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Accurate Solution of Some I-Beam Optimization Problems

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Abstract

The problem of proportioning an assembled steel I-beam made of three sheets is well-known. It is included in nearly every course of study where the analysis of steel components is involved. In the presented case we consider finding the depth of a beam with a predefined modulus of resistance, based on the condition that the cross sectional area should be minimum. Such a depth of the beam is generally called optimum. At the same time, the minimum depth of the beam, which can be found from the stiffness conditions, can be considered in addition to its optimum depth. Such a problem is generally solved by empirical approaches with no strict formulation nor a proper mathematical analysis. The paper poses the problem as that of parametric mathematical programming with inequality constraints which appear as the condition of strength in bending, the condition of strength in shear, and the stiffness condition. Several variations which differ in the functional relationships between the depth of the beam's web and its thickness have been analyzed.

Keywords: Steel I-beam, optimization, inequality constraints, active constraints, parametric mathematical programming, accurate solution, initial data space.

1. Introduction

The idea of an I-beam in bending as a rational-cross-section beam seems to have been uttered first in a paper by Hodgkinson [1] published in early half of XIX century. An optimization problem of distributing metal between the web and the flanges of a beam appeared in the same period of time. The profile seems to be more rational if more metal is used for the flanges and the web is made as thin as possible. However, there is a number of obstacles for the web to made thinner. It cannot be too thin because it has to withstand the lateral force, it should be stable, corrosion-proof and resistant against random damage.

The most important obstacle that does not permit the web to be made thinner is its stability. The stability is usually ensured by reinforcing the web with a set of stiffening ribs. It would be hardly possible to ever solve the optimal I-beam problem comprehensively once an in-depth consideration of the stiffening ribs was involved. The problem is usually simplified by introducing generalized parameters based on the actual design experience. A pretty good generalized parameter that takes account of the web stability measures is the ratio of the web's depth, h, to its thickness, δ:

$$k = h / \delta . \qquad (1)$$

The tentative value for this ratio can be taken as 50 to 100 for webs not reinforced by stiffening ribs, 100 to 150 for webs reinforced by lateral stiffeners only, and 150 to 200 for webs reinforced by a set of lateral and longitudinal stiffeners. Making use of the web's depth vs. thickness ratio permits to reduce the number of variables in the problem by one because the thickness of the web is thus determined via its depth using a simple formula.

The problem of an I-beam of the minimum cross-section area was solved for a fixed beam's web depth-to-thickness ratio by K.K. Mukhanov [6]. However, this approach works only within a limited range of the beam's depth values. The actual relationship between the beam's web depth and its thickness is more complicated. Some textbooks on steelwork design such as [4], [6], [7] recommend using an empirical formula to determine the web's thickness via the beam's depth:

$$\delta = 3 + 7 \cdot h; \tag{2}$$

where the beam's depth h is taken in meters while the web's thickness δ in millimeters.

V.M.Vakhurkin [3] suggested a power relationship between the beam's web depth and its thickness where two independent parameters, k and m, are involved:

$$\delta = h^m / k \tag{3}$$

By using this approach, he solved a problem where an I-beam of a minimum cross-section area with a given modulus of resistance had to be found. It should be noted that relationship (3) is equivalent to (1) at $m = 1$ where it defines beams with a fixed web's depth-to-thickness ratio, while at $m = 0$ it defines beams with a fixed web's thickness.

Approximate techniques for solving problems like this, which sometimes involve empirical coefficients, are presented in quite a few publications (see [4], [7] for examples). The book [5] already posed the problem of a minimum cross-section I-beam as a problem of mathematical programming with inequality constraints: ones imposed on the bending strength, on the shear strength and on the deflection. The problem was solved for some particular cases.

This report presents a solution of the minimum cross-section area I-beam problem under the above-said limitations where the power relationship (3) is used. The problem will be further referred to as V.M.Vakhurkin's problem. Results for beams with a fixed web's depth-to-thickness ratio and for ones with a fixed web thickness will be presented as particular cases of this problem's solution.

