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Exact manual analysis of space frames by decomposition of the loads

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Abstract

This paper presents a method of equilibrium that calculates the deflection of certain space frames in an exact manner, without needing to solve equation systems. For this purpose, the unknown quantity of the deflection is considered to be a combination of other partial deflections that can be immediately determined and that are subject to some partial conditions of the loads, the addition of which constitutes the load conditions of the problem. The procedure to calculate the partial conditions of the loads considers the hypotheses of manual calculations. An example is carried out to complete this exposition.

Keywords: Analysis, Space frames, Equilibrium, Manual, Exact

1. Introduction

Under a linear system and according to the first-order theory, the present manual method of equilibrium analyses rigid space frames having rigid nodes with an independent displacement when punctual static actions act on their nodes such as the example in fig.1*a*. The deflection of the structure has been considered as the addition of partial deflections, each one of them being determined by the independent movement of any of its nodes. The load condition that produces a partial deflection has been called "primary state", which has been defined by means of an active action and several restrictive ones. The active action applied on a node *N* can be a momentum M_N^a or a force F_N^a, which cause a rotation θ_N or displacement δ_N of the structure, the values of which in an *X* direction are respectively:

$$\theta_N^x = \frac{M_N^{ax}}{\sum\left(\frac{4EI}{L}\right)_i + \sum\left(\frac{GI}{L}\right)_j} \qquad \delta_N^x = \frac{F_N^{ax}}{\sum\left(\frac{12EI}{L^3}\right)_k} \tag{1}$$

The denominator in the first equation is the addition of the flexural and torsional rigidities of all the *i* and *j* beams that concur in *N*, respectively, in relation to *X*. The denominator in the second equation is the addition of the shear forces that are produced in the *k* beams that rotate due to δ_N^x. Fig.1*b* shows the rigidities (torsional and flexural) of three concurrent beams at any *N* node and the location of its three possible rotations. In accordance with fig.1*b,* fig.1*c* shows a sketch of the previous space frame (fig.1*a*) where the values of the expressions (1) in all the nodes have been calculated on the basis of the rigidities of the beams when the active actions are equal to one. In a primary state, the restrictive actions of *N* (M_N^r, F_N^r) depend numerically on the active action and prevent the movement of all the nodes, except for that produced by the active action (figs.2*b,c,d*). Therefore, an *A* load condition of the space frame is a *B* group of primary states expressed in function of the active actions, which when applied in (1), give the deflection. The objective of this exposition is to assess M_N^a and F_N^a. For this purpose, we begin with a *C* group of primary states that have been chosen specifically. Based on this, we calculate new *D* primary states, which, added to *C*, define a load condition that increasingly approaches the condition *A*. The *D* primary states are arranged into groups, each of which forms an iteration. The deflection is determined with the active actions of *C*+*D*.

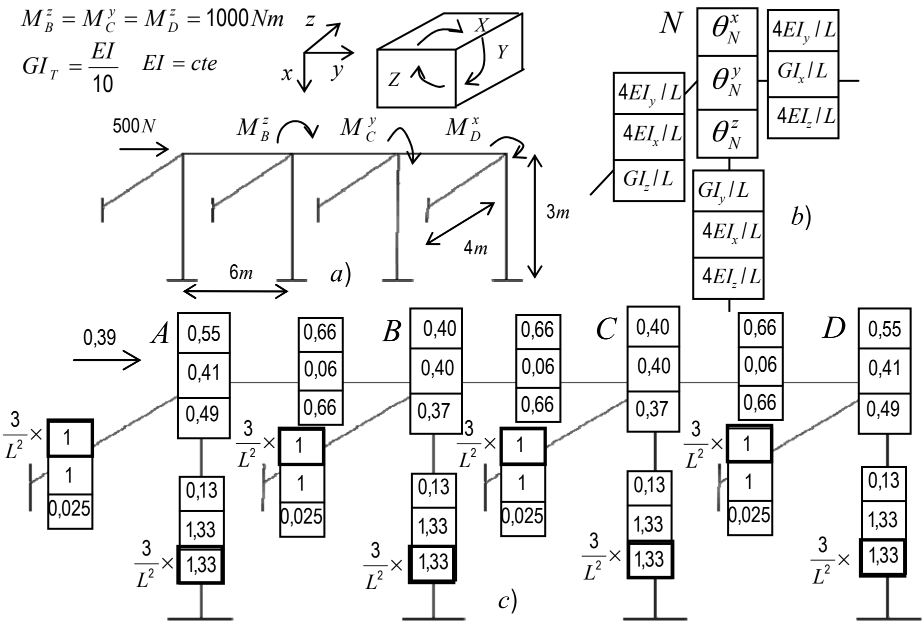

Figure 1. Space frame with beams of square section: *a*) Geometry, applied actions and criterion of positive signs; *b*) Diagram to calculate rotations on a node; *c*) Diagram of the frame with the rotations and the displacement in f(1/*EI*) by active unitary actions.

