GROWTH AT INFINITY AND INDEX OF POLYNOMIAL MAPS

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Abstract. Let $F : \mathbb{K}^n \to \mathbb{K}^n$ be a polynomial map such that $F^{-1}(0)$ is compact, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then we give a condition implying that there is a uniform bound for the Łojasiewicz exponent at infinity in certain deformations of F. This fact gives a result about the invariance of the global index of F.

1. Introduction

Given a polynomial map $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F^{-1}(0)$ is finite, in this article we study the problem of determining which monomials can be added to each component function of F leading to a map having the same global index than F. We recall that the *global index of* F, that we denote by $\mathrm{ind}(F)$, is defined as $\mathrm{ind}(F) = \sum_{x \in F^{-1}(0)} \mathrm{ind}_x(F)$, where $\mathrm{ind}_x(F)$ denotes the index, or topological degree, of F at each point $x \in F^{-1}(0)$. The local version of this question, which is analyzed in the articles [1], [8], [15] and [20], takes part in the wider problem of determining which monomials in the Taylor expansion of a smooth vector field determine the local phase portrait (see for instance [3] and [4]). The first step in this approach to the study of global indices is the result of Cima-Gasull-Mañosas [7, Proposition 2] on the index of maps whose monomials of maximum degree with respect to some vector of weights have an isolated zero. We call these maps pre-weighted homogeneous (see Definition 7.1 for a precise formulation of this concept).

Apart from [7], our motivation to study global indices comes from the estimation of the Łojasiewicz exponent at infinity of a given polynomial map $F : \mathbb{K}^n \to \mathbb{K}^p$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (see the article of Krasiński [17] for a detailed survey about Łojasiewicz exponents at infinity). This number, which is denoted by $\mathcal{L}_\infty(F)$, is defined as the supremum of those real numbers α such that there exist constants $C, M > 0$ such that

$$\|x\|^\alpha \leqslant C\|F(x)\| \tag{1}$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. It is known that this number exists if and only if $F^{-1}(0)$ is compact and, in this case, this is a rational number. The exact computation or the estimation of $\mathcal{L}_\infty(F)$ from below is a non-trivial problem [10], [17], [19], [24]. This number is intimately related with questions about the injectivity of polynomial maps [5] and the equivalence at

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infinity of polynomial vector fields [25]. We give a sufficient condition that implies that there is a uniform Łojasiewicz inequality associated to a homotopy of the form $F+tG$, $t\in[0,1]$, where G denotes another polynomial map, and this gives our result about the invariance of the index (Theorem 8.1). That condition is given in terms of Newton polyhedra and non-degeneracy conditions on maps. We point that inequality (1) can be generalized in many directions, as can be seen in [11], where Newton polyhedra and non-degeneracy are also applied to derive very interesting computations.

In this article we generalize the notion of pre-weighted homogeneous polynomial map $\mathbb{K}^n\to\mathbb{K}^p$ thus leading to the notion of *strongly adapted map* to a given convenient global Newton polyhedron in $\mathbb{R}^n$ (Section 4). This is the key idea that allows us to show one of the main results, Theorem 4.4, which gives an estimation of the region in $\mathbb{R}^n$ determined by the monomials that we call *special* with respect to F (Definition 4.1). These monomials play a role analogous to the monomials belonging to the integral closure of a given ideal of the ring $\mathcal{A}_n$ of analytic functions $f:(\mathbb{K}^n,0)\to\mathbb{K}$. Section 5 is devoted to the proof of this result.

In Sections 6 and 7 we apply Theorem 4.4 to establish a positive lower bound for $\mathcal{L}_\infty(F)$ (Corollary 6.3 and Proposition 7.3) and to derive a consequence about the injectivity of polynomial maps, which is Corollary 6.5. We remark that in [2] the first author developed a technique to obtain a lower bound for Łojasiewicz exponents at infinity that only works in the real case (see Remark 6.6). The proofs in the present paper are mostly self contained and work simultaneously for real and complex polynomial maps.