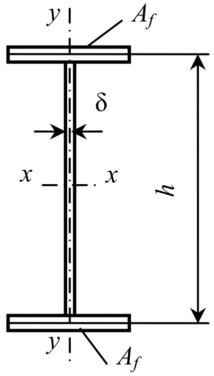

Figure 1: Cross-section of an I-beam

2. Object of optimization

An idealized I-beam is under consideration (Figure 1), which consists of a web and two flanges. It is subject to bending in its web's plane and restrained against buckling out of the same plane. The cross-section of the I-beam has two symmetry axes: $x-x$ and $y-y$. The flanges of the I-beam are assumed to have a small thickness comparing to the beam's depth; they are characterized by only one parameter, A_f, which is the chord's cross-section area. The web is defined by two parameters: depth h and thickness δ. The depth of the web, the distance between the centroids of the flanges, and the depth of the beam are assumed equal to one another.

The geometrical properties of the I-beam's cross-section are defined as:

$$A = 2 \cdot A_f + \delta \cdot h; \quad I = \frac{h^2}{4} \cdot \left(2 \cdot A_f + \frac{\delta \cdot h}{3} \right); \quad W = \frac{h}{2} \cdot \left(2 \cdot A_f + \frac{\delta \cdot h}{3} \right); \quad S = \delta \cdot h; \quad \mu = \frac{S}{A}; \qquad (4)$$

where A is the cross-section area;

I is the moment of inertia of the cross-section with respect to the $x-x$ axis;

W is the cross-section modulus with respect to the $x-x$ axis;

S is the web's cross-section area;

μ is the fraction of the web in the total cross-section area.

3. V.M.Vakhurkin's problem formulation

Let's consider relationship (3) in greater detail. The relationship contains empirical coefficients k and m which need to be specified. According to recommendations by V.M.Vakhurkin, the m coefficient should be taken from the range $0 \leq m \leq 1$. As was indicated, its value of 0 corresponds to beams with a fixed web's thickness while the value of 1 to beams with a fixed web's depth-to-thickness ratio. As for coefficient k, choosing a proper value for it is much more complicated. The reason for this is that coefficient k does not have an explicit physical meaning and, in addition, it is a dimensional value, the unit of length raised to the power of $m-1$.

This report presents relationship (3) in the form of

$$\frac{\delta}{\delta_0} = \left(\frac{h}{h_0} \right)^m \qquad (5)$$

where there are additional empirical coefficients in addition to the dimensionless one m: δ_0 and h_0 which have the dimension of length. When we consider the following formula, it becomes clear that relationships (3) and (5) are identical:

$$k = h_0^m / \delta_0. \tag{6}$$

The values of h_0 and δ_0 should be treated as the respective depth and thickness of the web of a particular beam from a set of beams which we search for the optimum one.

Further we will formulate and solve the problem of minimization of the cross-section area of a compound I-beam, A, where the web's depth and thickness are related through (5), where the stiffness condition: $I \geq I_r$; the bending strength condition: $W \geq W_r$, and the shear strength condition on the support: $S \geq S_r$ need to be met.

The given initial data include: m is the exponent in the power relationship; h_0 and δ_0 are the respective web's depth and thickness for a particular beam from the set of beams which we search for the optimum one; I_r is the required moment of inertia of the cross-section based on the stiffness condition; W_r is the required modulus of section based on the bending strength condition; S_r is the required web's cross-section area based on the shear strength condition on the support. We need to find the depth of the web of the optimum I-beam, h, and the cross-section area of its flange, A_f.

The formulation of the problem as one of mathematical programming is:

minimize

$$A = 2 \cdot A_f + \delta_0 \cdot h_0 \cdot \left(\frac{h}{h_0} \right)^{m+1} \tag{7}$$

under the constraints:

$$F_A = A_f \geq 0; \quad F_I = \frac{h^2}{4} \cdot \left(2 \cdot A_f + \frac{\delta_0 \cdot h_0}{3} \cdot \left(\frac{h}{h_0} \right)^{m+1} \right) - I_r \geq 0;$$
$$F_W = \frac{h}{2} \cdot \left(2 \cdot A_f + \frac{\delta_0 \cdot h_0}{3} \cdot \left(\frac{h}{h_0} \right)^{m+1} \right) - W_r \geq 0; \quad F_S = \delta_0 \cdot h_0 \cdot \left(\frac{h}{h_0} \right)^{m+1} - S_r \geq 0. \tag{8}$$

This is a problem of nonlinear mathematical programming in the space of two variables, h and A_f, with the objective function (7) and four inequality constraints (8). The first of the inequality constraints requires that the I-beam's flanges have a non-negative cross-section area, the second establishes a stiffness limitation, the third is based on the bending strength requirement, and the fourth demands a proper shear strength on the support.

