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Structural analysis of parabolic arches defined in Global Coordinates

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Abstract

This paper deals with curved beams, particularly, parabolic arches defined in global coordinates. The problem is approached differentially and expressed in a unique system of twelve linear ordinary differential equations given by Gimena *et al.* [2]. The lower-triangular form of the system of equations permits the determination of the exact analytical solution through successive integrations row by row. The system can also be solved using a numerical method with boundary conditions. Axial and shearing deformations, varying cross section area, non-symmetric section, generalized loads and different support conditions can be taken into account. In special cases, this system is particularized, and different subsystems arise, to model the mechanical behavior of special structural elements. This study presents the way to proceed with parabolic arches, which have an optimal response transmitting vertical load (Gimena *et al.* [3]). Analytic expressions and results of forces, moment, rotation and displacements for parabolic arches (spans, heights, load cases and supports) are given and plotted.

Keywords: Structural analysis, curved beam, parabolic arches, Frenet frame, global coordinates, differential system, transfer matrix.

1. Introduction

Many authors as for example Love [8], Parcel and Moorman 11], Washizu [12], Papangelis and Trahair [10], Haktanir and Kiral [6], Litewka and Rakowski [7], Murin and Kutis [9], Rajasekaran and Padmanabhan [13] and Yang [16], have deeply study on structural analysis of curved beams, offering different methodologies, resolution procedures and results. Former researches have approached this problem of twisted elements, expressing the functions in natural coordinates using the Frenet frame system of reference.

The authors that subscribe this research, presented a general formulation for curved beam elements expressed in Frenet frame (Gimena *et al.* [4], [5]), taking into account shearing deformations, varying cross section area, non-symmetric section, generalized loads and any support condition. Recently, these approaches have been reformulated in a Cartesian coordinate system (Gimena *et al.* [2]), which has several advantages from former models owing to its lower-triangular nature.

In this paper, this differential system of equations is used to solve the problem of arches, given a general procedure to obtain analytical results. This way to proceed is applied to examples of parabolic arches with different load and cross-sections. Examples are provided to compare results given in the literature for verification purposes.

2. Curved beam formula defined in Global Coordinates

A curved beam is generated by a plane cross-section which centroid P sweeps perpendicularly through all the points of an axis line. The vector radius $\mathbf{r}=\mathbf{r}(s)$ expresses this curved line, where s (m) length of the arc, is the independent variable. The reference coordinate system used to represent the intervening known and unknown functions of the problem is the Frenet frame P_{tnb} . Its unit vectors tangent $\mathbf{t}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$ are $\mathbf{t} = D\mathbf{r}$; $\mathbf{n} = D^2\mathbf{r}/|D^2\mathbf{r}|$; $\mathbf{b} = \mathbf{t}\wedge\mathbf{n}$: where, $D = d/ds$ is the derivative respect the parameter s.

The Frenet-Serret equations (see Sokolnikoff and Redeffer [14]) describe the movement of the frame system along the axis line. They are obtained with the versors tangent, normal and binormal derivates with respect to the arc length. Its matricial expression is:

$$D\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}0 & \chi(s) & 0\\ -\chi(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{bmatrix}\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} \tag{1}$$

where $\chi=\chi(s)$ (m^{-1}) and $\tau=\tau(s)$ (m^{-1}) are the flexure and torsion curvatures respectively.

Assuming the habitual principles and hypotheses of the strength of materials (Timoshenko [15]) and considering the stresses associated with the normal cross-section (σ, τ_n, τ_b) (N/m^2), the geometric characteristics of the section are: area $A(s)$ (m^2), shearing coefficients $\alpha_n(s)$, $\alpha_{nb}(s)$, $\alpha_{bn}(s)$, $\alpha_b(s)$ and moments of inertia $I_t(s)$, $I_n(s)$, $I_b(s)$, $I_{nb}(s)$ (m^4).

$E(s)$ (N/m^2) and $G(s)$ (N/m^2) are the longitudinal and transversal elastic moduli which give the elastic properties of the material.

Equilibrium and kinematics equations compose a system of twelve linear ordinary differential equations of a curved beam element (Gimena *et al.* [4]).

It is possible to apply a change of basis in the referenced equations and express the functions in a global coordinate system P_{xyz} with unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ instead of unit vectors tangent $\mathbf{t}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$, through the direction cosines:

$$\begin{bmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}\upsilon_{tx} & \upsilon_{ty} & \upsilon_{tz}\\ \upsilon_{nx} & \upsilon_{ny} & \upsilon_{nz}\\ \upsilon_{bx} & \upsilon_{by} & \upsilon_{bz}\end{bmatrix}\begin{bmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{bmatrix} \tag{2}$$

