (15) and, finally, an expression for the error depending on the parameters

$$\begin{aligned} e_{n+1} = x_{n+1} - z_n = & -\frac{\big((64s_2+117)c_2^3 - 81c_1c_3c_2 + 9c_1^2c_4\big)\left(t_1+t_2+t_3+t_4-1\right)}{81c_1^3}\, e_n^4 \\ & - \frac{2}{243c_1^4}\big(c_2^4(-(64s_2(12t_1+10t_2+14t_3+8t_4-11) + 9(99t_1+73t_2+125t_3+47t_4 \\ & - 86))) + 9c_1c_3c_2^2(64s_2(t_1+t_2+t_3+t_4-1) + 9(17t_1+15t_2+19t_3+13t_4-16)) \\ & - 9c_1^2c_4c_2(31t_1+29t_2+33t_3+27t_4-30) + 9c_1^2\big(4c_1c_5 - 27c_3^2\big)(t_1+t_2+t_3+t_4 \\ & - 1)\big)\, e_n^5 + \frac{1}{729c_1^5}\Big(-2c_2^5(64s_2(165t_1+107t_2+231t_3+57t_4-147) + 9(813t_1 \\ & + 287t_2+1443t_3-135t_4-693)) + 3c_1c_3c_2^3(64s_2(171t_1+139t_2+203t_3+107t_4 \\ & - 158) + 9(1431t_1+859t_2+2051t_3+335t_4-1250)) - 6c_1^2c_4c_2^2(832s_2(t_1+t_2+t_3 \\ & + t_4-1) + 3(707t_1+573t_2+849t_3+447t_4-651)) + 9c_1^2c_2\big(2c_1c_5(143t_1+127t_2 \\ & + 3(53t_3+37t_4-45)) - 3c_3^2(256s_2(t_1+t_2+t_3+t_4-1) + 9(75t_1+59t_2+91t_3 \\ & + 43t_4-70))\big) - 27c_1^3(14c_1c_6(t_1+t_2+t_3+t_4-1) - c_3c_4(199t_1+191t_2+3(69t_3 \\ & + 61t_4-66)))\Big)\, e_n^6 + O(e_n^7) \end{aligned}$$

Imposing the following conditions:

$$\begin{aligned}
C8 &: t_1 + t_2 + t_3 + t_4 - 1 = 0, \\
C9 &: 64s_2(12t_1 + 10t_2 + 14t_3 + 8t_4 - 11) + 9(99t_1 + 73t_2 + 125t_3 + 47t_4 - 86) = 0 \\
C10 &: 64s_2(t_1 + t_2 + t_3 + t_4 - 1) + 9(17t_1 + 15t_2 + 19t_3 + 13t_4 - 16) = 0 \\
C11 &: 31t_1 + 29t_2 + 33t_3 + 27t_4 - 30 = 0 \\
C12 &: 64s_2(165t_1 + 107t_2 + 231t_3 + 57t_4 - 147) + 9(813t_1 + 287t_2 + 1443t_3 - 135t_4 - 693) = 0 \\
C13 &: 64s_2(171t_1 + 139t_2 + 203t_3 + 107t_4 - 158) + 9(1431t_1 + 859t_2 + 2051t_3 + 335t_4 - 1250) = 0 \\
C14 &: 832s_2(t_1 + t_2 + t_3 + t_4 - 1) + 3(707t_1 + 573t_2 + 849t_3 + 447t_4 - 651) = 0 \\
C15 &: 143t_1 + 127t_2 + 3(53t_3 + 37t_4 - 45) = 0 \\
C16 &: 53t_3 + 37t_4 - 45 = 0 \\
C17 &: 256s_2(t_1 + t_2 + t_3 + t_4 - 1) + 9(75t_1 + 59t_2 + 91t_3 + 43t_4 - 70) = 0 \\
C18 &: 199t_1 + 191t_2 + 3(69t_3 + 61t_4 - 66) = 0
\end{aligned}$$

one finds the following solutions:

$$t_2 = \frac{1}{8}(-8t_1 - 3), t_3 = \frac{1}{24}(15 - 8t_1), t_4 = \frac{1}{12}(4t_1 + 9), \qquad \forall t_1 \in \mathbb{R}$$

For these values of the parameters, the error equation is given by

$$e_{n+1} = \frac{c_3\left(-(64s_2 + 117)c_2^3 + 81c_1c_3c_2 - 9c_1^2c_4\right)e_n^6}{81c_1^4} + O(e_n^7)$$

which proves that the method is of sixth order of convergence. We can see that the error expression depends on the parameter s_2 of the first steps of the iteration. □

