A way to optimally solve a time-dependent vehicle routing problem with time windows

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Abstract

In this paper we deal with a generalization of the Vehicle Routing Problem with Time Windows that considers time-dependent travel times and costs. Through several steps we transform this extension into an Asymmetric Capacitated Vehicle Routing Problem, so it can be solved both optimally and heuristically with known codes.

Keywords: Vehicle Routing, time window, time-dependent costs.

1 Introduction

The Vehicle Routing Problem with Time Windows (VRPTW) basically consists of finding a least cost set of routes made by a fleet of vehicles from a specified location (the depot) to a set of points geographically distributed (customers with a positive demand) such that: each route begins and ends at the depot, each customer is visited only once by exactly one vehicle within a given time window, all the vehicles have the same capacity and the total demand serviced by a vehicle must not exceed its capacity. The number of vehicles can be fixed a priori or left as a decision variable.

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Because of the great number of applications of the VRPTW in real-life distributions and scheduling problems, it has been widely studied, and due to its computational complexity, its study has been mainly focused on heuristic approaches. To be brief we will cite the relevant surveys [3], [4] and [6] and the recent papers [2] and [16].

Like in most routing problems studied in the OR literature, in the VRPTW the costs or times to go from one location to another are considered constant throughout the day. This assumption may be far from the reality in distribution problems inside big cities, where the time or cost of traversing some streets, like main avenues, depend on the moment of the day, for example the peak hours, with their corresponding traffic jams.

Routing problems with time-dependent costs have hardly been studied because they are more difficult to model and to solve. However, in the last few years several papers on vehicle routing problems have taken into account time-dependent travel costs. We may cite [5], [8],[10],[12],[13] and [17], that provide heuristic procedures for the solution of different time-dependent VRP models with different kinds of time-dependent travel times. Due to the characteristics of the problems (identical vehicles) most of these papers consider the "first-in-first-out" (FIFO) property, also called the "non-passing" property: if a vehicle leaves a vertex i for a vertex j at a given time, leaving vertex i for vertex j at a later time implies arriving later at vertex j.

In the case of a single vehicle problem, recently Albiach et al. ([1]) define the Asymmetric Traveling Salesman Problem with Time-Dependent Costs (ATSPTDC), an extension of the well-known Asymmetric Traveling Salesman Problem with Time Windows (that can be considered as a VRPTW with a single vehicle) in which the time and the cost of traversing an arc are time-dependent. The main difference with respect to the papers cited above is that Albiach et al. focus their work on optimal resolution; they optimally solve the ATSPTDC by its transformation into a classical Asymmetric TSP.

In this paper we present a generalization of the VRPTW that, following the names given in [5] and [8], we will call the Time-Dependent Vehicle Routing Problem with Time Windows (TDVRPTW), in which the time and the cost of traversing an arc depend on the period of time at which we start to traverse it, and in which some kinds of waiting times are allowed at the customer locations. Through several steps we transform the TDVRPTW into an Asymmetric Capacitated Vehicle Routing Problem (ACVRP), a VRP without time windows restrictions for which in addition to heuristic procedures like the one given in [18] or the more recent in [7], at least two exact procedures have been reported in [9] and [14]. Therefore, in contrast to the papers on time-dependent VRP cited above, we provide a way to optimally solve this problem, at least for small size instances due to its complexity.

To our aim we need to formally define the ACVRP:

Let $G = (V, A)$ *be a complete digraph,* $V = \{v_i\}_{i=0}^{n}$ *being its set of vertices, where* v_0 *is the depot vertex, each vertex* v_i *with* $i > 0$ *has associated a demand* $d_i > 0$, *and each arc* $(v_i, v_j) \in A$ *has associated a cost* $c_{i,j} \geq 0$. *Moreover, a fleet of vehicles with the same capacity* W *where* $W \geq d_i \ \forall i = 1, ..., n$ *is available at the depot.*

Find a set of shortest routes starting and ending at the depot such that each vertex $v_i \ \forall \ i \in \{1, ..., n\}$ *must be visited by one and only one vehicle and the sum of the demands of the vertices visited by each vehicle does not exceed* W.

To obtain our transformation, we will also use the Generalized Vehicle Routing Problem (GVRP), an extension of the ACVRP introduced by Ghiani and Improta ([11]) and defined as follows:

"*Let* $G = (V, A)$ *be a directed graph where the set of vertices* V *is divided into* $m+1$ *nonempty subsets* $S_0, S_1, ..., S_m$ *such that* S_0 *has only one vertex* v_0 *which represents the depot,* S_h $h = 1, ..., m$, *represents* $l(h)$ *possible locations of the same vertex which has associated a positive demand* d_i *and each arc* $(v_i, v_j) \in A$ *has associated a cost* $c_{i,j} \geq 0$. *Moreover, a fleet of vehicles with the same capacity* W *where* $W \geq d_i \ \forall i = 1, ..., m$ *is available at the depot.*

Find a set of shortest routes starting and ending at the depot such that in each subset S_i, $i = 1, ..., m$ *one and only one vertex is visited exactly once and the sum of the demands of every route does not exceed* W.