Previously, two other ways of calculating the active actions of condition *A* were proposed in [1] and [2]. These were calculated approximately, by means of iterations, as it is described in [1]. The proposal seemed adequate provided that there were few nodes in the space frame and that the frame was undisplaceable or had an only displacement.

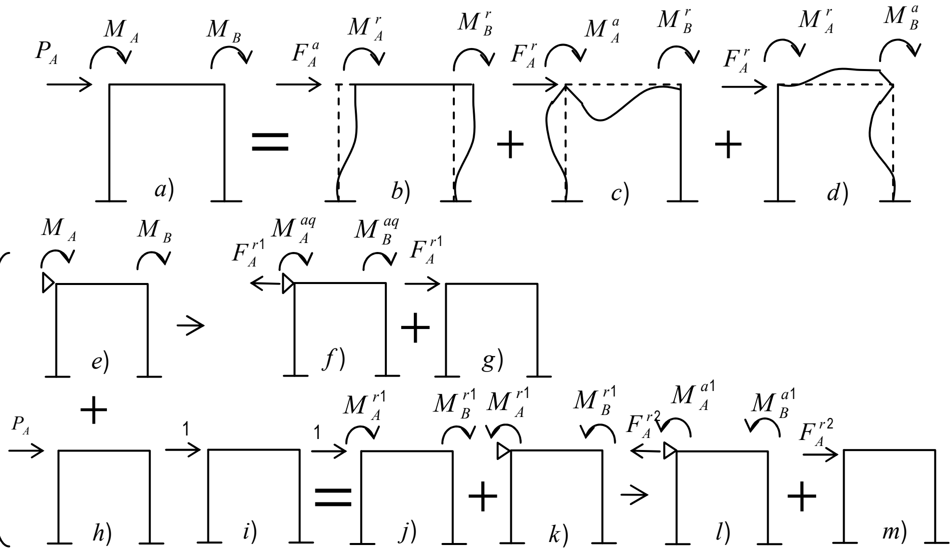

Figure 2. Analysis of a portico by means of decomposition of the loads: *a-d*) Decomposition under primary states; *e-m*) Calculation of the active actions.

A subsequent modification [2] allowed an exact calculation of the active forces on the basis of the active momentums, obtained approximately by means of a smaller number of iterations. This way, it was possible to analyse structures with several displacements and a greater number of nodes. The process applied to fig.2*a* consists of decomposing the load condition into two conditions (fig.2*e* and fig.2*h*). The momentums of fig.2*e* are replaced in fig.2*f* by its active momentums M_A^{aq}, M_B^{aq}. F_A^{r1} is applied on the fictitious support, which is eliminated adding the load condition in fig.2*g*. The active actions of fig.2*g* and fig.2*h* are calculated from fig.2*i*, which is decomposed into figs.2*j* and 2*k*. If we solve fig.2*k* like fig.2*e*, we obtain F_A^{r2} (fig.2*l*), which is smaller than the unit. F_A^{r2} is cancelled adding fig.2*m*, that is "F_A^{r2}" times fig.2*i*. Repeating these operations, we obtain the active force F_A^{a1} of fig.2*i*:

$$F_A^{a1} = 1 + F_A^{r2} + (F_A^{r2})^2 + ... = \frac{1}{1 - F_A^{r2}} \tag{2}$$

, from which are obtained the total active actions:

$$F_A^a = (P_A + F_A^{r1}).F_A^{a1}$$
$$M_A^a = M_A^{aq} + M_A^{a1}.F_A^{a1} \qquad (3)$$
$$M_B^a = M_B^{aq} + M_B^{a1}.F_A^{a1}$$

In this paper, a complementary methodology *a* [2] is presented that determines the exact values of the active momentums. To begin with, an explanation of how to obtain the active momentums of a continuous beam with four points of support (fig.3*a)* is given, since the typology of the space frames is comparable to a continuous beam with horizontal displacement. After this, the space frame in fig.1*a* is studied by means of some mnemotechnic rules different from to those used in [1] and [2]. The error that can be made is only caused by the rounding off resulting from the manual performance of the operations.

2. Exact calculation of the active momentums in a continuous fixed-ended beam with four points of support

They are obtained by adding the active momentums produced by each *M* momentum of the load condition. The active and restrictive momentums are designed, respectively as, M_{IJ}^a and M_{IJ}^r. They are caused by an *M* momentum on a generic point of support *I* and are applied on a node *J*. In sections 2.1 and 2.2 the calculation of M_{IJ}^a is explained in function of the position of *I*. In both sections, it has been considered that *M* is equal to the unit. In 2.3, a practical procedure is explained to calculate M_{IJ}^a.