4. Solution of the problem

The solution of the above problem can be represented in formulas. However, problems where inequality constraints participate are combinatorial in their nature, so the formulas for finding the desirable variables will be different for different sets of active constraints.

The set of active constraints for a particular problem depends on initial data, which consist of required values of I_r, W_r and S_r. The space of these parameters can be divided into areas such that each one will conform to a certain set of active constraints. Knowing which area the point with the I_r, W_r, S_r coordinates falls into will let us know the particular set of the active constraints and therefore the particular formulas for finding the sought-for values of h and A_f.

In order to divide the space of I_r, W_r, S_r into areas with fixed sets of active constrains, we will write the Kuhn-Tucker conditions [2] .

According to those, the coordinates of the point that represents the solution of optimization problem (7), (8) should satisfy the following set of equations and inequalities:

$$\lambda_A \cdot \frac{\partial F_A}{\partial h} + \lambda_I \cdot \frac{\partial F_I}{\partial h} + \lambda_W \cdot \frac{\partial F_W}{\partial h} + \lambda_S \cdot \frac{\partial F_S}{\partial h} = \frac{\partial A}{\partial h}; \tag{9}$$

$$\lambda_A \cdot \frac{\partial F_A}{\partial A_f} + \lambda_I \cdot \frac{\partial F_I}{\partial A_f} + \lambda_W \cdot \frac{\partial F_W}{\partial A_f} + \lambda_S \cdot \frac{\partial F_S}{\partial A_f} = \frac{\partial A}{\partial A_f}; \tag{10}$$

$$F_A \ge 0; \quad F_I \ge 0; \quad F_W \ge 0; \quad F_S \ge 0; \tag{11}$$

$$\lambda_A \ge 0; \quad \lambda_I \ge 0; \quad \lambda_W \ge 0; \quad \lambda_S \ge 0; \tag{12}$$

$$\lambda_A \cdot F_A = 0; \quad \lambda_I \cdot F_I = 0; \quad \lambda_W \cdot F_W = 0; \quad \lambda_S \cdot F_S = 0; \tag{13}$$

where λ_A, λ_I, λ_W, λ_S are Lagrangian multipliers yet to be found.

Equations (9) and (10) will look as follows, taking into account (7) and (8):

$$\begin{gathered} \lambda_I \cdot h \cdot \left(A_f + \frac{\delta_0 \cdot h_0 \cdot (m+3)}{12} \cdot \left(\frac{h}{h_0} \right)^{m+1} \right) + \lambda_W \cdot \left(A_f + \frac{\delta_0 \cdot h_0 \cdot (m+2)}{6} \cdot \left(\frac{h}{h_0} \right)^{m+1} \right) + \\ + \left(\lambda_S - 1 \right) \cdot \frac{\delta_0 \cdot h_0 \cdot (m+1)}{h} \cdot \left(\frac{h}{h_0} \right)^{m+1} = 0; \end{gathered} \tag{14}$$

$$\lambda_A + \lambda_I \cdot h^2 / 2 + \lambda_W \cdot h = 2; \tag{15}$$

Equalities (13) are referred to as complementary slackness conditions. Each one of them requires that at least one participating variable be equal to zero. If we choose a particular set of active constraints, it will become clear what inequalities of (11) actually hold true as equalities. Their respective constraints (w.r. to the index) from group (12) will remain inequalities. However, the rest of the constrains from group (12) will have to hold as equalities, and their respective constraints from group (11) as inequalities. This will derive a set of 6 equations and 4 inequalities from relationships (9) through (12). The six equations are to be used to find six unknowns: h, A_f, λ_A, λ_I, λ_W, λ_S, by expressing those via the given values of I_r, W_r, S_r. The other four inequalities define an area in the space of parameters I_r, W_r, S_r, which conforms to the selected set of active constraints. Any contradiction between the sets of equations and inequalities means that the particular selected active constraint set is not feasible.