Finally in Section 8 we apply the argument of the proof of Theorem 4.4 to obtain a result about the global index of polynomial maps.

2. Newton polyhedra at infinity. Preliminary concepts

In this section we expose some basic definitions and results that we will need in subsequent sections.

Definition 2.1. Let $\widetilde{\Gamma}_+\subseteq\mathbb{R}^n_{\geqslant 0}$. We say that $\widetilde{\Gamma}_+$ is a *global Newton polyhedron*, or a *Newton polyhedron at infinity*, if there exists some finite subset $A\subseteq\mathbb{Z}^n_{\geqslant 0}$ such that $\widetilde{\Gamma}_+$ is equal to the convex hull in $\mathbb{R}^n$ of $A\cup\{0\}$.

Let us fix a global Newton polyhedron $\widetilde{\Gamma}_+\subseteq\mathbb{R}^n_{\geqslant 0}$. If $w\in\mathbb{R}^n$ then we define

$$\ell(w,\widetilde{\Gamma}_+)=\min\big\{\langle w,k\rangle : k\in\widetilde{\Gamma}_+\big\} \tag{2}$$

$$\Delta(w,\widetilde{\Gamma}_+)=\big\{k\in\widetilde{\Gamma}_+ : \langle w,k\rangle=\ell(w,\widetilde{\Gamma}_+)\big\} \tag{3}$$

where we denote by $\langle\ ,\ \rangle$ the standard scalar product in $\mathbb{R}^n$. If $w\in\mathbb{R}^n\smallsetminus\{0\}$, then $\Delta(w,\widetilde{\Gamma}_+)$ is called a *face* of $\widetilde{\Gamma}_+$. The set $\Delta(w,\widetilde{\Gamma}_+)$ is also called the face of $\widetilde{\Gamma}_+$ *supported by* w. The hyperplane given by the equation $\langle w,k\rangle=\ell(w,\widetilde{\Gamma}_+)$ is called a *supporting hyperplane of* $\widetilde{\Gamma}_+$ (this concept can be extended naturally to any convex and closed subset of $\mathbb{R}^n$).

The *dimension* of a face Δ of Γ_+, denoted by $\dim(\Delta)$, is defined as the minimum among the dimensions of the affine subspaces containing Δ. The faces of $\widetilde{\Gamma}_+$ of dimension 0 are called the *vertices* of $\widetilde{\Gamma}_+$ and the faces of $\widetilde{\Gamma}_+$ of dimension $n-1$ are called *facets* of $\widetilde{\Gamma}_+$. We define the *dimension* of $\widetilde{\Gamma}_+$ as

$$\dim(\widetilde{\Gamma}_+) = \max\big\{\dim(\Delta) : \Delta \text{ is a face of } \widetilde{\Gamma}_+ \text{ such that } 0 \notin \Delta\big\}.$$

For any $w = (w_1, \dots, w_n) \in \mathbb{R}^n$, we denote by w_0 the minimum of the coordinates of w. Then we define $\mathbb{R}^n_0 = \{w \in \mathbb{R}^n : w_0 < 0\}$ and $\mathbb{R}^n_0(i) = \{w \in \mathbb{R}^n_0 : w_0 = w_i\}$, for all $i = 1, \dots, n$. Let us remark that if $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n$ is a global Newton polyhedron then

$$\widetilde{\Gamma}_+ = \{k \in \mathbb{R}^n_{\geqslant 0} : \langle k, w\rangle \geqslant \ell(w, \widetilde{\Gamma}_+), \text{ for all } w \in \mathbb{R}^n_0\}. \tag{4}$$

Let $w \in \mathbb{Z}^n$. We say that w is *primitive* when $w \neq 0$ and w is the vector of smallest length between all vectors of $\mathbb{Z}^n$ of the form λw, for some $\lambda > 0$.