2.1. Calculation of the M_{IJ}^a when *I* is a lateral support (fig.3)

The M_{IJ}^a momentums of fig.3*a* are calculated from the active momentums of the partial beams in figs.3*b,c,d* when some unitary momentums are applied on the lateral supports *B,C,* and *D*. By means of an iterative procedure, the exact value of M_{II}^a of each partial beam is calculated adding up all the possible iterations. As described in (2), said sum is $\frac{1}{1-r_I}$, where r_I is the ratio that relates two consecutive iterations. r_I is smaller than the unit and it is calculated applying the following load conditions to the unloaded beam:

-Load condition 1. It is a primary state formed by a unitary active momentum on *I* and by another restrictive momentum on the adjacent point of support.

-Load condition 2. It can be a primary state or not. It eliminates the restrictive momentums of load condition 1 and applies a restrictive momentum on *I*, with value r_I.

-Load condition 3. It is "r_I" times load condition 1, and, therefore:

$$M_{II}^a = 1 + r_I + r_I^2 + ... = \frac{1}{1-r_I} \qquad (4)$$

When *K* and *L* are any two contiguous nodes, *KL* is defined as the restrictive momentum due to a unitary momentum applied on a node *K* and is located on a node *L*. Its value is $\theta_K^1 \left(\frac{2EI}{L} \right)_{KL}$, θ_K^1 being the rotation of *K* by the unitary momentum. Next, we calculate the M_{IJ}^a of the beams in figs.3*b,c,d*:

2.1.1. Partial beam 1: Beam with a point of support having M=1 in B (fig.3b)

It is a particular case where $M = M_{BB}^a = 1$. M_{BC}^r is worth BC.

2.1.2. Partial beam 2: Beam with two points of support having M=1 in C (fig.3c)

-Load condition 1. It is defined with a unitary momentum on *C* and with a restrictive one *CB*

-Load condition 2. It has an active momentum CB that eliminates the restrictive one in load condition 1. It also has a restrictive momentum *CB.BC*.

-Load condition 3. It is "$CB.BC$" times load condition 1. $CB.BC = r_C$, and the M_{IJ}^a are:

$$M_{CC}^a = 1 + r_C + r_C^2 + ... = \frac{1}{1 - r_C}$$
$$M_{CB}^a = -(1 + r_C + r_C^2 + ...).CB = -M_{CC}^a.CB \tag{5}$$

M_{CB}^a is negative since it is the opposite of the restrictive momentum in *B*. The factors "CD" and "BA" multiplied by M_{CC}^a and M_{CB}^a are, respectively, M_{CD}^r and M_{BA}^r. These factors are represented under the directrix of the beam and, under the nodes, M_{IJ}^a.

2.1.3. Partial beam 3: Beam with three points of support having M=1 in D (fig.3d)

-Load condition 1. It is formed by a unitary momentum on *D* and by another restrictive one *DC*.

-Load condition 2. It is formed by a *DC* momentum and by another restrictive one *CD.DC*. In the position of equilibrium, this condition produces on the beam some active momentums that are "DC" times those of fig.1.*c*, and, the momentum in *D* is $DC(M_{CC}^a CD)$.

-Load condition 3. It is "$DC(M_{CC}^a CD)$" times load condition 1. $DC(M_{CC}^a CD) = r_D$ and, according to (4), $M_{DD}^a = \frac{1}{1 - r_D}$. The active momentums to the right of *D* are worth "$-M_{DD}^a DC$" times those in fig.1.*c*, whereas the restrictive momentum in *E* is worth $M_{DD}^a DE$.

2.1.4. Beam with four points of support having M=1 in E (fig.3e)

The approach is repeated in the same manner as in the previous cases. Load condition 3 is "$ED(M^{a}_{DD}DE)$" times load condition 1. Therefore, $ED(M^{a}_{DD}DE) = r_E$ and $M^{a}_{EE} = \dfrac{1}{1-r_E}$.

From M^{a}_{EE} we have calculated the active momentums in the rest of the points of support, which depend on those in fig.3*d*. When the unitary momentum operates on the extreme right point of support, the way of proceeding is the same (figs.3*f*-*i*).