3 Computational efficiency

In order to compare the different methods we have to study their efficiency. We use the efficiency index introduced in [5, 10], given by $E = \rho^{1/C}$, where ρ is the order of convergence and C is the computational cost per iteration. For a system of n nonlinear equations with n unknowns, C is obtained by:

$$C(\mu_0, \mu_1, n) = \mu_0 a_0 n + \mu_1 a_1 n^2 + P(n)$$

where a_0 and a_1 represent the number of evaluations of $F(x)$ and $F'(x)$ respectively, $P(n)$ is the number of products per iteration and μ_0 and μ_1 are the ratios between products and evaluations required to express the value of $C(\mu_0, \mu_1, n)$ in terms of products.

The best methods of the family defined by (1-5) from the point of view of computational efficiency are obtained for $s_2 = \frac{9}{8}$ and $s_2 = 0$, because in the first case the corresponding term of s_4, that involves more operations, is annihilated and in the second one two terms are annihilated. The first one is the method proposed in [8], which we denote by M1$_4$. The second one is a new method, denoted by M2$_4$.

We point out that for each particular method of the fourth order family, performing the new step (14-15), we obtain a different family of sixth order methods. For the comparisons, starting from $M1_4$ and $M2_4$ we choose for the new step value $t_1 = -\frac{9}{4}$ in both cases, and so, we obtain two new methods denoted by $M1_6$ and $M2_6$ respectively. Table 1 shows the values of the parameters of the four methods considered in the numerical experiments.

Method	s_1	s_2	s_3	s_4	t_1	t_2	t_3	t_4
$M1_4$	$-1/2$	$9/8$	$3/8$	0	-	-	-	-
$M2_4$	$5/8$	0	0	$3/8$	-	-	-	-
$M1_6$	$-1/2$	$9/8$	$3/8$	0	$-9/4$	$15/8$	$11/8$	0
$M2_6$	$5/8$	0	0	$3/8$	$-9/4$	$15/8$	$11/8$	0

Table 1: Parameters that define the selected methods.

We will express the computational cost per iteration with the same notation as in [5], where p_0 denotes the number of scalar products per iteration, p_1 the number of complete resolutions of the linear system (LU decomposition and resolution of two triangular systems) and p_2 the number of resolutions of two linear systems when LU decomposition is computed in another step in the same iteration.

Nevertheless, we need to introduce a new factor p_3 that is the number matrix by vector products per iteration. This adds a new term in the expression of the total number of products:

$$P(n) = \frac{n}{6}(2p_1 n^2 + (3p_1(k+1) + 6p_2))n + 6p_0 + p_1(3k-5) + 6p_2(k-1) + 6p_3 n)$$

where it is supposed that a quotient is equivalent to k products.

Table 2 shows the expression of the computational cost of the analyzed methods and compare it with Newton's method, that we denote by M_2.

Method	a_0	a_1	p_0	p_1	p_2	p_3	$C(\mu_0, \mu_1, n)$
M_2	1	1	0	1	0	0	$1/6n(-5 + 6\mu_0 + 3n + 6\mu_1 n + 2n^2 + 3k(1+n))$
$M1_4$	1	2	4	2	1	1	$1/3n(4 + 3\mu_0 + 9n + 6\mu_1 n + 2n^2 + 3k(2+n))$
$M2_4$	1	2	3	2	1	1	$1/3n(1 + 3\mu_0 + 9n + 6\mu_1 n + 2n^2 + 3k(2+n))$
$M1_6$	2	2	7	2	3	2	$1/3n(7 + 6\mu_0 + 18n + 6\mu_1 n + 2n^2 + 3k(4+n))$
$M2_6$	2	2	6	2	4	2	$1/3n(1 + 6\mu_0 + 21n + 6\mu_1 n + 2n^2 + 3k(5+n))$

Table 2: Computational cost for the different methods.

Assuming standard values for the parameters μ_0 and μ_1, one obtains the efficiencies of tables 3 and 4.

	M_2	$M1_4$	$M2_4$	$M1_6$	$M2_6$
$n=2$	1.05846	1.03818	1.0404	1.03116	1.03011
$n=4$	1.01292	1.00933	1.00959	1.00833	1.00789
$n=6$	1.00491	1.00377	1.00383	1.00356	1.00336
$n=9$	1.00177	1.00143	1.00145	1.00143	1.00135
$n=12$	1.00083	1.0007	1.0007	1.00072	1.00069
$n=15$	1.00045	1.00039	1.00039	1.00042	1.0004
$n=18$	1.00028	1.00024	1.00024	1.00026	1.00025

Table 3: Efficiency indexes for different values of n for $\mu_0 = 1.7$ and $\mu_1 = 0.7$.