The differential system in global Cartesian coordinates is (Gimena *et al.* [2]):

$$
\begin{array}{l}
DV_x + q_x = 0 \\
DV_y + q_y = 0 \\
DV_z + q_z = 0 \\
-\upsilon_{tz}V_y + \upsilon_{ty}V_z + DM_x + m_x = 0 \\
+\upsilon_{tz}V_x - \upsilon_{tx}V_z + DM_y + m_y = 0 \\
-\upsilon_{ty}V_x + \upsilon_{tx}V_y + DM_z + m_z = 0 \\
-\gamma_{xx}M_x - \gamma_{yx}M_y - \gamma_{zx}M_z + D\theta_x - \Theta_x = 0 \\
-\gamma_{xy}M_x - \gamma_{yy}M_y - \gamma_{zy}M_z + D\theta_y - \Theta_y = 0 \\
-\gamma_{xz}M_x - \gamma_{yz}M_y - \gamma_{zz}M_z + D\theta_z - \Theta_z = 0 \\
-\varepsilon_{xx}V_x - \varepsilon_{yx}V_y - \varepsilon_{zx}V_z - \upsilon_{tz}\theta_y + \upsilon_{ty}\theta_z + D\delta_x - \Delta_x = 0 \\
-\varepsilon_{xy}V_x - \varepsilon_{yy}V_y - \varepsilon_{zy}V_z + \upsilon_{tz}\theta_x - \upsilon_{tx}\theta_z + D\delta_y - \Delta_y = 0 \\
-\varepsilon_{xz}V_x - \varepsilon_{yz}V_y - \varepsilon_{zz}V_z - \upsilon_{ty}\theta_x + \upsilon_{tx}\theta_y + D\delta_z - \Delta_z = 0
\end{array}
\qquad (3)
$$

where,

$$\gamma_{xx} = \frac{\upsilon_{tx}^2}{GI_t} + \frac{\upsilon_{nx}^2 I_b + 2\upsilon_{nx}\upsilon_{bx}I_{nb} + \upsilon_{bx}^2 I_n}{E(I_n I_b - I_{nb}^2)}$$

$$\gamma_{yy} = \frac{\upsilon_{ty}^2}{GI_t} + \frac{\upsilon_{ny}^2 I_b + 2\upsilon_{ny}\upsilon_{by}I_{nb} + \upsilon_{by}^2 I_n}{E(I_n I_b - I_{nb}^2)}$$

$$\gamma_{zz} = \frac{\upsilon_{tz}^2}{GI_t} + \frac{\upsilon_{nz}^2 I_b + 2\upsilon_{nz}\upsilon_{bz}I_{nb} + \upsilon_{bz}^2 I_n}{E(I_n I_b - I_{nb}^2)}$$

$$\gamma_{xy} = \gamma_{yx} = \frac{\upsilon_{tx}\upsilon_{ty}}{GI_t} + \frac{\upsilon_{nx}\upsilon_{ny}I_b + (\upsilon_{nx}\upsilon_{by} + \upsilon_{bx}\upsilon_{ny})I_{nb} + \upsilon_{bx}\upsilon_{by}I_n}{E(I_n I_b - I_{nb}^2)}$$

$$\gamma_{xz} = \gamma_{zx} = \frac{\upsilon_{tx}\upsilon_{tz}}{GI_t} + \frac{\upsilon_{nx}\upsilon_{nz}I_b + (\upsilon_{nx}\upsilon_{bz} + \upsilon_{bx}\upsilon_{nz})I_{nb} + \upsilon_{bx}\upsilon_{bz}I_n}{E(I_n I_b - I_{nb}^2)}$$

$$\gamma_{yz} = \gamma_{zy} = \frac{\upsilon_{ty}\upsilon_{tz}}{GI_t} + \frac{\upsilon_{ny}\upsilon_{nz}I_b + (\upsilon_{ny}\upsilon_{bz} + \upsilon_{by}\upsilon_{nz})I_{nb} + \upsilon_{by}\upsilon_{bz}I_n}{E(I_n I_b - I_{nb}^2)}$$

$$\varepsilon_{xx} = \frac{\upsilon_{tx}^2}{EA} + \frac{\upsilon_{nx}^2\alpha_n + 2\upsilon_{nx}\upsilon_{bx}\alpha_{nb} + \upsilon_{bx}^2\alpha_b}{GA}$$

$$\varepsilon_{yy} = \frac{\upsilon_{ty}^2}{EA} + \frac{\upsilon_{ny}^2\alpha_n + 2\upsilon_{ny}\upsilon_{by}\alpha_{nb} + \upsilon_{by}^2\alpha_b}{GA}$$

$$\varepsilon_{zz} = \frac{\upsilon_{tz}^2}{EA} + \frac{\upsilon_{nz}^2\alpha_n + 2\upsilon_{nz}\upsilon_{bz}\alpha_{nb} + \upsilon_{bz}^2\alpha_b}{GA}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{\upsilon_{tx}\upsilon_{ty}}{EA} + \frac{\upsilon_{nx}\upsilon_{ny}\alpha_n + (\upsilon_{nx}\upsilon_{by} + \upsilon_{bx}\upsilon_{ny})\alpha_{nb} + \upsilon_{bx}\upsilon_{by}\alpha_b}{GA}$$

$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{\upsilon_{tx}\upsilon_{tz}}{EA} + \frac{\upsilon_{nx}\upsilon_{nz}\alpha_n + (\upsilon_{nx}\upsilon_{bz} + \upsilon_{bx}\upsilon_{nz})\alpha_{nb} + \upsilon_{bx}\upsilon_{bz}\alpha_b}{GA}$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{\upsilon_{ty}\upsilon_{tz}}{EA} + \frac{\upsilon_{ny}\upsilon_{nz}\alpha_n + (\upsilon_{ny}\upsilon_{bz} + \upsilon_{by}\upsilon_{nz})\alpha_{nb} + \upsilon_{by}\upsilon_{bz}\alpha_b}{GA}$$

This new general expression, which simulates the structural behaviour of the linear element, has a lower-triangular form. This important property permits to solve analytically the differential equation system using successive integrations row by row.