Let $\widetilde{\Gamma}_+$ be a global Newton polyhedron in $\mathbb{R}^n$ such that $\dim(\widetilde{\Gamma}_+) = n-1$. We denote by $\mathcal{F}(\widetilde{\Gamma}_+)$ the family of primitive vectors $w \in \mathbb{Z}^n$ such that $\dim \Delta(w, \widetilde{\Gamma}_+) = n-1$. Since $\widetilde{\Gamma}_+$ is a polytope, i.e. the convex hull of a finite subset of $\mathbb{R}^n$, and $\dim(\widetilde{\Gamma}_+) = n-1$ then $\mathcal{F}(\widetilde{\Gamma}_+)$ is finite and any face of $\widetilde{\Gamma}_+$ can be expressed as an intersection $\cap_{w\in J}\Delta(w, \widetilde{\Gamma}_+)$, for some subset $J \subseteq \mathcal{F}(\widetilde{\Gamma}_+)$ (see [14, p. 33]). We denote by $\mathcal{F}_0(\widetilde{\Gamma}_+)$ the set of vectors $w \in \mathcal{F}(\widetilde{\Gamma}_+)$ such that $\Delta(w, \widetilde{\Gamma}_+)$ does not contain the origin.

Lemma 2.2. *Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a global Newton polyhedron. Let J be a subset of $\mathcal{F}(\widetilde{\Gamma}_+)$. Then the following conditions are equivalent:*

(i) $\cap_{w\in J}\Delta(w, \widetilde{\Gamma}_+) \neq \emptyset$;
(ii) $\cap_{w\in J}\Delta(w, \widetilde{\Gamma}_+) = \Delta\big(\sum_{w\in J} w, \widetilde{\Gamma}_+\big)$;
(iii) $\ell(\sum_{w\in J} w, \widetilde{\Gamma}_+) = \sum_{w\in J} \ell(w, \widetilde{\Gamma}_+)$.

Proof. The result follows as a direct consequence of the definition of $\ell(w, \widetilde{\Gamma}_+)$ and $\Delta(w, \widetilde{\Gamma}_+)$, for a given vector $w \in \mathbb{R}^n$. □

Let $\widetilde{\Gamma}^1_+, \dots, \widetilde{\Gamma}^p_+$ be global Newton polyhedra in $\mathbb{R}^n$. Then the *Minkowski sum of* $\widetilde{\Gamma}^1_+, \dots, \widetilde{\Gamma}^p_+$ is defined as $\widetilde{\Gamma}^1_+ + \dots + \widetilde{\Gamma}^p_+ = \{k_1 + \dots + k_p : k_i \in \widetilde{\Gamma}^i_+, \text{ for all } i = 1, \dots, p\}$. It is well known that $\widetilde{\Gamma}^1_+ + \dots + \widetilde{\Gamma}^p_+$ is again a global Newton polyhedron. The following lemma is also known.

Lemma 2.3. *Let $\widetilde{\Gamma}^1_+, \dots, \widetilde{\Gamma}^p_+$ be global Newton polyhedra in $\mathbb{R}^n$. Let $\widetilde{\Gamma}_+ = \widetilde{\Gamma}^1_+ + \dots + \widetilde{\Gamma}^p_+$ and let $w \in \mathbb{R}^n \smallsetminus \{0\}$. Then*

(i) $\ell(w, \widetilde{\Gamma}_+) = \ell(w, \widetilde{\Gamma}^1_+) + \dots + \ell(w, \widetilde{\Gamma}^p_+)$
(ii) $\Delta(w, \widetilde{\Gamma}_+) = \Delta(w, \widetilde{\Gamma}^1_+) + \dots + \Delta(w, \widetilde{\Gamma}^p_+)$.