	M_2	$M1_4$	$M2_4$	$M1_6$	$M2_6$
$n=2$	1.02123	1.02377	1.02462	1.01808	1.01772
$n=4$	1.0071	1.00703	1.00717	1.00591	1.00569
$n=6$	1.00329	1.00309	1.00313	1.00279	1.00266
$n=8$	1.00179	1.00164	1.00166	1.00156	1.00148
$n=12$	1.00069	1.00063	1.00064	1.00064	1.00061
$n=16$	1.00034	1.00031	1.00031	1.00033	1.00031
$n=20$	1.00019	1.00017	1.00017	1.00019	1.00018

Table 4: Efficiency indexes for different values of n for a $\mu_0 = 11.5$ and $\mu_1 = 1$.

4 Numerical Examples

In this section, we applied methods presented above in order to solve to the following integral equation:

$$y(t) = \frac{t}{e} + \int_0^1 2ts\mathrm{e}^{-y(s)^2}\mathrm{d}s$$

By discretizing this equation we obtain the following nonlinear system:

$$y_i = \frac{t_i}{\mathrm{e}} + 2t_i \sum_{j=1}^{n} p_j t_j \mathrm{e}^{-y_j^2} \tag{16}$$

where $t_i \in [0,1],\ y_i = y(t_i),\ p_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$.

We now apply to the nonlinear system (16) Newton's method, and the most efficient of our methods with two different starting points: $x_{0a} = (0.5, 0.5, \ldots, 0.5)$ and $x_{0b} = (0.5, -0.5, 0.5, -0.5, \ldots)$. We discretize the integral by Simpson's formula with 30 subintervals. The computations have been performed in variable precision arithmetics with 1000 digits mantissa. Table 5 shows the increments of the iterates, the computational order of convergence $ACOC$, see [14], and the number of iterations to converge with tolerance 10^{-125}.

		$M2$	$M1_4$	$M2_4$	$M1_6$	$M2_6$
x_{0a}	$\|x_{k+1} - x_k\|$	2.1225e-214	3.4406e-272	3.8187e-272	4.3999e-206	4.2176e-206
	ACOC	2.0000	4.0000	4.0000	6.0000	6.0000
	iter	8	5	5	4	4
x_{0b}	$\|x_{k+1} - x_k\|$	5.1892e-222	1.5384e-162	6.6729e-180	4.8624e-152	8.1328e-163
	ACOC	2.0000	4.0000	4.0000	5.9999	6.0000
	iter	9	5	5	4	4

Table 5: Numerical results with two different starting points.

There are no relevant differences between the two methods of the same order of convergence for the first starting point, neither in the number of iterations nor in the increment. However, method $M2_4$ behaves slightly better than $M1_4$ for the second starting point and so do their sixth order extensions.

5 Dynamics of the methods

In this section we study the dynamics of the iterative methods $M1_4$, $M2_4$, $M1_6$ and $M2_6$ when applied to the solution of a 2×2 nonlinear system and compare them with the dynamics of Newton's method. We show that the methods are generally convergent and depict their attraction basins.

Let us first recall some dynamical concepts. Consider a Frechet differentiable function $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. For $x \in \mathbb{R}^n$, we define the orbit of x as the set $x, G(x), G^2(x), \ldots, G^p(x), \ldots$. A point x_f is a fixed point of G if $G(x_f) = x_f$. The basin of attraction of a fixed point x_f is the set of points whose orbit tends to this fixed point

$$\mathcal{A}(x_f) = \{x \in \mathbb{R}^n : G^p(x) \longrightarrow x_f \text{ for } p \longrightarrow \infty\}$$

The dynamics of Newton's method and higher order iterative methods has been widely studied [15, 16, 17, 18]. In these references the method is applied to simple polynomial equations in the complex domain. Our purpose here is to show the aspect of the basins of attraction of the above mentioned methods applied to a system of nonlinear equations in $\mathbb{R}^2$, because we are mainly interested in the behavior of the methods for solving systems of nonlinear equations in the real n-dimensional space.