3. Arches formula defined in Global Coordinates

In this section it is exposed the procedure to follow and the formulation to analyze arches.

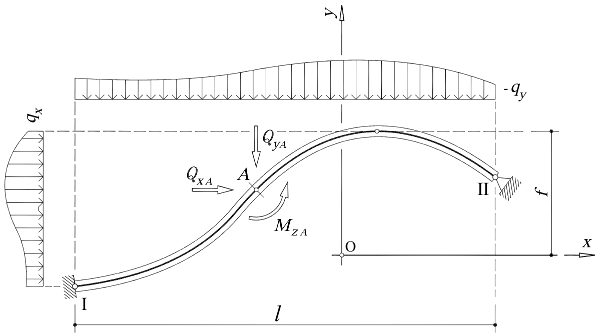

Figure 1: Generic arch with punctual and distributed load.

Particularizing the differential system (Eq. 3) for plane curves loaded in its plane (arches):

$$\begin{array}{llllll} DV_x & & & & & +q_x=0 \\ & DV_y & & & & +q_y=0 \\ -\upsilon_{ty}V_x & +\upsilon_{tx}V_y & +DM_z & & & +m_z=0 \\ & & -\dfrac{M_z}{EI_z} & +D\theta_z & & -\Theta_z=0 \\ -\dfrac{\upsilon_{tx}^2 V_x}{EA} & -\dfrac{\upsilon_{tx}\upsilon_{ty}V_y}{EA} & & +\upsilon_{ty}\theta_z+D\delta_x & & -\Delta_x=0 \\ -\dfrac{\upsilon_{tx}\upsilon_{ty}V_x}{EA} & -\dfrac{\upsilon_{ty}^2 V_y}{EA} & & -\upsilon_{tx}\theta_z & +D\delta_y & -\Delta_y=0 \end{array} \tag{4}$$

In matricial notation, it is obtained:

$$D\mathbf{e}(s)=\left[\mathbf{T_D}(s)\right]\mathbf{e}(s)+\mathbf{q_D}(s) \tag{5}$$

Integrating the above system directly, row by row, the general solution can be written:

$$\mathbf{e}(s)=\left[\mathbf{T_T}(s)\right]\mathbf{C}+\mathbf{q_T}(s) \tag{6}$$

where, $\mathbf{C}=\left\{C_1,C_2,C_3,C_4,C_5,C_6\right\}^T$ is the vector of arbitrary coefficients.

With the proper change of variable of the arc length *s* by the parameter λ, Eq. (5), yields:

$$D_\lambda\mathbf{e}(\lambda)=\left[\mathbf{T_D}(\lambda)D_\lambda s\right]\mathbf{e}(\lambda)+\mathbf{q_D}(\lambda)D_\lambda s \tag{7}$$

In the same manner, integrating the above system Eq. (7), general solution is as follows:

$$\mathbf{e}(\lambda)=\left[\mathbf{T_T}(\lambda)\right]\mathbf{C}+\mathbf{q_T}(\lambda) \tag{8}$$

Let suppose a punctual load applied at a generic point A (see Figure 1); The equilibrium and kinematics relate the effects (forces and displacements) at this point:

$$-\mathbf{e}\left(\lambda_{\mathbf{A,I}}\right)+\mathbf{e}\left(\lambda_{\mathbf{A,II}}\right)+\mathbf{Q_A}=\mathbf{0} \tag{9}$$

been $\mathbf{Q_A}=\left\{Q_{x\mathbf{A}},Q_{y\mathbf{A}},M_{z\mathbf{A}},0,0,0\right\}^T$ the load vector.