This paper is organized as follows. Section 2 gives the definition of the TDVRPTW as well as the construction of auxiliary digraphs from a TDVRPTW instance. Section 3 transforms the TDVRPTW first into a GVRP and then into an ACVRP, and Section 4 presents some final remarks about this work.

2 Definition and auxiliary digraphs

2.1 Definition of the TDVRPTW

We define the TDVRPTW as follows:

Let $G = (V, A)$ *be a directed complete graph,* $V = \{v_i\}_{i=0}^{n}$ *being its set of vertices, where* v_0 *is the depot vertex, each vertex* v_i *has associated a time window* $[a_i, b_i]$ *verifying that* $a_i, b_i \in \mathbb{Z}^+ \cup \{0\}$ *and* $[a_i, b_i] \subseteq [a_0, b_0]$ $\forall$ $i \in \{1, ..., n\}$*. Given* $p_i = b_i - a_i$*, the time window* $[a_i, b_i]$ *has associated* $p_i + 1$ *instants of time* $\{a_i + k\}_{k=0}^{p_i}$*. For simplicity we will denote* $t_i^k = a_i + k$ *(therefore* $t_i^k \in \mathbb{Z}^+ \cup \{0\}$*). Each instant of time* t_i^k *with* $i > 0$ *has associated a waiting time window* $[w_i^k, t_i^k]$*,* $w_i^k \in (\mathbb{Z}^+ \cup \{0\}) \cap [a_0, t_i^k]$ *and each vertex* v_i *has also associated a demand* $d_i > 0$ $\forall i > 0$*.*

On the other hand, the time and the cost of traversing an arc $(v_i, v_j) \in A$ *depend on the instant of time* t_i^k $(k \in \{0, 1, ..., p_i\})$ *at which we start traversing it. Let us denote by* $t_{i,j}^k \in \mathbb{Z}^+$ *and* $c_{i,j}^k \geq 0$ *the time and the cost respectively of traversing the arc* (v_i, v_j) *starting at instant* t_i^k*. Moreover, each waiting time* $t \in \mathbb{Z}^+$ *at each vertex* v_i *has associated a cost* $cwt_i(t) \geq 0$ *and the traversing arc times satisfy the FIFO property.*

For a fixed number of vehicles r *with identical capacity* $W > 0$ *such that* $W \geq d_i$ $\forall i$ *and* $rW \geq \sum_{i=1}^{n} d_i$*, the goal of the TDVRPTW is to find* r *cycles in* G *such that:*

- Each cycle starts and ends at the depot at integer instants of time inside $[a_0, b_0]$*. Starting a circuit at time* $t_0^k \geq a_0$ *involves a waiting time cost* $cwt_0(t_0^k - a_0)$ *with* $cwt_0(0) = 0$*.*

- Every vertex v_i *with* $i \in \{1, \ldots, n\}$ *must be visited by one and only one vehicle, that must leave vertex* v_i *inside its associated time window. If a circuit arrives at vertex* v_i *with* $i > 0$ *at time* $t \in \mathbb{Z}^+ \cap [w_i^k, t_i^k]$*, it is allowed a waiting time* $t_i^k - t$ *with cost* $cwt_i(t_i^k - t) \geq 0$ *(*$cwt_i(0) = 0$*) for all* $k \in \{0, 1, ..., p_i\}$ *if the circuit leaves* v_i *at time* t_i^k*.*

- The sum of the demands of the vertices visited by each vehicle does not exceed its capacity W.

- The sum of the costs of the r cycles be minimum, where the cost of a cycle is defined as the sum of its arc costs and of its waiting time costs.

Some relevant aspects of this definition are:

- This definition allows a vehicle to start its route after instant a_0 with a waiting time cost. This is very important to minimize the cost; for instance, if a_0 belongs to a peak hour, if possible, the worker can work for a short period of time inside the warehouse until the traffic be moving quite freely.

- In the same way, this definition also allows a waiting time at each customer location v_i, if due to the traffic conditions, it is preferable to wait in order to minimize the cost of the circuit. This waiting time has an associated cost which normally is given by a non-decreasing linear function. Note that in this case, $t_i^k - w_i^k$ indicates the maximum waiting time allowed if we want to leave vertex i at time t_i^k. Therefore, if we do not want to wait when $t_i^k > a_i$ (as it happens in the VRPTW), we only have to do $w_i^k = t_i^k$.

- As usual in routing problems, we suppose that if a service time is necessary at a vertex i with $i > 0$, this time is included in the travel times t_{ij}^k for all $j \neq i$ and for all k.

- From a practical point of view, the fact that the travel times must take integer values does not involve a strong restriction with respect to the continuous case, because we can define an appropriate and as-small-as required unit of time for each instance.

- In contrast to the papers on time-dependent vehicle routing problems cited in Section 1, this definition distinguishes between two magnitudes: the time-dependent travel time and the time-dependent cost, focusing on cost minimization.

- Due to the characteristics of the problem (identical vehicles), this definition assumes that the travel times satisfy the FIFO property, in spite of the fact that this assumption does not affect to the transformation discussed in this paper.