Solutions of the equation/inequality set (9) – (13) for all feasible active constraint sets are presented in Table 1. The constraints are denoted as A, I, W, S in accordance with the subscripts in the constraint formulas (8). The h column gives formulas to find the optimum depth of the beam's web, the A_f column gives those for finding the optimum area of the I-beam's flange, and the μ column gives formulas for finding the fraction of the web in the total area of the I-beam's cross-section.

Table 1. Formulas for finding optimum parameters of an I-beam in V.M.Vakhurkin's problem

Act. constr.	h	A_f	μ
I	$h_0 \cdot \left(\frac{12 \cdot I_r}{\delta_0 \cdot h_0^3 \cdot (m+1)} \right)^{\frac{1}{m+3}}$	$\frac{\delta_0 \cdot h_0 \cdot m}{6} \cdot \left(\frac{12 \cdot I_r}{\delta_0 \cdot h_0^3 \cdot (m+1)} \right)^{\frac{m+1}{m+3}}$	$\frac{3}{m+3}$
W	$h_0 \cdot \left(\frac{3 \cdot W_r}{\delta_0 \cdot h_0^2 \cdot (m+1)} \right)^{\frac{1}{m+2}}$	$\frac{\delta_0 \cdot h_0 \cdot (2 \cdot m+1)}{6} \cdot \left(\frac{3 \cdot W_r}{\delta_0 \cdot h_0^2 \cdot (m+1)} \right)^{\frac{m+1}{m+2}}$	$\frac{3}{2 \cdot (m+2)}$
AS	$h_0 \cdot \left(\frac{S_r}{\delta_0 \cdot h_0} \right)^{\frac{1}{m+1}}$	0	1
IW	$\frac{2 \cdot I_r}{W_r}$	$\frac{W_r^2}{2 \cdot I_r} - \frac{\delta_0 \cdot h_0}{6} \cdot \left(\frac{2 \cdot I_r}{h_0 \cdot W_r} \right)^{m+1}$	
IS	$h_0 \cdot \left(\frac{S_r}{\delta_0 \cdot h_0} \right)^{\frac{1}{m+1}}$	$\frac{2 \cdot I_r}{h_0^2} \cdot \left(\frac{\delta_0 \cdot h_0}{S_r} \right)^{\frac{2}{m+1}} - \frac{S_r}{6}$	
WS	$h_0 \cdot \left(\frac{S_r}{\delta_0 \cdot h_0} \right)^{\frac{1}{m+1}}$	$\frac{W_r}{h_0} \cdot \left(\frac{\delta_0 \cdot h_0}{S_r} \right)^{\frac{1}{m+1}} - \frac{S_r}{6}$	

Inequalities which define areas conforming to specific sets of active constraints are listed in Table 2. The "Space of I_r, W_r, S_r" column of this table gives inequalities for each set of active constraints which bound an area in the said three-dimensional space of parameters. However, a transition to variables κ_I and κ_S can be made, which makes it possible to reduce the dimensionality of the space to two.

Table 2. Inequalities which define areas for possible sets of active constraints in V.M.Vakhurkin's problem