Solution given in Eq. (8) is particularized in the extremes of both parts:

$$\mathbf{e}\left(\lambda_{\mathbf{A,I}}\right)=\left[\mathbf{T_T}(\lambda_\mathbf{A},\lambda_\mathbf{I})\right]\left(\mathbf{e}(\lambda_\mathbf{I})-\mathbf{q_T}(\lambda_\mathbf{I})\right)+\mathbf{q_T}(\lambda_\mathbf{A}) \text{ for the first part } \lambda_\mathbf{I}\geq\lambda\geq\lambda_\mathbf{A}$$

$$\text{where } \left[\mathbf{T_T}(\lambda_\mathbf{A},\lambda_\mathbf{I})\right]=\left[\mathbf{T_T}(\lambda_\mathbf{A})\right]\left[\mathbf{T_T}(\lambda_\mathbf{I})\right]^{-1}$$

$$\mathbf{e}\left(\lambda_{\mathbf{A,II}}\right)=\left[\mathbf{T_T}(\lambda_\mathbf{A},\lambda_\mathbf{II})\right]\left(\mathbf{e}(\lambda_\mathbf{II})-\mathbf{q_T}(\lambda_\mathbf{II})\right)+\mathbf{q_T}(\lambda_\mathbf{A}) \text{ for the second part } \lambda_\mathbf{A}\geq\lambda\geq\lambda_\mathbf{II}$$

$$\text{where } \left[\mathbf{T_T}(\lambda_\mathbf{A},\lambda_\mathbf{II})\right]=\left[\mathbf{T_T}(\lambda_\mathbf{A})\right]\left[\mathbf{T_T}(\lambda_\mathbf{II})\right]^{-1}$$

Substituting former values in Eq. (9), it is obtained:

$$-\left[\mathbf{T_T}(\lambda_\mathbf{A},\lambda_\mathbf{I})\right]\left(\mathbf{e}(\lambda_\mathbf{I})-\mathbf{q_T}(\lambda_\mathbf{I})\right)+\left[\mathbf{T_T}(\lambda_\mathbf{A},\lambda_\mathbf{II})\right]\left(\mathbf{e}(\lambda_\mathbf{II})-\mathbf{q_T}(\lambda_\mathbf{II})\right)+\mathbf{Q_A}=\mathbf{0} \tag{10}$$

Latter, support conditions are introduced in Eq. 10 to obtain values at both extremes initial $\mathbf{e}(\lambda_\mathbf{I})$ and final $\mathbf{e}(\lambda_\mathbf{II})$.

Once knowing these values, the exact solution in both parts is written:

$$\mathbf{e}(\lambda)=\left[\mathbf{T_T}(\lambda,\lambda_\mathbf{I})\right]\left(\mathbf{e}(\lambda_\mathbf{I})-\mathbf{q_T}(\lambda_\mathbf{I})\right)+\mathbf{q_T}(\lambda) \text{ for } \lambda_\mathbf{I}\geq\lambda\geq\lambda_\mathbf{A} \tag{11}$$

$$\mathbf{e}(\lambda)=\left[\mathbf{T_T}(\lambda,\lambda_\mathbf{II})\right]\left(\mathbf{e}(\lambda_\mathbf{II})-\mathbf{q_T}(\lambda_\mathbf{II})\right)+\mathbf{q_T}(\lambda) \text{ for } \lambda_\mathbf{A}\geq\lambda\geq\lambda_\mathbf{II} \tag{12}$$

4. Examples. Parabolic arches

In the previous section, a general procedure has been presented for analyzing arches with distributed and a punctual load.

In the following examples, the geometry of these arches is restrained to parabolic shaped arches, with variable cross-section and different heights and spans.

4.1. Parabolic arch with variable cross-section

The Cartesian equation that represent the geometry of a parabolic arch in terms of the height *f* and span *l* is given by $y=4fx^2/l^2$.

In parametric equations: $x(\lambda)=p\lambda$; $y(\lambda)=p\lambda^2/2$; $z(\lambda)=0$.

being $p=l^2/8f$ and $\lambda=8fx/l^2$, in this example for $-1=\lambda_\mathbf{I}\geq\lambda\geq\lambda_\mathbf{II}=1$ (see Figure 2).

A punctual load $\mathbf{Q}_0=\left\{0,Q_{y0},0,0,0,0\right\}^T$ is applied at the point $\lambda_\mathbf{A}=\lambda_\mathbf{0}=0$.

Direction cosines of the curve are $\upsilon_{tx} = 1/\sqrt{\lambda^2+1}$ and $\upsilon_{ty} = \lambda/\sqrt{\lambda^2+1}$.

The derivative of the arc length *s* with respect to the parameter λ is $D_\lambda s = p\sqrt{\lambda^2+1}$.

Properties of the variable cross-section (with height $h(\lambda) = h$ and width $b(\lambda) = b_0\sqrt{\lambda^2+1}$) are: $A_0 = b_0 h$, $I_{z0} = b_0 h^3/12$, $A(\lambda) = A_0\sqrt{\lambda^2+1}$, $I_z(\lambda) = I_{z0}\sqrt{\lambda^2+1}$ and $i_0^2 = I_{z0}/A_0$.