- In the particular case of the TDVRPTW in which $t_{ij}^k = t_{ij}^s = c_{ij}^k = c_{ij}^s$

$\forall k, s \in \{0, 1, ..., p_i\}$ and $\forall (v_i, v_j) \in A$, $c^k_{0,j} = \infty$ $\forall k > 0$ and $\forall j > 0$ (each circuit must start at time a_0), $\forall i > 0$ $[w^k_i, t^k_i] = [a_0, a_i]$ if $k = 0$ and $w^k_i = t^k_i$ if $k > 0$ (no waiting time), and all waiting time costs equal to zero, we have a VRPTW with a fixed number of vehicles.

- In the particular case $r = 1$, if $\forall i > 0$ $[w^k_i, t^k_i] = [a_0, a_i]$ if $k = 0$ and $w^k_i = t^k_i$ if $k > 0$ we have the ATSPTDC, studied in [1]. Thus, we also generalize the ATSPTDC in two ways: first by allowing waiting times at the customer locations even when arriving inside the customer time window, and then by extending the first generalization to the multivehicle case.

2.2 Auxiliary digraph

Due to the fact that traversing arc costs are not constant, we can not work directly with the digraph $G = (V, A)$ in a classical way, because each arc would have many associated costs. To avoid this handicap, we will work with an auxiliary digraph, in which basically a vertex v^k_i corresponds with a customer i (or with the depot) at time t^k_i, such that arc (v^k_i, v^h_j) exists if and only if a vehicle leaving customer i at time t^k_i arrives at customer j at time t^h_j or before (if allowed) and leaves customer j at time t^h_j. Arc (v^k_i, v^h_j) will have associated a single cost consisting of the travel cost c^k_{ij} plus maybe a waiting time cost if it arrives at v_j before t^h_j. In this way we can use the classical properties of the graphs.

Consider then a TDVRPTW defined on graph $G = (V, A)$ with all the corresponding data. We construct a directed auxiliary graph $G' = (V', A')$ as follows:

- For each vertex v_i with $i \in \{0, ..., n\}$ and for each instant of time t^k_i for all $k \in \{0, 1, ..., p_i\}$ create a vertex v^k_i.

- For each pair of vertices $v^k_i, v^l_j \in V'$ with $i \neq j$ and such that $t^k_i + t^k_{ij} \in [w^l_j, t^l_j]$ if $j \neq 0$ and $t^k_i + t^k_{ij} = t^l_j$ if $j = 0$, add to G' an arc (v^k_i, v^l_j) with cost equal to $c^k_{i,j} + cwt_j(t^l_j - (t^k_i + t^k_{ij}))$. Note that $w^l_j \leq t^k_i + t^k_{ij} < t^l_j$ implies a waiting time at vertex $v_j \in G$ if a cycle takes arc (v_i, v_j) at time t^k_i and leaves v_j at time t^l_j.

- Divide $\{0, 1, ..., p_0\}$ into four subsets I_1, I_2, I_3, I_4 in the following way:

1) $k \in I_1$ if v_0^k has only leaving arcs in G'. Replace this vertex with r copies of the same one that we will denote $v_{0_{s1}}^k, ..., v_{0_{sr}}^k$ (a starting vertex for each vehicle), and from each one of them the same set of arcs must exit, that is, each arc (v_0^k, v_i^q) will be replaced in G' with one copy for each vehicle, $(v_{0_{sj}}^k, v_i^q)$ $j = 1, ..., r$, with the same cost.

2) $k \in I_2$ if v_0^k has only entering arcs in G'. Replace this vertex with r copies of the same one that we will denote $v_{0_{e1}}^k, ..., v_{0_{er}}^k$(an ending vertex for each vehicle) and for each one of them the same set of arcs must enter, that is, each arc (v_i^q, v_0^k) will be replaced in G' with one copy for each vehicle, $(v_i^q, v_{0_{ej}}^k)$ $j = 1, ..., r$, with the same cost.

3) $k \in I_3$ if v_0^k has both entering and leaving arcs in G'. In this case split v_0^k into $2r$ vertices: $v_{0_{s1}}^k, ..., v_{0_{sr}}^k, v_{0_{e1}}^k, ..., v_{0_{er}}^k$, so that the first r vertices only have leaving arcs (the same as v_0^k) in G' and the other r vertices only have entering arcs (the same as v_0^k) in G', in the same way as in the two previous cases.

4) $k \in I_4$ if v_0^k has neither entering arcs nor leaving arcs in G'. Then, delete v_0^k from G' for all $k \in I_4$.

- Add to G' a new vertex v_d, that will be the depot, with the next arcs, all of them with zero cost:

For each $k \in I_1 \cup I_3$ and for each $j \in \{1, ..., r\}$, an arc $(v_d, v_{0_{sj}}^k)$.

For each $k \in I_2 \cup I_3$ and for each $j \in \{1, ..., r\}$, an arc $(v_{0_{ej}}^k, v_d)$.