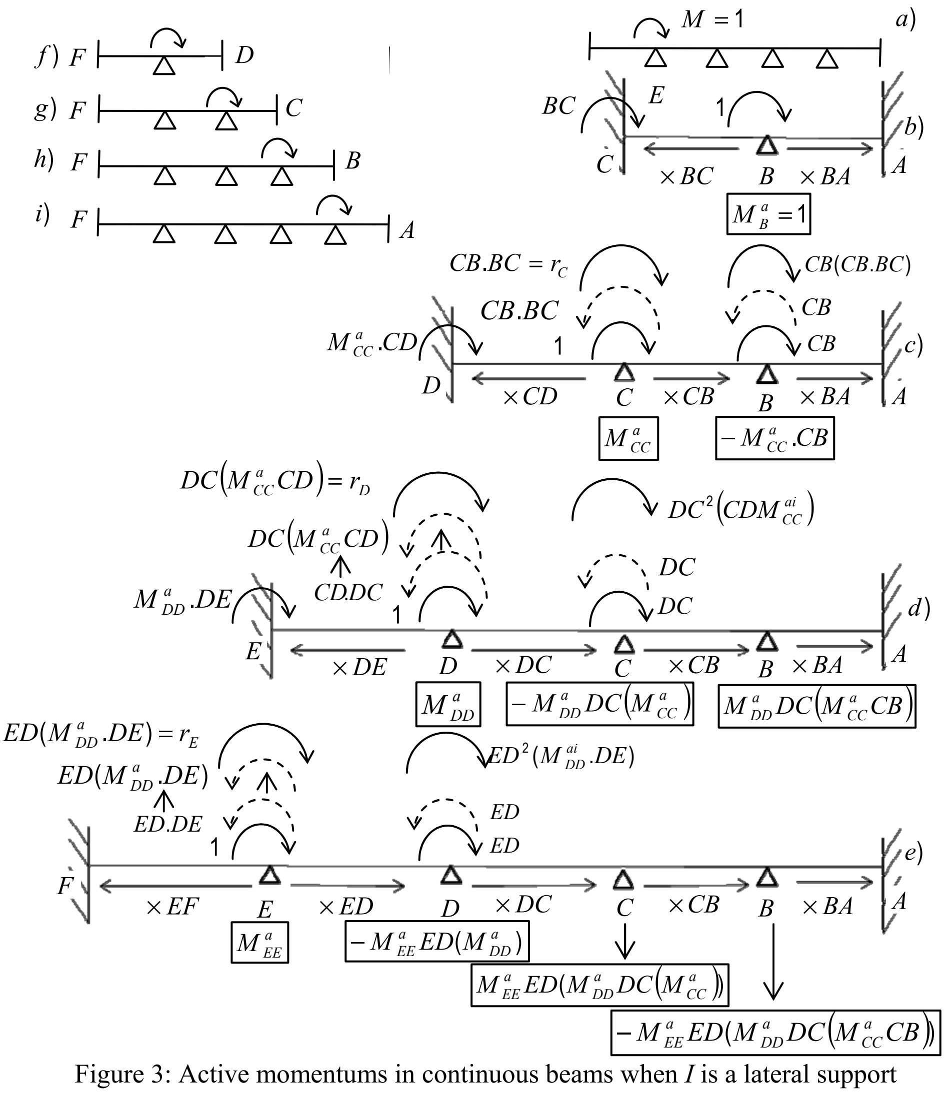

Figure 3: Active momentums in continuous beams when *I* is a lateral support

2.2. Calculation of the M^a_{IJ} when *I* is an intermediate point of support (fig.4)

The M^a_{IJ} are obtained from the active momentums calculated in 2.1. For example, to determine the M^a_{IJ} produced by a unitary momentum on *D* (fig.4*a*), the following load conditions are applied on the unloaded beam (fig.4*b*):

-Load condition 1. It is formed by a unitary active momentum on *D* y and by two restrictive ones *DE* and *DC*.

-Load condition 2. It is not a primary state. It cancels the restrictive momentums of load condition 1 and applies a restrictive momentum on *D,* which is the R_D addition of momentums R^i_D and R^d_D that are produced in figs.4*c*,*d*, calculated, respectively, from figs.3*f*,*c*.

-Load condition 3. It is "R_D" times load condition 1 and therefore: $M^a_{DD} = \dfrac{1}{1-R_D}$.

The active momentums to the right of *D* are "$-M^a_{DD}.DC$" times those in fig.3*c*, and the active momentum on *E* is "$-M^a_{DD}.DE$" times the one in fig.3*f*.