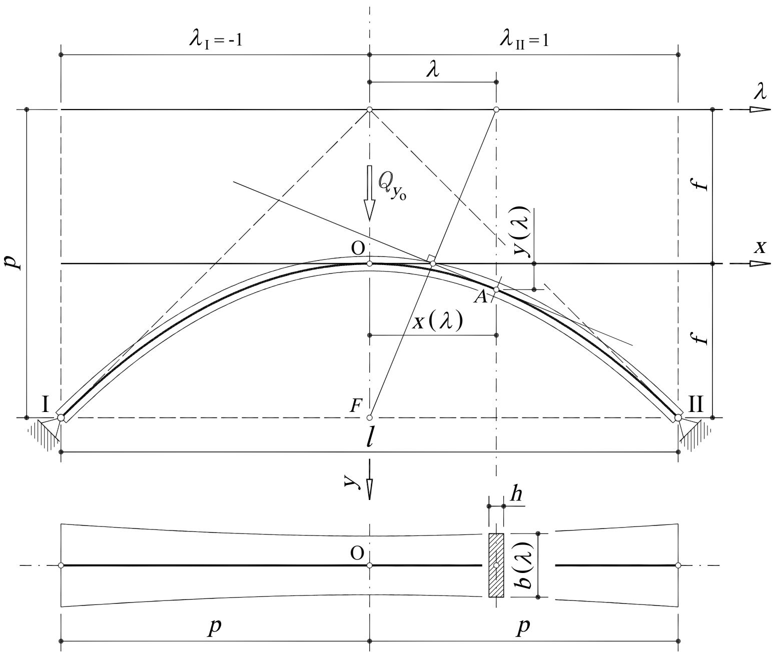

Figure 2: Hinged parabolic arch with hyperbolic variable cross-section.

The differential system Eq. (4) with these geometry, will be:

$$\begin{array}{rcl}
D_\lambda V_x & = & 0 \\
D_\lambda V_y & = & 0 \\
-\,p\lambda V_x + pV_y + D_\lambda M_z & = & 0 \\
-\dfrac{p}{EI_{z0}} M_z + D_\lambda \theta_z & = & 0 \\
-\dfrac{p}{(\lambda^2+1)EA_0} V_x - \dfrac{p\lambda}{(\lambda^2+1)EA_0} V_y + p\lambda\theta_z + D_\lambda \delta_x & = & 0 \\
-\dfrac{p\lambda}{(\lambda^2+1)EA_0} V_x - \dfrac{p\lambda^2}{(\lambda^2+1)EA_0} V_y - p\theta_z + D_\lambda \delta_y & = & 0
\end{array} \qquad (13)$$

Its direct integration, row by row, gives:

$$
\begin{aligned}
V_x(\lambda) &= C_1 \\
V_y(\lambda) &= C_2 \\
M_z(\lambda) &= \frac{p\lambda^2}{2}C_1 - p\lambda C_2 + C_3 \\
\theta_z(\lambda) &= \frac{p^2\lambda^3}{6EI_{z0}}C_1 - \frac{p^2\lambda^2}{2EI_{z0}}C_2 + \frac{p\lambda}{EI_{z0}}C_3 + C_4 \\
\delta_x(\lambda) &= \frac{p\left[30i_0^2\arctan\lambda - p^2\lambda^5\right]}{30EI_{z0}}C_1 + \frac{p\left[4i_0^2\ln\left(\lambda^2+1\right)+p^2\lambda^4\right]}{8EI_{z0}}C_2 - \frac{p^2\lambda^3}{3EI_{z0}}C_3 - \frac{p\lambda^2}{2}C_4 + C_5 \\
\delta_y(\lambda) &= \frac{p\left[12i_0^2\ln\left(\lambda^2+1\right)+p^2\lambda^4\right]}{24EI_{z0}}C_1 + \frac{p\left[6i_0^2(\lambda-\arctan\lambda)-p^2\lambda^3\right]}{6EI_{z0}}C_2 + \frac{p^2\lambda^2}{2EI_{z0}}C_3 + p\lambda C_4 + C_6
\end{aligned}
\tag{14}
$$

Following the procedure explained in section 3, considering the parabolic arch hinged in both extremes, values at both ends of the first part of the curve are given:

$$
\mathbf{e}(\lambda_1) = \left\{ \frac{5\left[12i_0^2\ln 2 - 5p^2\right]}{4\left[15i_0^2\pi + 8p^2\right]}Q_{y0}, \frac{1}{2}Q_{y0}, 0, \frac{p^2\left[15i_0^2(4\ln 2 + 3\pi) - p^2\right]}{12EI_{z0}\left[15i_0^2\pi + 8p^2\right]}Q_{y0}, 0, 0 \right\}^T
$$

$$
\begin{aligned}
\mathbf{e}(\lambda_{0,1}) = &\left\{ \frac{5\left[12i_0^2\ln 2 - 5p^2\right]}{4\left[15i_0^2\pi + 8p^2\right]}Q_{y0}, \frac{1}{2}Q_{y0}, -\frac{p\left[60i_0^2(\ln 2 + \pi) + 7p^2\right]}{8\left[15i_0^2\pi + 8p^2\right]}Q_{y0}, \right. \\
&\left. 0, 0, \frac{p\left[p^4 + 8p^2i_0^2(6\pi + 25\ln 2 + 16) - 60i_0^4\left(\pi^2 - 4\pi + 4(\ln 2)^2\right)\right]}{32EI_{z0}\left[15i_0^2\pi + 8p^2\right]}Q_{y0} \right\}^T
\end{aligned}
$$