An example illustrates the construction of this auxiliary digraph: a TD-VRPTW with $n = 4$, $r = 2$ and $W = 9$, that is, with four customers and the depot, and two vehicles with a capacity of 9 units. In order to clearly show the whole transformed directed graph, in this example we will suppose that $\forall i > 0$ $w_i^k = a_0$ if $k = 0$ and $w_i^k = t_i^k$ if $k > 0$, that is, a waiting time at v_i is only allowed if we arrive at v_i before time a_i and we leave v_i at time a_i. Like in [1], in this example we will also suppose that waiting times have zero cost and that the travel cost is proportional to the travel time except for a little deviation due, for instance, to the characteristics of the different routes. More specifically, the travel cost is about 40 times the travel time with 5% maximum deviation, that is, $c_{ij}^t \in [38t_{ij}^t, 42t_{ij}^t]$. The time window and the demand of each vertex are given in Figure 1.

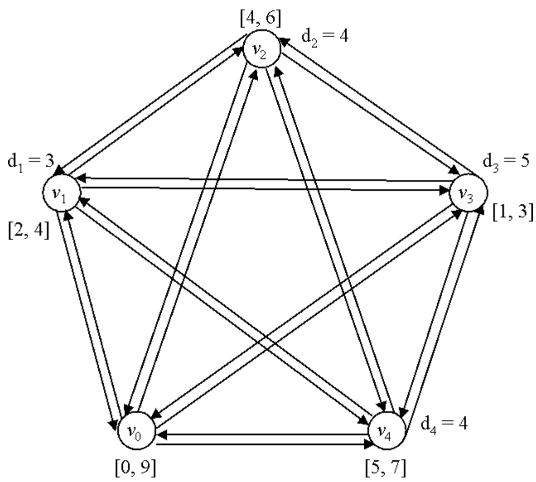

Figure 1. Graph G.

Table 1 shows the time-dependent travel times and costs corresponding to this example, in a particular format to simplify the construction of the auxiliary digraph. Each t_i^k shows in brackets its corresponding time instant. For example, the ordered pair corresponding to the row t_0^0 and to the column t_1^1 means that if we traverse arc (v_0, v_1) starting at time t_0^0, which corresponds to instant 0, $t_{0,1}^0 = 3$ and $c_{0,1}^0 = 125$. A dash inside the cell corresponding to row t_i^k and column t_j^l means that if we traverse arc (v_i, v_j) starting at time t_i^k we will not arrive at v_j at time t_j^l if $l > 0$ or that we will arrive after a_j if $l = 0$. Note that the table does not include the rows and columns with not possible paths, and that a waiting time at the customer location only occurs going from v_3 to v_4 at time $t = 1$, being $t_3^0 + t_{34}^0 = 4$ and $t_4^0 = 5$ (one unit of waiting time).

Figure 2 shows the corresponding auxiliary digraph G' in which, to simplify, the vertices v_i^k are denoted only by their superindex k and they have been clustered into subsets S_i corresponding to customers or vehicles. Arc costs have been omitted in Figure 2; they can be easily obtained from Table 1.

	$t_1^1(3)$	$t_1^2(4)$	$t_2^0(4)$	$t_2^1(5)$	$t_3^2(3)$	$t_4^0(5)$	$t_4^1(6)$	$t_4^2(7)$
$t_0^0(0)$	(3,125)	-	(4,163)	-	(3,115)	(5,195)	-	-
$t_0^1(1)$	-	(3,120)	-	(4,162)	-	-	(5,205)	-
$t_0^2(2)$	-	-	-	-	-	-	-	(5,200)

	$t_0^5(5)$	$t_0^6(6)$	$t_0^8(8)$	$t_2^0(4)$	$t_2^1(5)$	$t_2^2(6)$	$t_4^1(6)$
$t_1^0(2)$	(3,123)	-	-	(2,81)	-	-	(4,160)
$t_1^1(3)$	-	(3,115)	-	-	(2,83)	-	-
$t_1^2(4)$	-	-	(4,160)	-	-	(2,79)	-

	$t_0^4(4)$	$t_0^5(5)$	$t_0^6(6)$	$t_2^1(5)$	$t_4^0(5)$	$t_4^1(6)$
$t_3^0(1)$	(3,120)	-	-	(4,161)	(3,125)	-
$t_3^1(2)$	-	(3,122)	-	-	(3,121)	-
$t_3^2(3)$	-	-	(3,120)	-	-	(3,115)

	$t_0^5(5)$	$t_0^6(6)$	$t_0^7(7)$
$t_2^0(4)$	(1,40)	-	-
$t_2^1(5)$	-	(1,41)	-
$t_2^2(6)$	-	-	(1,39)

	$t_0^6(6)$	$t_0^7(7)$	
$t_4^0(5)$	(1,41)	-	-
$t_4^1(6)$	-	(1,38)	
$t_4^2(7)$	-	-	

Table 1. Time-dependent travel times and costs (t_{ij}^k, c_{ij}^k) of graph G.