Proof. It arises as a consequence of the definition of Minkowski sum. □

Let $\{e_1, \dots, e_n\}$ denote the canonical basis in $\mathbb{R}^n$. Given a global Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n$, we say that $\widetilde{\Gamma}_+$ is *convenient* if $\widetilde{\Gamma}_+$ intersects each coordinate axis in a point different from the origin, that is, if for any $i \in \{1, \dots, n\}$ there exists some $r > 0$ such that

$re_i \in \widetilde{\Gamma}_+$. In this case we define

$$r_i(\widetilde{\Gamma}_+) = \max\{r > 0 : re_i \in \widetilde{\Gamma}_+\},\ i = 1, \dots, n \tag{5}$$

$$r_0(\widetilde{\Gamma}_+) = \min\left\{r_1(\widetilde{\Gamma}_+), \dots, r_n(\widetilde{\Gamma}_+)\right\}. \tag{6}$$

Lemma 2.4. *Let $\widetilde{\Gamma}_+$ be a convenient global Newton polyhedron in $\mathbb{R}^n$. Let $w \in \mathbb{R}^n \smallsetminus \{0\}$. Then the following conditions are equivalent:*

(i) $0 \notin \Delta(w, \widetilde{\Gamma}_+)$;
(ii) $\ell(w, \widetilde{\Gamma}_+) < 0$;
(iii) $w_0 < 0$.

Proof. It is analogous to [2, Lemma 4.2]. □

Lemma 2.5. *Let $\widetilde{\Gamma}_+$ be a convenient global Newton polyhedron. Then*

$$r_i(\widetilde{\Gamma}_+) = \min\left\{\frac{\ell(w, \widetilde{\Gamma}_+)}{w_i} : w \in \mathbb{R}^n_0(i)\right\},\ \textit{for all } i = 1, \dots, n \tag{7}$$

Proof. Equality (7) follows as an immediate consequence of Lemma 2.4 and relation (4). □

Let us fix coordinates $x_1, \dots, x_n$ in $\mathbb{K}^n$ and let $k \in \mathbb{Z}_{\geqslant 0}$. Then we write x^k to denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$.

Definition 2.6. Let $h \in \mathbb{K}[x_1, \dots, x_n]$, $h \neq 0$. Let us suppose that h is written as $h = \sum_k a_k x^k$. Then the *support of* h, denoted by $\mathrm{supp}(h)$, is defined as the set

$$\mathrm{supp}(h) = \left\{k \in \mathbb{Z}^n_{\geqslant 0} : a_k \neq 0\right\}. \tag{8}$$

The *Newton polyhedron at infinity of* h is defined as the convex hull of $\mathrm{supp}(h) \cup \{0\}$ and is denoted by $\widetilde{\Gamma}_+(h)$. If we denote the vector $(1, \dots, 1) \in \mathbb{R}^n_{\geqslant 0}$ by e, then we observe that $\ell(-e, h) = -\deg(h)$.

If $h = 0$, then we set $\mathrm{supp}(h) = \emptyset$ and $\widetilde{\Gamma}_+(h) = \emptyset$. If we consider a map $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$, then the *Newton polyhedron at infinity of* F, that we denote by $\widetilde{\Gamma}_+(F)$, is defined as the convex hull of $\widetilde{\Gamma}_+(F_1) \cup \cdots \cup \widetilde{\Gamma}_+(F_p)$. We say that F is *convenient* when $\widetilde{\Gamma}_+(F)$ is convenient.