Neglecting the axial deformation, the differential system Eq. (13), yields:

$$
\begin{aligned}
D_\lambda V_x &= 0 \\
D_\lambda V_y &= 0 \\
-p\lambda V_x + pV_y + D_\lambda M_z &= 0 \\
-\frac{p}{EI_{z0}}M_z + D_\lambda \theta_z &= 0 \\
+p\lambda\theta_z + D_\lambda \delta_x &= 0 \\
-p\theta_z + D_\lambda \delta_y &= 0
\end{aligned}
\tag{15}
$$

Its direct integration, row by row, gives:

$$
\begin{aligned}
V_x(\lambda) &= C_1 \\
V_y(\lambda) &= C_2 \\
M_z(\lambda) &= \frac{p\lambda^2}{2}C_1 - p\lambda C_2 + C_3 \\
\theta_z(\lambda) &= \frac{p^2\lambda^3}{6EI_{z0}}C_1 - \frac{p^2\lambda^2}{2EI_{z0}}C_2 + \frac{p\lambda}{EI_{z0}}C_3 + C_4 \\
\delta_x(\lambda) &= -\frac{p^3\lambda^5}{30EI_{z0}}C_1 + \frac{p^3\lambda^4}{8EI_{z0}}C_2 - \frac{p^2\lambda^3}{3EI_{z0}}C_3 - \frac{p\lambda^2}{2}C_4 + C_5 \\
\delta_y(\lambda) &= \frac{p^3\lambda^4}{24EI_{z0}}C_1 - \frac{p^3\lambda^3}{6EI_{z0}}C_2 + \frac{p^2\lambda^2}{2EI_{z0}}C_3 + p\lambda C_4 + C_6
\end{aligned}
\tag{16}
$$

In the same way, values at both ends of the first part of the curve are obtained:

$$\mathbf{e}(\lambda_1)=\left\{-\frac{25}{32}Q_{y0},\frac{1}{2}Q_{y0},0,\frac{-p^2}{96EI_{z0}}Q_{y0},0,0\right\}^T;\ \mathbf{e}(\lambda_{0,1})=\left\{-\frac{25}{32}Q_{y0},\frac{1}{2}Q_{y0},-\frac{7p}{64}Q_{y0},0,0,\frac{p^3}{256EI_{z0}}Q_{y0}\right\}^T$$

In next Figure 3, results of forces, moment, rotation and displacements, particularized for data given in Benedetti and Tralli [1],

$l=42\text{m}$; $f=10.5\text{m}$; $b_0=4\text{m}$; $h=1\text{m}$; $E=10000\text{MPa}$; $Q_{y0}=200\text{MN}$,

are plotted with or without axial deformation:

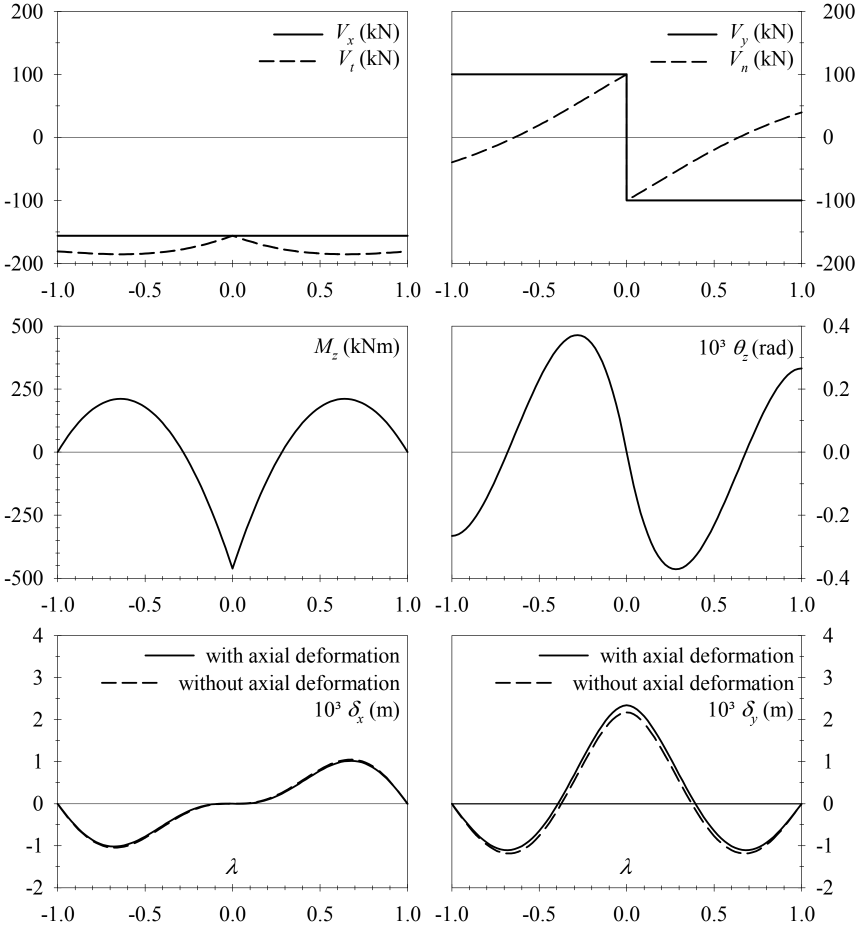

Figure 3: Forces, moment, rotation and displacements of the parabolic arch.