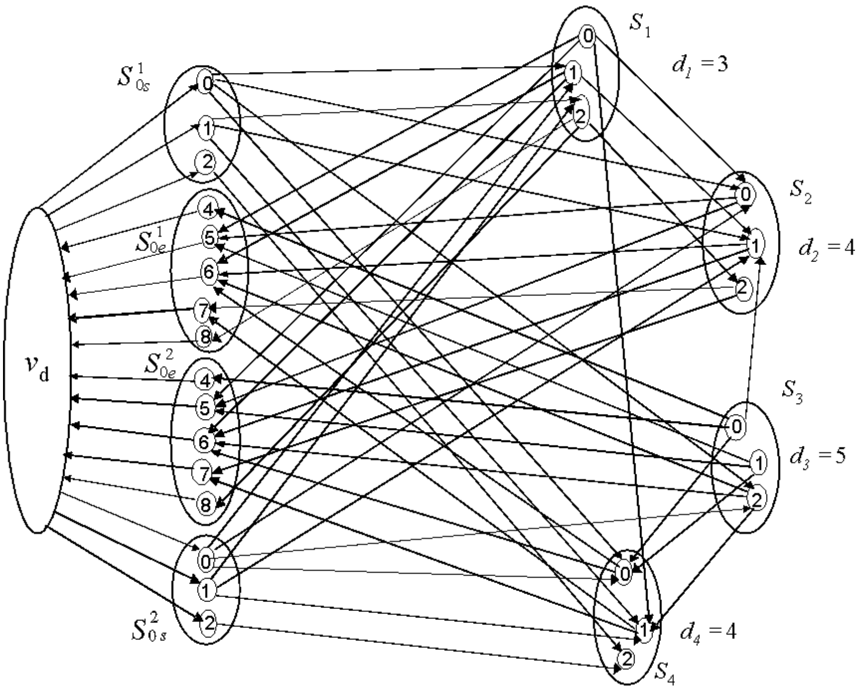

Figure 2. Auxiliary digraph G' from G.

2.2.1 Reduced auxiliary digraph

Given the auxiliary digraph defined above, it is logical to think that its size could be a handicap to apply any kind of exact procedure to solve an NP-hard routing problem on it. In fact, in real distribution problems inside big cities, the servicing time windows of the customers could have a relative small size (for example one hour) after preliminary studies and negotiations, but the depot time window should be opened during the entire working day. If there are customers to be serviced early and customers to be serviced at the end of the working day, each one of the sets $S^j_{0_s}$ and $S^j_{0_e}$ $j = 1, ..., r$, will contain about $b_0 - a_0$ vertices. For example, in this condition, in a 8-hour working day with a time unit equal to 1 minute and a fleet of 5 vehicles, would imply that the depot gives rise to about $8 \times 60 \times 5 \times 2 = 4800$ vertices in the auxiliary graph.

We show next that the size of the auxiliary graph can be considerably reduced to make this transformation more competitive. In fact, in the "reduced" auxiliary digraph, the number of vertices generated from the depot will always be $2r + 1$, independently of the size of the depot time window. Thus, in the example given above, we would only have 11 vertices vs the about 4800 vertices (a very considerable reduction), and in our example of Figure 1, we would have 5 vertices vs the 17 vertices in Figure 2.

Let then $G' = (V', A')$ be the auxiliary digraph obtained from the original TDVRPTW instance. Consider in V' the vertex subsets $S^j_{0_s} = \{v^k_{0_{sj}}\}_{k \in I_1 \cup I_3}$ $j = 1, ..., r$, $S^j_{0_e} = \{v^k_{0_{ej}}\}_{k \in I_2 \cup I_3}$ $j = 1, ..., r$ and $S_i = \{v^k_i\}^{p_i}_{k=0} \forall i \in \{1, ..., n\}$. From G' we construct a reduced auxiliary digraph $G" = (V", A")$ in the following way:

- For each subset $S^j_{0_s}$ with $j = 1, ..., r$ create a single vertex s^j_s.

- For each subset $S^j_{0_e}$ with $j = 1, ..., r$ create a single vertex s^j_e.

- Maintain the rest of vertices of G' including v_d.

-For every vertex $v \in G"$ different from those created in the first two steps, and for each $j \in \{1, ..., r\}$ do $cost(s^j_s, v) = \min_k \{cost(v^k_{0_{sj}}, v)\}$.

- For every arc $(v, v^k_{0_{ej}})$ with finite cost in G' do $cost(v, s^j_e) = cost(v, v^k_{0_{ej}})$.

- Do $cost(s^j_e, v_d) = cost(v_d, s^j_s) = 0 \;\; \forall j \in \{1, ..., r\}$.

- Maintain the arc costs between vertices belonging to different sets S_i with $i \in \{1, ..., n\}$.

- Remove all vertices $v_i^k \in G$" with $i \neq 0$ verifying that $d^+(v_i^k) = 0$ or $d^-(v_i^k) = 0$. To do this, we understand that an arc (u, v) exists in G" if it has been assigned before a finite value to $cost(u, v)$.

Figure 3 shows the reduced auxiliary digraph G" from G' in Figure 2 corresponding to our example. As we have said, the number of vertices generated from the depot is 5 vs 17 vertices in Figure 2, and vertices v_1^0, v_3^0, v_3^1 and v_4^2 have been removed, so G" has 13 vertices while G' has 29 vertices.