Lemma 2.7. *Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map such that $F(0) = 0$ and $\#F^{-1}(0)$ is compact. Then F is convenient.*

Proof. Let $F_1, \dots, F_p$ denote the component functions of F. If F is not convenient, then there exists some $i \in \{1, \dots, n\}$ such that $\widetilde{\Gamma}_+(F_j)$ does not intersect the x_i-axis, for all $j = 1, \dots, p$. In particular we have that F vanishes on the x_i-axis, since $F(0) = 0$, and hence $\#F^{-1}(0)$ is not compact. □

There is a notion of Newton polyhedron associated to germs of analytic functions $(\mathbb{K}^n, 0) \to \mathbb{K}$. If $f : (\mathbb{K}^n, 0) \to \mathbb{K}$ is an analytic function germ and $f = \sum_k a_k x^k$ is the Taylor expansion of f around the origin, then the *Newton polyhedron of* f, which is denoted by $\Gamma_+(f)$, is defined as the convex hull of $\{k + v : a_k \neq 0,\ v \in \mathbb{R}^n_+\}$ (see [2, Section 4]). If $h \in \mathbb{K}[x_1, \dots, x_n]$ then we can also attach to h the Newton polyhedron of the function $\mathrm{G}(h) = \sum_k a_k x^k \|x\|^{2(d-|k|)}$, where d denotes the degree of h. In general, the set of compact faces of dimension $n-1$ of $\Gamma_+(\mathrm{G}(h))$ is not bijective with the set of facets of $\widetilde{\Gamma}_+(h)$ not passing through the origin, as can be seen in [2, Example 4.8]. The set $\Gamma_+(\mathrm{G}(h))$ is applied in [2] to obtain information about the Łojasiewicz exponent at infinity of real polynomial maps.

3. Maps adapted to Newton polyhedra

Let us fix a convenient global Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n$. In this section we will expose a condition on a given polynomial map $F : \mathbb{K}^n \to \mathbb{K}^p$ that allows us to obtain information about $\mathcal{L}_\infty(F)$ in terms of $\widetilde{\Gamma}_+$.

Definition 3.1. Let $h \in \mathbb{K}[x_1, \dots, x_n]$. Let us suppose that h is written as $h = \sum_k a_k x^k$. If $w \in \mathbb{R}^n \smallsetminus \{0\}$, then we define

$$\ell(w, h) = \min\{\langle w, k\rangle : k \in \mathrm{supp}(h)\}$$
$$\Delta(w, h) = \{k \in \mathrm{supp}(h) : \langle w, k\rangle = \ell(w, h)\}.$$

We define the *principal part of h with respect to w*, denoted by $\mathrm{p}_w(h)$, as the sum of those terms $a_k x^k$ such that $\langle k, w\rangle = \ell(w, h)$. We observe that if h denotes a monomial x^k then $\mathrm{p}_w(h) = h$, for any $w \in \mathbb{R}^n \smallsetminus \{0\}$. If $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ is a polynomial map, then we denote the map $(\mathrm{p}_w(F_1), \dots, \mathrm{p}_w(F_p)) : \mathbb{K}^n \to \mathbb{K}^p$ by $\mathrm{p}_w(F)$.

Example 3.2. Let $h \in \mathbb{K}[x, y]$ be the polynomial given by $h(x, y) = x^2 + x^2y + xy^2$. Then $\mathrm{supp}(h) = \{(2, 0), (2, 1), (1, 2)\}$. Let $w = (3, -1)$, then we have $\ell(w, h) = 1$ and this minimum is attained only at the point $(1, 2) \in \mathrm{supp}(h)$. Then $\Delta(w, h) = \{(1, 2)\}$ and $p_w(h) = xy^2$. Let us remark that $\ell(w, \widetilde{\Gamma}_+(h)) = 0$ and $\Delta(w, \widetilde{\Gamma}_+(h)) = \{(0, 0)\}$. In general it is immediate to see that, if $g \in \mathbb{K}[x_1, \dots, x_n]$ and $v \in \mathbb{R} \smallsetminus \{0\}$ then $\ell(v, \widetilde{\Gamma}_+(g)) \leqslant \ell(v, g)$ and equality holds if and only if $\ell(v, g) \leqslant 0$.