Act. con.	Space of I_r, W_r, S_r	Space of κ_I, κ_S	Act. con.	Space of I_r, W_r, S_r	Space of κ_I, κ_S
I	$\frac{\delta_0 h_0 m}{6}\left(\frac{12\cdot I_r}{\delta_0 h_0^3 (m+1)}\right)^{\frac{m+1}{m+3}} \ge 0$	–	W	$\frac{\delta_0 h_0 (2m+1)}{6}\left(\frac{3\cdot W_r}{\delta_0 h_0^2 (m+1)}\right)^{\frac{m+1}{m+2}} \ge 0$	–
	$\frac{4}{h_0^2}\left(\frac{12\cdot I_r}{\delta_0 h_0^3 (m+1)}\right)^{-\frac{2}{m+3}} > 0$	–		$\frac{\delta_0 h_0^3 (m+1)}{6}\left(\frac{3\cdot W_r}{\delta_0 h_0^2 (m+1)}\right)^{\frac{m+3}{m+2}} \ge I_r$	$2^{\frac{1}{m+3}} \ge \kappa_I$
	$\frac{2I_r}{h_0}\left(\frac{12\cdot I_r}{\delta_0 h_0^3 (m+1)}\right)^{-\frac{1}{m+3}} \ge W_r$	$\kappa_I \ge 2^{\frac{1}{m+2}}$		$\frac{2}{h_0}\cdot\left(\frac{3\cdot W_r}{\delta_0\cdot h_0^2\cdot(m+1)}\right)^{-\frac{1}{m+2}} > 0$	–
	$\delta_0 h_0\left(\frac{12\cdot I_r}{\delta_0 h_0^3 (m+1)}\right)^{\frac{m+1}{m+3}} \ge S_r$	$\kappa_I \ge \kappa_S$		$\delta_0\cdot h_0\cdot\left(\frac{3\cdot W_r}{\delta_0\cdot h_0^2\cdot(m+1)}\right)^{\frac{m+1}{m+2}} \ge S_r$	$1 \ge \kappa_S$
AS	$2 \ge 0$	–	IW	$W_r \ge \frac{\delta_0\cdot h_0^2}{6}\cdot\left(\frac{2\cdot I_r}{h_0\cdot W_r}\right)^{m+2}$	–
	$\frac{\delta_0\cdot h_0^3}{12}\cdot\left(\frac{S_r}{\delta_0\cdot h_0}\right)^{\frac{m+3}{m+1}} \ge I_r$	$\kappa_S \ge (m+1)^{\frac{1}{m+3}}\kappa_I$		$\frac{\delta_0 h_0^2 (m+1)}{3}\left(\frac{2I_r}{h_0 W_r}\right)^{m+2} > W_r$	$\kappa_I > 2^{\frac{1}{m+3}}$
	$\frac{\delta_0\cdot h_0^2}{6}\cdot\left(\frac{S_r}{\delta_0\cdot h_0}\right)^{\frac{m+2}{m+1}} \ge W_r$	$\kappa_S \ge (2m+2)^{\frac{1}{m+2}}$		$W_r > \frac{\delta_0 h_0^2 (m+1)}{6}\left(\frac{2I_r}{h_0 W_r}\right)^{m+2}$	$2^{\frac{1}{m+2}} > \kappa_I$
	$1 \ge 0$	–		$\delta_0\cdot h_0\cdot\left(\frac{2\cdot I_r}{h_0\cdot W_r}\right)^{m+1} \ge S_r$	$\kappa_I^{m+3} \ge \kappa_S$
IS	$I_r \ge \frac{\delta_0\cdot h_0^3}{12}\cdot\left(\frac{S_r}{\delta_0\cdot h_0}\right)^{\frac{m+3}{m+1}}$	$(m+1)^{\frac{1}{m+3}}\kappa_I \ge \kappa_S$	WS	$W_r \ge \frac{\delta_0\cdot h_0^2}{6}\cdot\left(\frac{S_r}{\delta_0\cdot h_0}\right)^{\frac{m+2}{m+1}}$	$(2m+2)^{\frac{1}{m+2}} \ge \kappa_S$
	$\frac{4}{h_0^2}\cdot\left(\frac{S_r}{\delta_0\cdot h_0}\right)^{-\frac{2}{m+1}} > 0$	–		$W_r \ge \frac{2\cdot I_r}{h_0}\cdot\left(\frac{S_r}{\delta_0\cdot h_0}\right)^{-\frac{1}{m+1}}$	$2\cdot\kappa_S \ge \kappa_I^{m+3}$
	$\frac{2\cdot I_r}{h_0}\cdot\left(\frac{S_r}{\delta_0\cdot h_0}\right)^{-\frac{1}{m+1}} \ge W_r$	$\kappa_I^{m+3} \ge 2\cdot\kappa_S$		$\frac{2}{h_0}\cdot\left(\frac{S_r}{\delta_0\cdot h_0}\right)^{-\frac{1}{m+1}} > 0$	–
	$\frac{\delta_0 h_0^3 (m+1)}{12}\left(\frac{S_r}{\delta_0 h_0}\right)^{\frac{m+3}{m+1}} \ge I_r$	$\kappa_S > \kappa_I$		$\frac{\delta_0 h_0^2 (m+1)}{3}\left(\frac{S_r}{\delta_0 h_0}\right)^{\frac{m+2}{m+1}} \ge W_r$	$\kappa_S > 1$