This graph shows how the hypothesis of neglecting axial deformation is acceptable.

4.2. Parabolic arch with constant cross-section

The Cartesian equation that represents the geometry of a parabolic arch in terms of the height *f* and span *l* is given by: $y = f/l^2\left[l^2 - 4x^2\right]$ (see Figure 4).

In parametric equations: $x(\lambda) = p\lambda$, $y(\lambda) = f - p\lambda^2/2$ and $z(\lambda) = 0$.

being $p = l^2/8f$ and $\lambda = 8fx/l^2$.

Direction cosines of the curve are $\upsilon_{tx} = 1/\sqrt{\lambda^2+1}$ and $\upsilon_{ty} = -\lambda/\sqrt{\lambda^2+1}$.

The derivative of the arc length *s* with respect to the parameter λ is $D_\lambda s = p\sqrt{\lambda^2+1}$.

A distributed load is applied q_y in the y direction and the section remains constant.

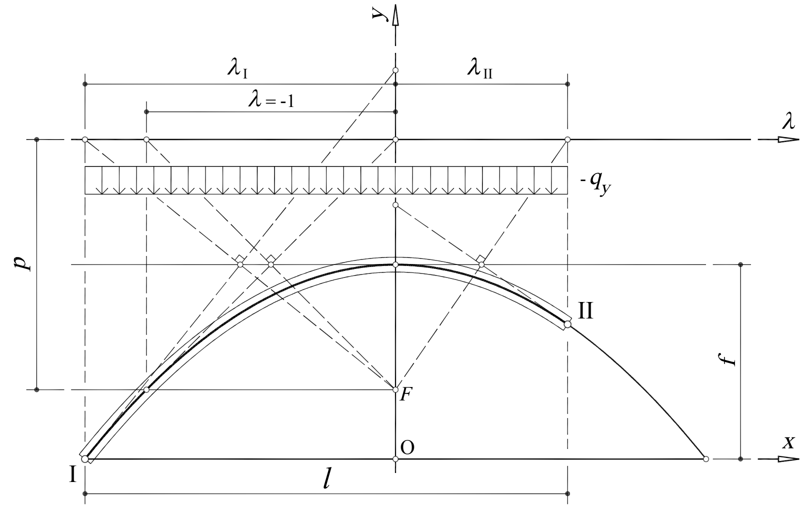

Figure 4: Parabolic arch with constant cross-section vertically loaded.

The differential system Eq. (4) in this example, will be:

$$\begin{array}{llllll}
D_\lambda V_x & & & & & = 0 \\
 & D_\lambda V_y & & & - pq_y & = 0 \\
- p\lambda V_x & - pV_y & + D_\lambda M_z & & & = 0 \\
 & & - \dfrac{p\sqrt{\lambda^2+1}}{EI_z} M_z + D_\lambda \theta_z & & & = 0 \\
- \dfrac{p}{\sqrt{\lambda^2+1}EA} V_x & + \dfrac{p\lambda}{\sqrt{\lambda^2+1}EA} V_y & & - p\lambda\theta_z + D_\lambda \delta_x & & = 0 \\
+ \dfrac{p\lambda}{\sqrt{\lambda^2+1}EA} V_x & - \dfrac{p\lambda^2}{\sqrt{\lambda^2+1}EA} V_y & & - p\theta_z & + D_\lambda \delta_y & = 0
\end{array} \tag{17}$$

Its direct integration, row by row, gives:

$$\begin{aligned}
V_x(\lambda) &= C_1 \\
V_y(\lambda) &= C_2 + p\lambda q_y \\
M_z(\lambda) &= -\frac{p\lambda^2}{2}C_1 - p\lambda C_2 + C_3 + \frac{p^2\lambda^3}{3}q_y \\
\theta_z(\lambda) &= -\frac{p^2\alpha_1(\lambda)}{16EI_z}C_1 - \frac{p^2\alpha_2(\lambda)}{3EI_z}C_2 + \frac{p\alpha_3(\lambda)}{2EI_z}C_3 + C_4 \quad \frac{p^3\alpha_{10}(\lambda)}{45EI_z}q_y \\
\delta_x(\lambda) &= \frac{p^3\alpha_4(\lambda)}{192EI_z}C_1 - \frac{p^3\alpha_5(\lambda)}{15EI_z}C_2 + \frac{p^2\alpha_6(\lambda)}{16EI_z}C_3 + \frac{p\lambda^2}{2}C_4 + C_5 + \frac{p^4\alpha_{11}(\lambda)}{630EI_z}q_y \\
\delta_y(\lambda) &= -\frac{p^3\alpha_7(\lambda)}{240EI_z}C_1 + \frac{p^3\alpha_8(\lambda)}{24EI_z}C_2 + \frac{p^2\alpha_9(\lambda)}{6EI_z}C_3 + p\lambda C_4 + C_6 - \frac{p^4\alpha_{12}(\lambda)}{720EI_z}q_y
\end{aligned} \qquad (18)$$