Given a subset $J \subseteq \mathcal{F}(\widetilde{\Gamma}_+)$ and $h \in \mathbb{K}[x_1, \dots, n]$, we denote by $\Delta_J(h)$ the intersection $\cap_{w\in J}\Delta(w, h)$. We define the *principal part of h with respect to J*, which we will denote by $\mathrm{p}_J(h)$, as the sum of all terms $a_k x^k$ such that $k \in \Delta_J(h)$. If $\Delta_J(h) = \emptyset$, then we set $\mathrm{p}_J(h) = 0$.

We denote by $|A|$ the cardinal of a given finite set A. If Δ is a face of $\widetilde{\Gamma}_+$, then we denote by $\mathcal{J}(\Delta)$ the family of those subsets $J \subseteq \mathcal{F}(\widetilde{\Gamma}_+)$ such that $\Delta = \cap_{w\in J}\Delta(w, \widetilde{\Gamma}_+)$ and $\dim \Delta = n - |J|$. Then we observe that $\mathcal{J}(\Delta)$ is formed by all subsets $J \subseteq \mathcal{F}(\widetilde{\Gamma}_+)$ that minimally satisfy the condition $\Delta = \cap_{w\in J}\Delta(w, \widetilde{\Gamma}_+)$. In particular, if Δ is a vertex of $\widetilde{\Gamma}_+$ then $|J| = n$, for all $J \in \mathcal{J}(\Delta)$.

Definition 3.3. Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. We say that F is *adapted to* $\widetilde{\Gamma}_+$ when for any face Δ of $\widetilde{\Gamma}_+$ such that $0 \notin \Delta$ and for all $J \in \mathcal{J}(\Delta)$ we have

$$\big\{x \in \mathbb{K}^n : \mathrm{p}_J(F_1)(x) = \cdots = \mathrm{p}_J(F_p)(x) = 0\big\} \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\}.$$

We will also refer to the above inclusion as the *condition* $(\mathrm{C}_{F,J})$. We will denote the map $(\mathrm{p}_J(F_1), \dots, \mathrm{p}_J(F_p))$ by $\mathrm{p}_J(F)$.

The previous definition is motivated by the notion of pre-weighted homogeneous map (see Definition 7.1) and the *Newton non-degeneracy* condition on germs of analytic functions $(\mathbb{K}^n, 0) \to \mathbb{K}$ studied by Kouchnirenko [16] and Yoshinaga [27].

Remark 3.4. Let us consider a polynomial map $F : \mathbb{K}^n \to \mathbb{K}^p$ such that some of the component functions of F is a monomial x^k, for some $k \in \mathbb{Z}^n_{\geqslant 0}$, $k \neq 0$. Since $\mathrm{p}_J(x^k) = x^k$, for any $J \subseteq \mathcal{F}(\widetilde{\Gamma}_+)$, then F is automatically adapted to $\widetilde{\Gamma}_+$. This fact suggests that we need to strengthen the above definition in order to obtain a sufficiently restrictive class of polynomials $F : \mathbb{K}^n \to \mathbb{K}^p$ for which it is possible to obtain a lower bound for $\mathcal{L}_\infty(F)$.

Let $\mathtt{I} \subseteq \{1, \dots, n\}$, $\mathtt{I} \neq \emptyset$. We define $\mathbb{K}^n_{\mathtt{I}} = \{x \in \mathbb{K}^n : x_i = 0, \text{ for all } i \notin \mathtt{I}\}$ and we denote by $\pi_{\mathtt{I}}$ the natural projection $\mathbb{R}^n \to \mathbb{R}^n_{\mathtt{I}}$.