These variables are dimensionless and are defined as:

$$\kappa_I = h_I / h_W; \quad \kappa_S = h_S / h_W; \tag{16}$$

where $h_I = h_0 \cdot \left(\frac{12 \cdot I_r}{\delta_0 \cdot h_0^3 \cdot (m+1)} \right)^{\frac{1}{m+3}}; \quad h_W = h_0 \cdot \left(\frac{3 \cdot W_r}{\delta_0 \cdot h_0^2 \cdot (m+1)} \right)^{\frac{1}{m+2}}; \quad h_S = h_0 \cdot \left(\frac{S_r}{\delta_0 \cdot h_0} \right)^{\frac{1}{m+1}}.$ (17)

The geometrical meaning of variables h_I, h_W, h_S is that they are optimum depths of the beam with respect to stiffness, to the strength in bending, and to the strength in shear on the support.

The "Space of κ_I, κ_S" column of Table 2 presents inequalities which bound areas in this two-dimensional space for the same sets of active constrains. Dashes replace inequalities which are always true or follow from the others.

5. The case of a fixed web's depth-to-thickness ratio

Such a problem is a particular case of the previous one when $m = 1$ is assumed. Table 3 presents relationships for the optimum parameters where:

$$h_0 / \delta_0 = k. \tag{18}$$

Table 3. Formulas for finding optimum parameters of I-beams where the web's depth to its thickness ratio is fixed

Act. con.	h	A_f	μ	Act. con.	h	A_f	μ
I	$\sqrt[4]{6 \cdot I_r \cdot k}$	$\frac{\sqrt{6 \cdot I_r \cdot k}}{6 \cdot k}$	$\frac{3}{4}$	W	$\frac{\sqrt[3]{12 \cdot W_r \cdot k}}{2}$	$\frac{\sqrt[3]{18 \cdot W_r^2 \cdot k^2}}{4 \cdot k}$	$\frac{1}{2}$
AS	$\sqrt{S_r \cdot k}$	0	1	IW	$2 \cdot \frac{I_r}{W_r}$	$\frac{W_r^2}{2 \cdot I_r} - \frac{2 \cdot I_r^2}{3 \cdot k \cdot W_r^2}$	
IS	$\sqrt{S_r \cdot k}$	$\frac{2 \cdot I_r}{S_r \cdot k} - \frac{S_r}{6}$		WS	$\sqrt{S_r \cdot k}$	$\frac{W_r \cdot \sqrt{S_r \cdot k}}{S_r \cdot k} - \frac{S_r}{6}$	

Table 4 presents inequalities for possible sets of active constraints in the space of dimensionless parameters κ_I, κ_S determined from (16) where

$$h_I = \sqrt[4]{6 \cdot I_r \cdot k}; \quad h_W = \frac{\sqrt[3]{12 \cdot W_r \cdot k}}{2}; \quad h_S = \sqrt{S_r \cdot k}. \tag{19}$$

Table 4. Inequalities which define areas in the space of dimensionless parameters for possible sets of active constraints in the case of a fixed web's depth-to-thickness ratio

Act. con.	Space of κ_I, κ_S	Act. con.	Space of κ_I, κ_S	Act. con.	Space of κ_I, κ_S
I	$\kappa_I \geq \sqrt[3]{2}$	*W*	$\sqrt[4]{2} \geq \kappa_I$	*AS*	$\kappa_S \geq \sqrt[4]{2} \cdot \kappa_I$
	$\kappa_I \geq \kappa_S$		$1 \geq \kappa_S$		$\kappa_S \geq \sqrt[3]{4}$
IW	$\kappa_I > \sqrt[4]{2}$	*IS*	$\sqrt[4]{2} \cdot \kappa_I \geq \kappa_S$	*WS*	$\sqrt[3]{4} \geq \kappa_S$
	$\sqrt[3]{2} > \kappa_I$		$\kappa_I^4 \geq 2 \cdot \kappa_s$		$2 \cdot \kappa_S \geq \kappa_I^4$
	$\kappa_I^4 \geq 2 \cdot \kappa_S$		$\kappa_S > \kappa_I$		$\kappa_S > 1$

The graphical representation of the areas is shown in Figure 2.