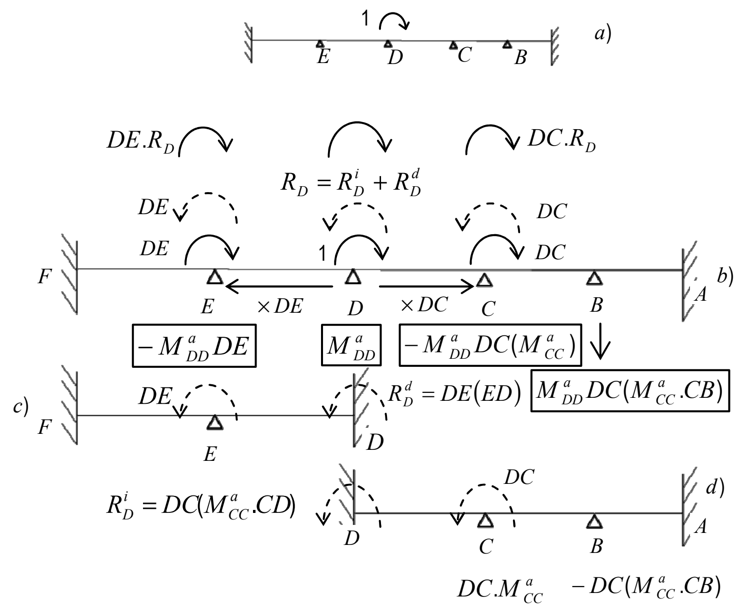

Figure 4. Active momentums on continuous beams if *I* is an intermediate point of support

2.3. Systematic calculation of the M^a_{IJ}.

The operations described in 2.1 and 2.2 can be performed systematically using a calculation table together with some mnemotechnic rules. They have been used to calculate the active momentums of the beam with four points of support. The following steps have been carried out:

2.3.1. Representation of the beam and of the calculation table (fig.5)

-Draw the beam, indicating its flexural rigidities under the sections.

-Calculate $-0{,}5\theta_N^1$ under each N node, using (1).

-Draw a table under $-0{,}5\theta_N^1$ (fig.5*a*). The contents of the boxes/cells are described to the right of each row. Fig.5*b* shows the numbering of the rows and columns in the table.

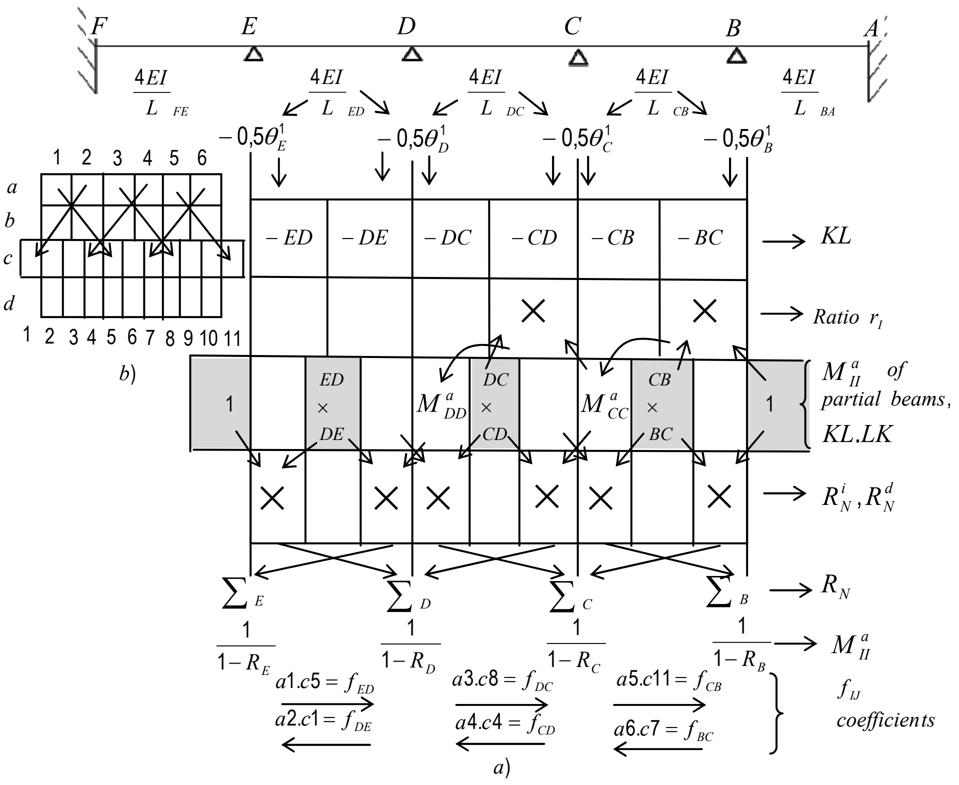

Figure 5. Calculation table of active momentums of a continuous beam: *a*) Diagram of the table and contents of the boxes; *b*) Numbering of the rows and columns and situation of the boxes that calculate the *f* coefficients

2.3.2. Filling in the table using mnemotechnic rules

-Arrange the boxes in row *a*, multiplying the rigidity of each beam by $-0{,}5\theta_N^1$, according to what is indicated by the arrows.