Let $h \in \mathbb{K}[x_1, \dots, x_n]$ and let us suppose that h is written as $h = \sum_k a_k x^k$. Then we denote by $h^{\mathtt{I}}$ the sum of all terms $a_k x^k$ such that $k \in \mathrm{supp}(h) \cap \mathbb{R}^n_{\mathtt{I}}$. If $\mathrm{supp}(h) \cap \mathbb{R}^n_{\mathtt{I}} = \emptyset$ then we set $h^{\mathtt{I}} = 0$. If $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ is a polynomial map then we define $F^{\mathtt{I}} = (F_1^{\mathtt{I}}, \dots, F_p^{\mathtt{I}}) : \mathbb{K}^n_{\mathtt{I}} \to \mathbb{K}^p$. Let us denote by $\widetilde{\Gamma}^{\mathtt{I}}_+$ the projection $\pi_{\mathtt{I}}(\widetilde{\Gamma}_+ \cap \mathbb{R}^n_{\mathtt{I}})$. It is easy to find examples of polyhedrons $\widetilde{\Gamma}_+$ such that $\widetilde{\Gamma}^{\mathtt{I}}_+$ is not equal to $\pi_{\mathtt{I}}(\widetilde{\Gamma}_+)$.

Definition 3.5. Let $F : \mathbb{K}^n \to \mathbb{K}^p$ denote a polynomial map. We say that F is *strongly adapted to* $\widetilde{\Gamma}_+$ when the map $F^{\mathtt{I}} : \mathbb{K}^n_{\mathtt{I}} \to \mathbb{K}^p$ is adapted to $\widetilde{\Gamma}^{\mathtt{I}}_+$, for any non-empty subset $\mathtt{I} \subseteq \{1, \dots, n\}$.

We will characterize the above notion in the next result. Let $w \in \mathcal{F}(\widetilde{\Gamma}_+)$ and let $h \in \mathbb{K}[x_1, \dots, x_n]$. Then we define

$$\ell^*(w, h) = \begin{cases} \ell(w, h), & \text{if } \ell(w, \widetilde{\Gamma}_+) < 0 \\ 0, & \text{if } \ell(w, \widetilde{\Gamma}_+) = 0. \end{cases}$$

Let us suppose that h is written as $h = \sum_k a_k x^k$. If $J \subseteq \mathcal{F}(\widetilde{\Gamma}_+)$, then we denote by $\mathrm{p}^*_J(h)$ the sum of all terms $a_k x^k$ such that $\langle k, w\rangle = \ell^*(w, h)$, for all $w \in J$. If the set of such terms $a_k x^k$ is empty, then we set $\mathrm{p}^*_J(h) = 0$.

Let us observe that, since $\widetilde{\Gamma}_+$ is convenient, then $\mathcal{F}(\widetilde{\Gamma}_+) = \mathcal{F}_0(\widetilde{\Gamma}_+) \cup \{e_1, \dots, e_n\}$. Therefore, if $w \in \mathcal{F}(\widetilde{\Gamma}_+)$, then the condition $\ell(w, \widetilde{\Gamma}_+) = 0$ is equivalent to saying that w is equal to some vector e_i. Thus if $J \cap \{e_1, \dots, e_n\} = \emptyset$, then $\mathrm{p}_J(h) = \mathrm{p}^*_J(h)$. If $J \cap \{e_1, \dots, e_n\} \neq \emptyset$ then

$$\mathrm{p}^*_J(h) = \begin{cases} \mathrm{p}_J(h), & \text{if } \ell(w, h) = 0, \text{ for all } w \in J \cap \{e_1, \dots, e_n\} \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Proposition 3.6. *Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. Then the following conditions are equivalent:*

(i) *F is strongly adapted to $\widetilde{\Gamma}_+$.*

(ii) *For any face* Δ *of* $\widetilde{\Gamma}_+$ *such that* $0 \notin \Delta$ *and for all* $J \in \mathcal{J}(\Delta)$ *we have*

$$\{x \in \mathbb{K}^n : \mathrm{p}^*_J(F_1)(x) = \cdots = \mathrm{p}^*_J(F_p)(x) = 0\} \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\}. \tag{10}$$