where,

$$\alpha_1(\lambda) = \left(2\lambda^2+1\right)\lambda\sqrt{\lambda^2+1} - \ln\left(\lambda+\sqrt{\lambda^2+1}\right)$$

$$\alpha_2(\lambda) = \sqrt{(\lambda^2+1)^3}$$

$$\alpha_3(\lambda) = \lambda\sqrt{\lambda^2+1} + \ln\left(\lambda+\sqrt{\lambda^2+1}\right)$$

$$\alpha_4(\lambda) = -\left(4\lambda^4+4\lambda^2+3\right)\lambda\sqrt{\lambda^2+1} + 3\left(2\lambda^2+64\,i_z^2/p^2+1\right)\ln\left(\lambda+\sqrt{\lambda^2+1}\right)$$

$$\alpha_5(\lambda) = \left(\lambda^4+2\lambda^2+15\,i_z^2/p^2+1\right)\sqrt{\lambda^2+1}$$

$$\alpha_6(\lambda) = \left(2\lambda^2-1\right)\lambda\sqrt{\lambda^2+1} + \left(4\lambda^2+1\right)\ln\left(\lambda+\sqrt{\lambda^2+1}\right)$$

$$\alpha_7(\lambda) = \left(6\lambda^4+7\lambda^2+16\left(15\,i_z^2/p^2+1\right)\right)\sqrt{\lambda^2+1} - 15\lambda\ln\left(\lambda+\sqrt{\lambda^2+1}\right)$$

$$\alpha_8(\lambda) = \left(-2\lambda^2+12\,i_z^2/p^2-5\right)\lambda\sqrt{\lambda^2+1} - \left(12\,i_z^2/p^2+3\right)\ln\left(\lambda+\sqrt{\lambda^2+1}\right)$$

$$\alpha_9(\lambda) = \left(\lambda^2-2\right)\sqrt{\lambda^2+1} + 3\lambda\ln\left(\lambda+\sqrt{\lambda^2+1}\right)$$

$$\alpha_{10}(\lambda) = \left(3\lambda^2-2\right)\sqrt{(\lambda^2+1)^3}$$

$$\alpha_{11}(\lambda) = \left(6\lambda^4-2\lambda^2-315\,i_z^2/p^2\,\lambda-8\right)\sqrt{\lambda^2+1} + 315\,i_z^2/p^2\ln\left(\lambda+\sqrt{\lambda^2+1}\right)$$

$$\alpha_{12}(\lambda) = \left(8\lambda^5+6\lambda^3-240\,i_z^2/p^2\,\lambda^2-17\lambda+480\,i_z^2/p^2\right)\sqrt{\lambda^2+1} - 15\ln\left(\lambda+\sqrt{\lambda^2+1}\right)$$

with $i_z^2 = I_z/A$.

The general solution in function of the arbitrary coefficients is obtained (Eq. (18) equivalent to Eq. (8)) integrating row by row the differential system (Eq. (17) equivalent to Eq. (4)). Then, support conditions could be applied to determine the particular solution, as it happens in a straight beam.

4. Conclusions

Normally, authors use the mobile Frenet system of reference with flexure and torsion curvatures to approach the structural problem of curved beams elements, when trying to reach exact analytical results. Otherwise, they can use different numerical approximations or simplifications in geometry (considering the curved composed of straight beams) to obtain acceptable results.

In this article, this problem is approached analytically with differential equations and solved using the global Cartesian reference system. The system of equations to determine the internal forces and displacements of curved elements is presented. The method considers in general, a twisted element with varying cross-sectional area with generalized loads and different boundary conditions.

Frenet frame P_{tnb} moves along the curve and changes its direction, which makes the equations of the differential system to be assembled (Gimena *et al.* [3]).

Directions in the global reference system P_{xyz} do not vary. Equilibrium, constitutive and kinematics relations permit to obtain a unique system of twelve linear ordinary differential equations (Eq. (3)). It is important to note the strict order of the twelve functions: forces produce moments, moments produce rotations and rotations produce displacements, in terms of the load applied. All functions are interconnected.

The principal advantage of the proposed model is that this differential system has lower-triangular form (see Eq. 3, 4, 13, 15 and 17), which permits its successive direct integration, row by row, obtaining the general solution straightforward.

Traditional analytical method can be applied to obtain exact results, and solve the problem irrespective of the boundary conditions applied, that could be statically indeterminate or not. No need to define or use energy procedures to formulate or solve this structural problem was necessary.

Analytical transfer matrix is derived directly. Displacements are obtained in global coordinates δ_x, δ_y and δ_z, which have more physical sense than displacement components in natural coordinates δ_t, δ_n and δ_b.

The general arch formulation is given and then particularized for parabolic arches.

Exact analytical solutions are given and can be computed to compare different shapes and conditions of arches. Graphs of accurate results of components of internal forces, moments, rotations and displacements presented in the literature are provided for verification purposes.

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