-Multiply between themselves the terms of row *a* that are comprised under each beam and arrange them in boxes *c3*, *c6* and *c9*. Place a "1" in cells *c*1 and *c*11. Multiply box *c*11 by *c*9 and put the result in *b*6, according to the indications of the arrows. Calculate $\frac{1}{1-b6} = M^{a}_{CC}$ and put the result in *c*8. Repeat this operation multiplying *c*8 by *c*6 and place it in *b*4. Calculate $\frac{1}{1-b4} = M^{a}_{DD}$ and put the result in *c*5. Repeat this operation, parting from *c*1 and *c*3, until reaching box *c*7.

-Multiply between themselves the *c* boxes, according to the arrows and place the results in cells *d*2,4,5,7,8 and 10.

-Add the *d* boxes related to the diagonals in the diagram.

-Calculate with each R_N the expression $\frac{1}{1-R_N}$ and put the result under R_N. These values are the M^{a}_{II} in any of the beam's point of support.

The M^{a}_{IJ} are calculated with M^{a}_{II} and with the f_{IJ} coefficients, by means of the expression $M^{a}_{II} \cdot f_{IJ} = M^{a}_{IJ}$. The f_{IJ} coefficients are obtained multiplying between themselves the boxes of rows *a* and *c*, which are related by arrows in the diagram of fig.5*b*. The following $[A]$ matrix is calculated with the values of M^{a}_{II} and f_{IJ}:

$$[A] = \begin{bmatrix} M^{a}_{EE} & M^{a}_{EE} f_{ED} & M^{a}_{EE} f_{ED} f_{DC} & M^{a}_{EE} f_{ED} f_{DC} f_{CB} \\ M^{a}_{DD} f_{DE} & M^{a}_{DD} & M^{a}_{DD} f_{DC} & M^{a}_{DD} f_{DC} f_{CB} \\ M^{a}_{CC} f_{CD} f_{DE} & M^{a}_{CC} f_{CD} & M^{a}_{CC} & M^{a}_{CC} f_{CB} \\ M^{a}_{BB} f_{BC} f_{CD} f_{DE} & M^{a}_{BB} f_{BC} f_{CD} & M^{a}_{BB} f_{BC} & M^{a}_{BB} \end{bmatrix} = \begin{bmatrix} M^{a}_{EE} & M^{a}_{ED} & M^{a}_{EC} & M^{a}_{EB} \\ M^{a}_{DE} & M^{a}_{DD} & M^{a}_{DC} & M^{a}_{DB} \\ M^{a}_{CE} & M^{a}_{CD} & M^{a}_{CC} & M^{a}_{CB} \\ M^{a}_{BE} & M^{a}_{BD} & M^{a}_{BC} & M^{a}_{BB} \end{bmatrix} \quad (6)$$

Each row represents the active momentums produced on the points of support by a unitary momentum applied on one of them, and each column represents the total active momentum that is produced on a point of support. If we multiply the load matrix $[B]$ by $[A]$, the exact active momentums are finally obtained:

$$[B] \times [A] = [M_E \quad M_D \quad M_C \quad M_B] \times \begin{bmatrix} M^{a}_{EE} & M^{a}_{ED} & M^{a}_{EC} & M^{a}_{EB} \\ M^{a}_{DE} & M^{a}_{DD} & M^{a}_{DC} & M^{a}_{DB} \\ M^{a}_{CE} & M^{a}_{CD} & M^{a}_{CC} & M^{a}_{CB} \\ M^{a}_{BE} & M^{a}_{BD} & M^{a}_{BC} & M^{a}_{BB} \end{bmatrix} = \left[M^{a}_{E} \quad M^{a}_{D} \quad M^{a}_{C} \quad M^{a}_{B}\right] \quad (7)$$