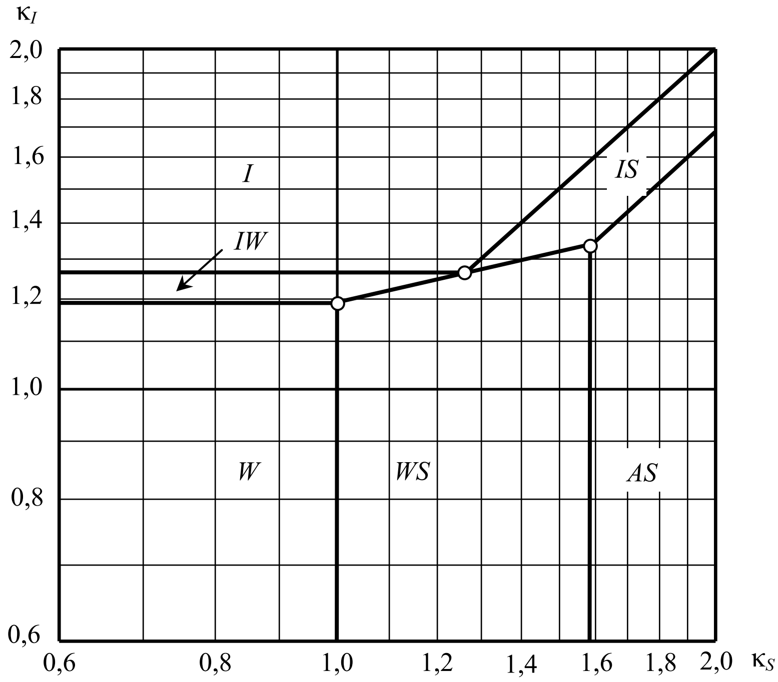

Figure 2: Division of the κ_I, κ_S plane into areas *I*, *IW*, *W*, *IS*, *WS*, *AS* for the case of a fixed web's depth-to-thickness ratio

It should be noted that the plots are built in logarithmic scales, so the boundaries of all areas prove to be straight lines.

6. A fixed web thickness case

This problem is a particular case of V.M.Vakhurkin's problem which appears at $m = 0$. Table 5 presents formulas for the optimum parameters, where $\delta_0=\delta$ is assumed.

Table 5. Formulas for finding optimum parameters of I-beams in the fixed-web-thickness case

Act. constr.	h	A_f	μ	Act. constr.	h	A_f	μ
I	$\frac{\sqrt[3]{12\cdot\delta^2\cdot I_r}}{\delta}$	0	1	W	$\frac{\sqrt{3\cdot\delta\cdot W_r}}{\delta}$	$\frac{\sqrt{3\cdot\delta\cdot W_r}}{6}$	$\frac{3}{4}$
AS	$\frac{S_r}{\delta}$	0	1	IW	$\frac{2\cdot I_r}{W_r}$	$\frac{W_r^2}{2\cdot I_r}-\frac{\delta\cdot I_r}{3\cdot W_r}$	
				WS	$\frac{S_r}{\delta}$	$\frac{\delta\cdot W_r}{S_r}-\frac{S_r}{6}$	

Table 6 presents inequalities for possible sets of active constraints in the space of dimensionless parameters κ_I, κ_S found from (16), where

$$h_I=\frac{\sqrt[3]{12\cdot\delta^2\cdot I_r}}{\delta};\quad h_W=\frac{\sqrt{3\cdot\delta\cdot W_r}}{\delta};\quad h_S=\frac{S_r}{\delta}. \tag{20}$$