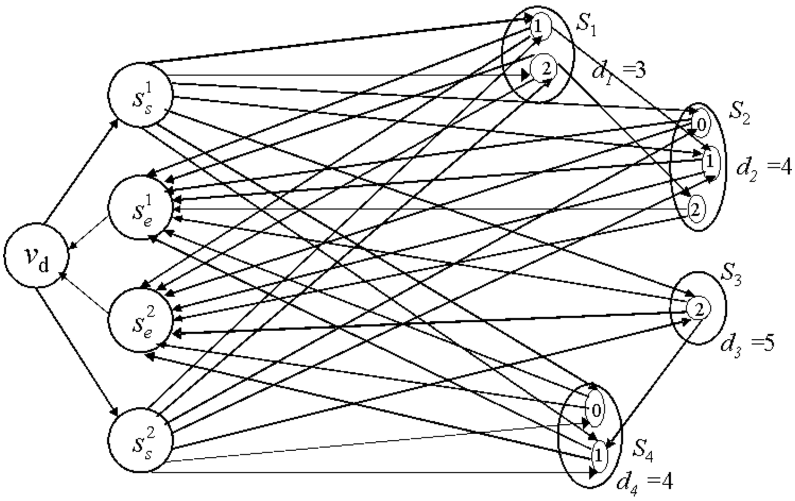

Figure 3. Reduced auxiliary digraph G".

3 Transformation of the TDVRPTW into an ACVRP

Once defined the graphs G' and G", we present a way to solve the TDVRPTW by transforming it into a GVRP and then transforming the obtained GVRP into an ACVRP. Thus, we can solve the TDVRPTW with existing algorithms for the ACVRP (as those cited in Section 1) both heuristically and optimally, in this last case at least for small size instances.

Let $G = (V, A)$ be the digraph where a TDVRPTW is defined. We consider in its auxiliary digraph $G' = (V', A')$ a GVRP corresponding to the partition of V' into the following subsets: $S_d = \{v_d\}$, $S_{0_s}^j = \{v_{0_{sj}}^k\}_{k \in I_1 \cup I_3}$ $j = 1, ..., r$, $S_{0_e}^j =$

$\{v^k_{0_{ej}}\}_{k\in I_2\cup I_3}$ $j = 1, ..., r$, and $S_i = \{v^k_i\}^{p_i}_{k=0} \forall i \in \{1, ..., n\}$, that is, $n + 2r + 1$ subsets, where S_d represents the depot vertex, $S^j_{0_s}$ and $S^j_{0_e}$ with $j = 1, ..., r$ have demand 1, each subset S_i $i \in \{1, ..., n\}$ has associated the demand d_i of the costumer i in the TDVRPTW, and the capacity of each vehicle is increased in 2 units (W+2).

We also consider in its reduced auxiliary digraph $G" = (V", A")$ a GVRP in the same terms as the GVRP defined in G', except that the subsets $S^j_{0_s}$ and $S^j_{0_e}$ have now a single element, that is, $S^j_{0_s} = \{s^j_s\}$ and $S^j_{0_e} = \{s^j_e\}$ $\forall j \in \{1, ..., r\}$, and that some S_i may contain fewer elements than in G'.

We have the following results:

Theorem 1 *The TDVRPTW can be transformed into the GVRP defined in the auxiliary digraph.*

Proof. Let us see that there is a one to one correspondence between the set of feasible GVRP solutions in G' and the set of feasible TDVRPTW solutions in G:

Let T' be a feasible GVRP solution in G'. By construction of G', T' has exactly r routes, one for each vehicle, and without loss of generality, we can suppose that the routes have the following structure:

$\left\{v_d, v^{k^1_0}_{0_{s1}}, v^{k^1_1}_{i_1}, ..., v^{k^1_a}_{i_a}, v^{k^1_{a+1}}_{0_{e1}}, v_d\right\}$ for vehicle 1,. . .,$\left\{v_d, v^{k^r_0}_{0_{sr}}, v^{k^r_1}_{i_b}, ..., v^{k^r_c}_{i_c}, v^{k^r_{c+1}}_{0_{er}}, v_d\right\}$ for vehicle r.

Note that if a vehicle does not return to the depot by a vertex $v^k_{0_{et}}$ with its corresponding index, a posteriori we can permute the names of the subsets S^j_{0e} with $j = 1, ..., r$ in order T' to have the previous structure.

We can easily identify the GVRP solution T' in G' with the feasible TDVRPTW solution H in G consisting of the following set of r cycles:

$\{v_0, v_{i_1}, v_{i_2}, ..., v_{i_a}, v_0\}$ for vehicle 1,. . .,$\{v_0, v_{i_b}, v_{i_{b+1}}, ..., v_{i_c}, v_0\}$ for vehicle r, where vehicle 1 leaves the depot vertex v_0 at time $k^1_0 \in [a_0, b_0]$, leaves each v_{i_l} in its route at instant $a_{i_l} + k^1_l \in [a_{i_l}, b_{i_l}]$ and ends at v_0 at time $a_0 + k^1_{a+1} \in [a_0, b_0]$, and so on for the rest of vehicles. Both solutions T' and H have the same cost and service the same demand except for the two units serviced by each vehicle j at S^j_{0e} and S^j_{0s} in the GVRP. In a similar way it is easy to see that a feasible TDVRPTW solution in G gives rise to a feasible GVRP solution in