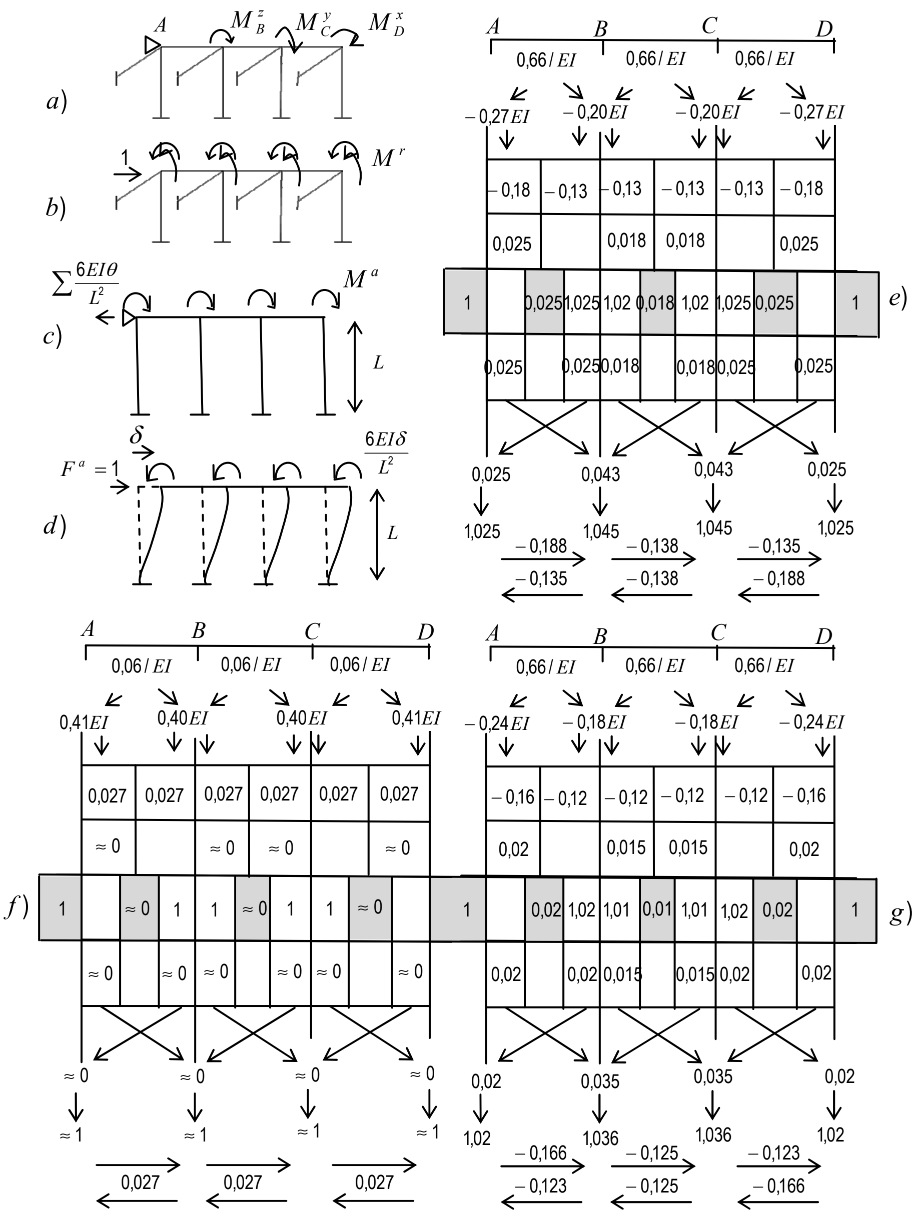

Figure 6. Calculation of the active actions in a space frame: *a,b*) Initial conditions; *c,d*) Restrictive actions on a plane from M^a and F^a defined with mnemotechnic rules; *e,f,g*) Calculation tables of active momentums on planes *OX*, *OY* y *OZ*, respectively

3. Calculation of the exact deflection of a space frame (fig.6)

The deflection in fig.1*a* is calculated like in fig.2. According to figs.2.*e,j*, we start from figs.6.*a,b*. With the mnemotechnic rule described in fig.6.*c*, and using (1), we obtain the restrictive action in *A* by generic active momentums applied on a plane of the space. In (8) we see the results when the active momentums are found, respectively, on planes *OX* and *OZ*.

$$F_A^{r1} = \begin{bmatrix} M_A^a & M_B^a & M_C^a & M_D^a \end{bmatrix}^x \times \begin{bmatrix} 0{,}20 \\ 0{,}15 \\ 0{,}15 \\ 0{,}20 \end{bmatrix} \qquad F_A^{r2} = \begin{bmatrix} M_A^a & M_B^a & M_C^a & M_D^a \end{bmatrix}^z \times \begin{bmatrix} 0{,}32 \\ 0{,}24 \\ 0{,}24 \\ 0{,}32 \end{bmatrix} \tag{8}$$

Fig.6.*d* describes a mnemotechnic rule that obtains, using (1), the restrictive momentums on a plane caused by an active unitary force applied on *A*. The restrictive momentums obtained in fig.6*b* are the following:

$$\begin{bmatrix} \begin{bmatrix} M_A^r & M_B^r & M_C^r & M_D^r \end{bmatrix}^x \\ \begin{bmatrix} M_A^r & M_B^r & M_C^r & M_D^r \end{bmatrix}^z \end{bmatrix} = \begin{bmatrix} [0{,}14 & 0{,}14 & 0{,}14 & 0{,}14] \\ [0{,}26 & 0{,}26 & 0{,}26 & 0{,}26] \end{bmatrix} \tag{9}$$

Figs 6.*e,f,g* are tables to determine the exact active momentums on the directions of the space *X*, *Y*, *Z*. The torsion has been considered in figure 6*f.* The data obtained are collected in the matrixes $[A]^x, [A]^y, [A]^z$ of (10). The active momentums produced by the momentums of figs.6*a,b* have been calculated in (10), in accordance with (7).