Consider the following quadratic system representing the intersection of two hyperbolas

$$\left.\begin{aligned} (x-3)^2 - 16y^2 &= 1 \\ x^2 - y^2 &= 1 \end{aligned}\right\}$$

In this system the axes of one hyperbola are parallel to the asymptotes of the other. This system presents four simple real roots. One solution is near

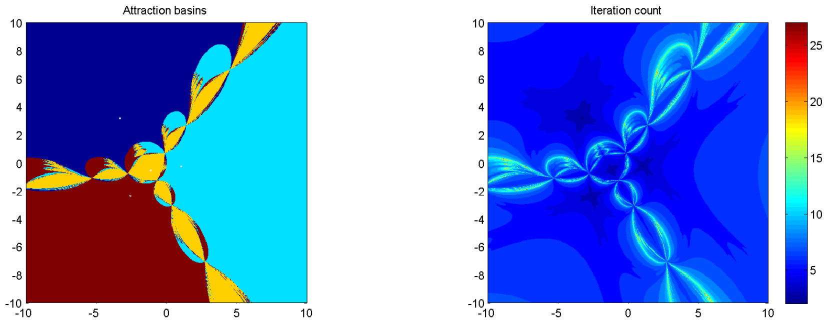

Figure 1: Attraction basins and iteration count for Newton's method

the barycenter of the other three. When there are less or multiple solutions, the convergence order is lower, as expected, and even the convergence fails in certain regions of the plane.

For the comparisons, we have run the methods iterating with tolerance 10^{-6} performing a maximum of 100 iterations. The starting points form a uniform grid of 512×512 in a rectangle of the real plane. The attraction basins have been colored according to the corresponding fixed point.

Figure 1 shows the attraction basins and the number of iterations for Newton's method. Figures 2, and 3 show the attraction basins of methods $M1_4$, $M2_4$, $M1_6$, and $M2_6$, respectively.

Observe that the complexity of the basins increases with the order, but the convergence regions cover almost all the plane. Methods $M2_4$ and $M2_6$ have slightly more complex basins than their counterparts $M1_4$ and $M1_6$. The four roots are superattracting for all the analyzed methods. In a further study we will consider the existence of periodic orbits and the convergence in case of double or missing roots.

6 Conclusions

As it can be observed in tables 3 and 4, Newton's method, M_2, maintains higher efficiency index than the other methods. Our method $M2_4$ always gets better indexes than $M1_4$ due to the fewer number of operations. However, $M2_6$ does not reach the efficiency of $M1_6$. Although the fourth order methods are good for systems with a reduced number of equations, as more complex a systems is, more advantages we get using the methods of order six. So, the fourth order methods are as good as the sixth order ones for systems between nine to twelve equations. From this point on, these last methods exceed lower order methods. In particular $M1_6$ goes closer than the others to the efficiency index of Newton's method. The dynamical experiment shows that the global convergence properties are not worsened by the increase of the order of the method.

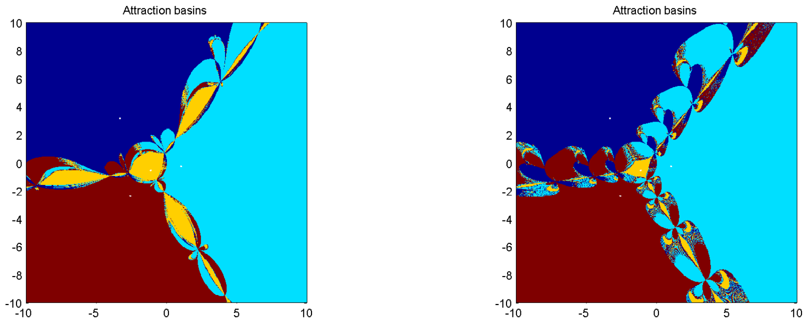

Figure 2: Attraction basins for methods $M1_4$ (left) and $M2_4$ (right)

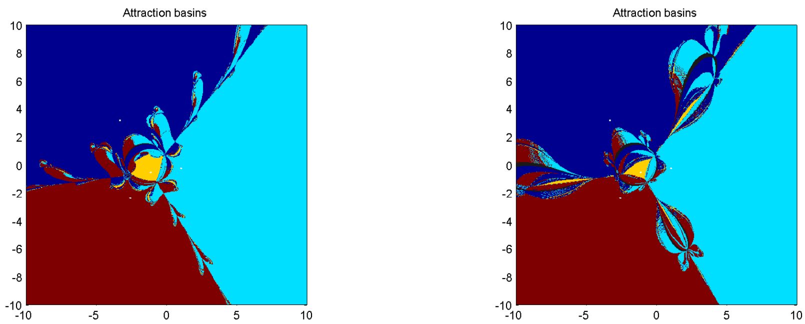

Figure 3: Attraction basins for method $M1_6$ (left) and $M2_6$ (right)

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