Table 6. Inequalities which define areas in the dimensionless parameter space for possible sets of active constrains in the fixed-web-thickness case

Act. con.	Space of κ_I, κ_S	Act. con.	Space of κ_I, κ_S	Act. con.	Space of κ_I, κ_S
I	$\kappa_I\ge\sqrt{2}$	W	$\sqrt[3]{2}\ge\kappa_I$	AS	$\kappa_S\ge\kappa_I$
	$\kappa_I\ge\kappa_S$		$1\ge\kappa_S$		$\kappa_S\ge\sqrt{2}$
IW	$\kappa_I^3\ge 2\cdot\kappa_S$			WS	$\kappa_S>1$
	$\sqrt{2}\ge\kappa_I$				$\sqrt{2}\ge\kappa_S$
	$\kappa_I>\sqrt[3]{2}$				$2\cdot\kappa_S\ge\kappa_I^3$

The graphical representation of the areas is given in Figure 3.

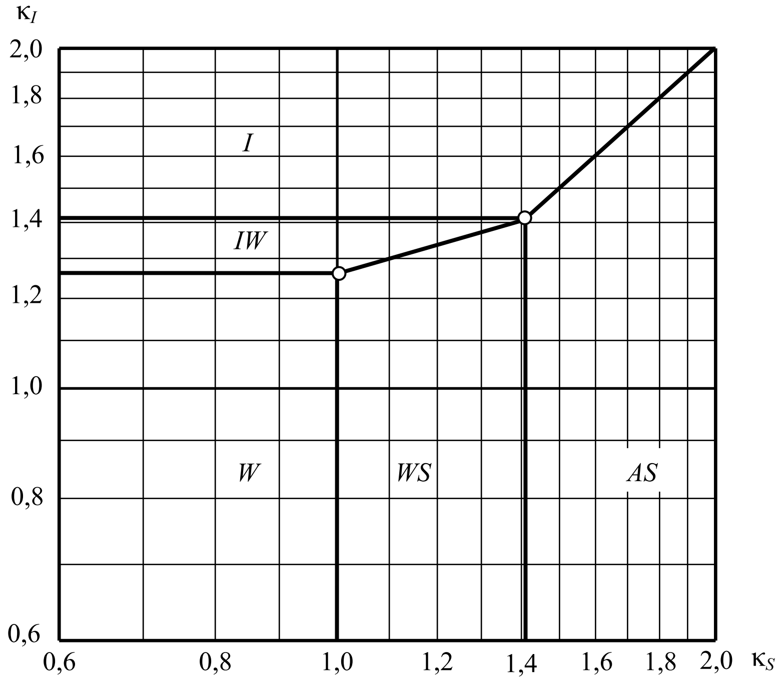

Figure 3: Dividing the κ_I, κ_S plane into areas *I*, *IW*, *W*, *WS*, *AS* for fixed-web-thickness beams

7. Solution steps

Here follows the sequence of steps that should be taken to solve the optimum I-beam problem:

- choose a relationship between the web's depth and its thickness by setting parameters m, h_0, δ_0, or k, or δ;
- use initial data: I_r, W_r, S_r to calculate the auxiliary values h_I, h_W and h_S from (17), (19), or (20);
- calculate parameters κ_I and κ_S using (16);
- find out, by using the inequalities from Table 2, or the plots from Figure 2 or Figure 3, which area the (κ_I, κ_S) point falls into;
- calculate the optimum values for h and A_f by using formulas from appropriate cells of Table 1, 3, or 5;
- calculate other characteristic numbers for the optimum I-beam using (4).

Conclusion

This paper formulates the problem of finding an optimum I-beam as a parametric mathematical programming problem that has inequality constraints. An exact solution is given for the problem in the form of formula sets. Appropriate formulas should be selected on the basis of imposed active constraints which are defined by the initial data of a particular problem.

A significant attention is paid to the initial data space, in particular, the division of that into areas where particular sets of active constraints are in effect. Results of such a division for commonly known particular cases are presented both in formulas and in plots. A qualitative distribution of the optimum solutions for I-beams over limit state types is demonstrated.

The results of this work can be used both in practical design activities and in courses of study dedicated to metalwork design or mathematical programming.

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