$$\begin{aligned}
[B]^x \times [A]^x &= \begin{bmatrix} 0 & 0 & 0 & 1000 \\ 0{,}14 & 0{,}14 & 0{,}14 & 0{,}14 \end{bmatrix} \times \begin{bmatrix} 1{,}2 & -0{,}2 & 0{,}02 & 0 \\ -0{,}14 & 1{,}04 & -0{,}14 & 0{,}02 \\ 0{,}02 & -0{,}14 & 1{,}04 & -0{,}14 \\ 0 & 0{,}02 & -0{,}19 & 1{,}02 \end{bmatrix} = \begin{bmatrix} -3 & 26 & -192 & 1025 \\ 0{,}13 & 0{,}10 & 0{,}10 & 0{,}13 \end{bmatrix} = \begin{bmatrix} [M^{aq}] \\ [M^{a1}] \end{bmatrix}^x \\
[B]^y \times [A]^y &= [0 \quad 0 \quad 1000 \quad 0] \times \begin{bmatrix} 1 & 0{,}02 & 0 & 0 \\ 0{,}02 & 1 & 0{,}02 & 0 \\ 0 & 0{,}02 & 1 & 0{,}02 \\ 0 & 0 & 0{,}02 & 1 \end{bmatrix} = [0 \quad 27 \quad 1000 \quad 27] = [M^{aq}]^y \\
[B]^z \times [A]^z &= \begin{bmatrix} 1 & 1000 & 0 & 0 \\ 0{,}26 & 0{,}26 & 0{,}26 & 0{,}26 \end{bmatrix} \times \begin{bmatrix} 1{,}02 & -0{,}17 & 0{,}02 & 0 \\ -0{,}12 & 1{,}03 & -0{,}13 & 0{,}01 \\ 0{,}01 & -0{,}13 & 1{,}03 & -0{,}12 \\ 0 & 0{,}02 & -0{,}17 & 1{,}02 \end{bmatrix} = \begin{bmatrix} -127 & 1036 & -129 & 16 \\ 0{,}23 & 0{,}2 & 0{,}2 & 0{,}23 \end{bmatrix} = \begin{bmatrix} [M^{aq}] \\ [M^{a1}] \end{bmatrix}^z
\end{aligned} \tag{10}$$

If we apply the results obtained in (10) in (8), we obtain some reactions in *A*. Their value is:

$$\begin{bmatrix} -3 & 26 & -192 & 1025 \\ 0{,}13 & 0{,}10 & 0{,}10 & 0{,}13 \end{bmatrix} \times \begin{bmatrix} 0{,}20 \\ 0{,}15 \\ 0{,}15 \\ 0{,}20 \end{bmatrix} = \begin{bmatrix} 187{,}5 \\ 0{,}088 \end{bmatrix} = \begin{bmatrix} F_A^{r1} \\ F_A^{r2} \end{bmatrix} \qquad \begin{bmatrix} -127 & 1036 & -129 & 16 \\ 0{,}23 & 0{,}2 & 0{,}2 & 0{,}23 \end{bmatrix} \times \begin{bmatrix} 0{,}32 \\ 0{,}24 \\ 0{,}24 \\ 0{,}32 \end{bmatrix} = \begin{bmatrix} 187{,}5 \\ 0{,}255 \end{bmatrix} = \begin{bmatrix} F_A^{r3} \\ F_A^{r4} \end{bmatrix} \tag{11}$$

From here we obtain the total active force:

$$F_A^a = \left(F_A^{r1} + F_A^{r3} + 500\right) \frac{1}{1-(F_A^{r2} + F_A^{r4})} = 1333{,}53N \tag{12}$$

Applying F_A^a and the data of (11) in (3), we obtain the following results:

	$M_N^{ax}(N.m)$	$M_N^{ay}(N.m)$	$M_N^{az}(N.m)$	$\theta_N^x(1/EI)$	$\theta_N^y(1/EI)$	$\theta_N^z(1/EI)$
A	174,62	0	191,1	97,01	0	94,37
B	171,06	27	1302,1	69,34	10,94	483,78
C	-46,93	1000	137,1	-19,0	405,4	50,97
D	1202,62	27	334,1	668,1	11,2	164,98

Table 1. Final results: active momentums and rotations

4. Conclusions

It is possible to determine the exact deflection of certain space frames without having to solve equation systems and without doing a limited number of iterations. Comparing with previous methods, this approach described here is more adequate when the number of nodes is elevated.

5. Acknowledgement

To my friend and work colleague, Paula Macicior Ferrando, for her invaluable translation work of this paper into a concise technical English.

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