G' with the same cost and therefore, an optimal GVRP solution in G' gives us an optimal TDVRPTW in G and vice versa.▌

Theorem 2 *Solving the GVRP in G' is equivalent to solving the GVRP in G".*

Proof. Given a feasible GVRP solution in G', if we change the initial sequence for each vehicle j $\{v_d, v_{0sj}^k, v\}$ for the sequence $\{v_d, s_s^j, v\}$ in G" and we change its final sequence $\{v, v_{0ej}^k, v_d\}$ for the sequence $\{v, s_e^j, v_d\}$ in G", it is evident that we have a feasible GVRP solution in G". Moreover, the costs of $\{v, v_{0ej}^k, v_d\}$ and $\{v, s_e^j, v_d\}$ are the same and in an optimal solution in G', the cost of $\{v_d, v_{0sj}^k, v\}$ must necessarily be $\min_k\{cost(v_{0sj}^k, v)\}$, which is the cost of $\{v_d, s_s^j, v\}$. On the other hand, it is clear that there is no solution in G' containing a vertex which is impossible to reach from another vertex or from which another vertex will be impossible to be reached.

Thus, an optimal GVRP solution in G' gives rise to a feasible GVRP solution in G" with the same cost, and with the same reasoning, a feasible GVRP solution in G" gives rise to a feasible GVRP solution in G' with the same cost, and so, an optimal GVRP solution in G' gives rise to an optimal GVRP solution in G" and vice versa.▌

Next we present a transformation of the GVRP defined in G" into an ACVRP, that can be considered an extension to a multivehicle case of the one given by Noon and Bean ([15]) for a single vehicle problem.

From $G" = (V", A")$ we construct a digraph $G^* = (V^*, A^*)$ as follows:

- $V^* = V"$

- For each subset R_i with $|R_i| > 1$ in which V" has been partitioned to define the GVRP, order its vertices consecutively in an arbitrary way $\{v_i^{t_1}, \ldots, v_i^{t_{l(i)}}\}$; then, for $j = 1, \ldots, l(i) - 1$, define the cost of arc $(v_i^{t_j}, v_i^{t_{j+1}}) \in A^*$ as zero; also define the cost of arc $(v_i^{t_{l(i)}}, v_i^{t_1})$ as zero.

- For every $v_i^{t_j} \in R_i$ and every $w \notin R_i$, if $|R_i| > 1$ define the cost of arc $(v_i^{t_j}, w) \in A^*$ equal to the cost in G" of the arc $(v_i^{t_{j+1}}, w)$ ($(v_i^{t_1}, w)$ if $j = l(i)$) plus a fixed positive large quantity M, and if $|R_i| = 1$ define the cost of $(v_i^{t_j}, w) \in A^*$ equal to the cost in G" of the arc $(v_i^{t_j}, w)$ plus M.

- Any other arc in A^* has infinite cost.

- In V^* assign positive demands having sum equal to d_i to the vertices in R_i $\forall$ i except for the depot subset.

Theorem 3 *The GVRP defined in digraph G" can be solved by transforming it into an ACVRP in the digraph G^*.*

Proof. If we identify each arc $(v_i^{t_j}, w)$ in G" $w \notin R_i$ (w can be the depot) with the path $(v_i^{t_j}, v_i^{t_{j+1}}, \ldots, v_i^{t_{l(i)}}, v_i^{t_1}, \ldots, v_i^{t_{j-1}}, w)$ in G^* if $j \neq 1$ and $|R_i| > 1$, or the path $(v_i^{t_1}, \ldots, v_i^{t_{l(i)}}, w)$ if $j = 1$ and $|R_i| > 1$ or $(v_i^{t_j}, w)$ if $|R_i| = 1$ (in this last case $v_i^{t_j}$ can be the depot), it is easy to see that a feasible GVRP solution S in G" gives rise to a feasible ACVRP solution H_S in G^* with cost $c(H_S) = c(S) + M(m + r)$ m being the number of subsets R_i different from the depot in G", and due to the arc costs in G^*, it is evident that an optimal ACVRP solution in G^* has the same structure than H_S: if a vehicle services a vertex $v_i^{t_j} \in R_i$, it services all vertices in R_i, and if $v_i^{t_j}$ is the first vertex serviced by the vehicle in R_i, it services the vertices of R_i consecutively and in the order $v_i^{t_j}, v_i^{t_{j+1}}, \ldots, v_i^{t_{l(i)}}, v_i^{t_1}, \ldots, v_i^{t_{j-1}}$ if $j \neq 1$ or $v_i^{t_1}, \ldots, v_i^{t_{l(i)}}$ if $j = 1$.

On the other hand, based on the same identification, a feasible ACVRP solution H in G^* with the structure cited above, gives rise to a feasible GVRP solution S_H in G" with cost $c(S_H) = c(H) - M(m + r)$.

Therefore, if H^* is an optimal ACVRP solution in G^*, S_{H^*} is an optimal GVRP solution in G". Otherwise, let S" be an optimal GVRP solution in G" with $c(S") < c(S_{H^*})$, then $H_{S"}$ is a feasible ACVRP solution in G^* with $c(H_{S"}) = c(S") + M(m + r) < c(S_{H^*}) + M(m + r) = (c(H^*) - M(m + r)) + M(m + r) = c(H^*)$ which is impossible due to the optimality of H^*.▌

Following with our example, once we have the reduced auxiliary digraph G", we construct G^* and we solve the corresponding ACVRP in G^*. Its optimal solution is given in Figure 4, where $\sum_j d_{ij} = d_i$. From this optimal solution we obtain the optimal solution to the GVRP defined in G"(see Figure 5): the cycles $\{v_d, s_s^1, v_1^2, v_2^2, s_e^1, v_d\}$ servicing a demand of 7 units with travel time 6 and cost 238 and $\{v_d, s_s^2, v_3^2, v_4^1, s_e^2, v_d\}$ servicing a demand of 9 units with travel time 7 and cost 268. Finally, Figure 6 shows the optimal solution to the TDVRPTW in G, with total cost 506. Note that the first circuit does not start at time a_0 (there is a waiting time at the depot).

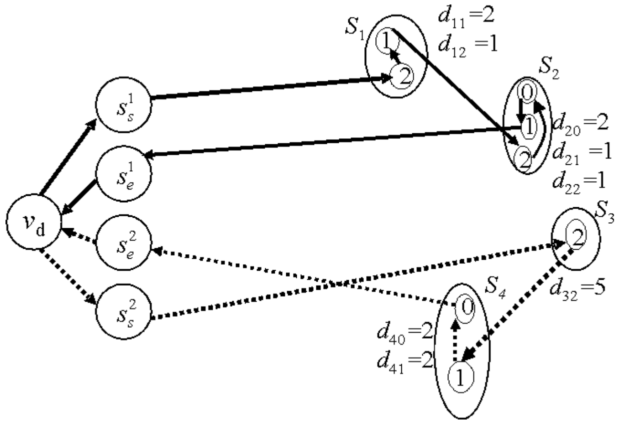

Figure 4. Optimal ACVRP solution in G^*.

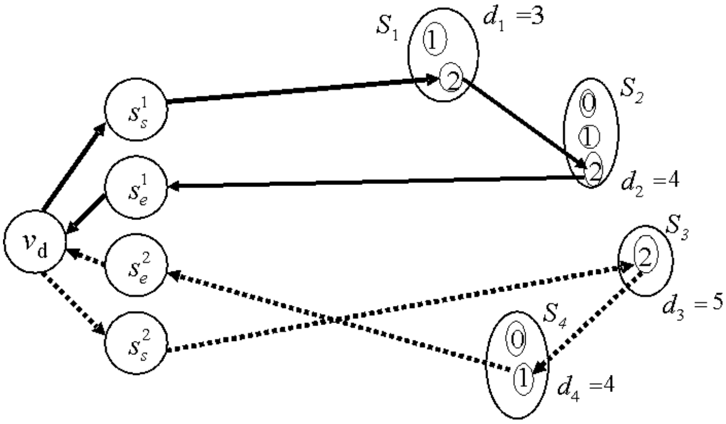

Figure 5. Optimal GVRP solution in G".

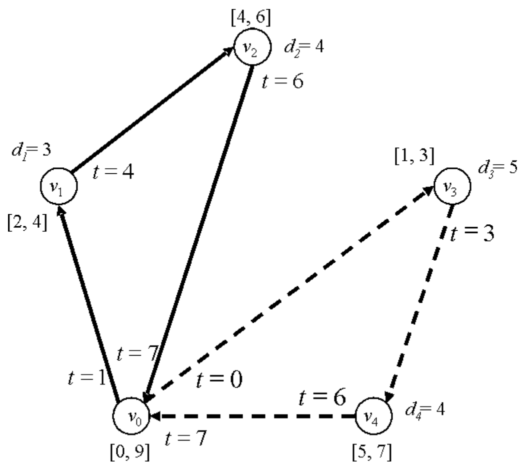

Figure 6. Optimal TDVRPTW solution in G.

4 Conclusion

More and more researches on routing problems are taking into account time-dependent costs in order to move the mathematical models closer to real-world problems inside big cities, where the costs of traversing some streets depend on the moment of the day, for example peak hours. We have cited seven of these recent papers and we are convinced that as computer power and speed increase, the number of researches on this topic will also increase.

By other hand, due to the fact that vehicle routing problems with time-dependent costs are more difficult to model and to solve, these studies have been focused on heuristic approaches. In contrast, we have presented here a way to optimally solve a time-dependent vehicle routing problem with time windows (at least for small size instances due to its complexity), by transforming it through several steps into an ACVRP, a well-known routing problem for which several heuristic and exact procedures exist.

This is a theoretical work whose aim is that its results can be used in the future as ideas or tools to test the efficiency of specific procedures for vehicle routing problems with time-dependent costs, with computational experiments through benchmark instances. This is the challenge for our future research in this topic.

Acknowledgements

This work has been supported by the Ministerio de Educación y Ciencia, D. G. Investigación of Spain (project TIN2005-07705-C02-01).

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