Proof. Let $J \subseteq \mathcal{F}(\widetilde{\Gamma}_+)$. Then we can express J as $J = J_1 \cup J_2$, where $J_2 = J \cap \{e_1, \dots, e_n\}$ and $J_1 = J \smallsetminus J_2$. Let $\mathtt{I} \subseteq \{1, \dots, n\}$, $\mathtt{I} \neq \emptyset$. Using the definition of principal part with respect to J it is immediate to see that, for any polynomial $h \in \mathbb{K}[x_1, \dots, x_n]$, we have

$$\mathrm{p}^*_J(h) = \mathrm{p}_{\pi_{\mathtt{I}}(J_1)}(h^{\mathtt{I}}). \tag{11}$$

Let Δ be a face of $\widetilde{\Gamma}_+$. By the definition of $\mathcal{J}(\Delta)$, we deduce that if $\mathtt{I}$ denotes the minimal subset of $\{1, \dots, n\}$ such that Δ is contained in the coordinate subspace $\mathbb{R}^n_{\mathtt{I}}$, then

$$\mathcal{F}(\widetilde{\Gamma}^{\mathtt{I}}_+) = \{\pi_{\mathtt{I}}(w) : w \in \mathcal{F}(\widetilde{\Gamma}_+),\ \Delta(w, \widetilde{\Gamma}_+) \cap \mathbb{R}^n_{\mathtt{I}} \neq \emptyset \text{ and } \dim \Delta(\pi_{\mathtt{I}}(w), \widetilde{\Gamma}^{\mathtt{I}}_+) = |\mathtt{I}| - 1\} \tag{12}$$

$$\mathcal{J}(\pi_{\mathtt{I}}(\Delta)) = \{\pi_{\mathtt{I}}(J_1) : J \in \mathcal{J}(\Delta)\}. \tag{13}$$

Then the equivalence between (i) and (ii) follows as an immediate application of (11), (12) and (13). □

We will refer to inclusion (10) as the *condition* $(\mathrm{C}^*_{F,J})$.

Corollary 3.7. *Let us suppose that* F_i *is convenient, for all* $i = 1, \dots, p$. *Then the following conditions are equivalent:*

(i) F *is adapted to* $\widetilde{\Gamma}_+$.
(ii) F *is strongly adapted to* $\widetilde{\Gamma}_+$.

Proof. It follows as an immediate application of (9) and Proposition 3.6. □

The following definition is concerned only with polynomial maps. That is, it is not applied to pairs $(F, \widetilde{\Gamma}_+)$ formed by a polynomial map F and a Newton polyhedron $\widetilde{\Gamma}_+$ (see Definitions 3.3 and 3.5). Thus, once we fix coordinates in $\mathbb{K}^n$, it can be considered as an intrinsic property of polynomial maps $\mathbb{K}^n \to \mathbb{K}^p$.

Definition 3.8. Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. We say that F is *non-degenerate* when for all $w \in \mathbb{R}^n_0$ we have

$$\Big\{x \in \mathbb{K}^n : \mathrm{p}_w(F_1)(x) = \cdots = \mathrm{p}_w(F_p)(x) = 0\Big\} \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\}.$$

We will refer to the above inclusion as the *condition* $(\mathrm{C}_{F,w})$.

Proposition 3.9. *Let* $F : \mathbb{K}^n \to \mathbb{K}^p$ *be a polynomial map such that* F_i *is convenient, for all* $i = 1, \dots, p$. *Then the following conditions are equivalent:*

(i) F *is non-degenerate.*
(ii) F *is adapted to the Minkowski sum* $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(F_1) + \cdots + \widetilde{\Gamma}_+(F_p)$.

Proof. Let us see (i) $\Rightarrow$ (ii). Let Δ be a face of $\widetilde{\Gamma}_+$ such that $0 \notin \Delta$. Let $J \in \mathcal{J}(\Delta)$. In particular $\Delta = \cap_{w\in J}\Delta(w, \widetilde{\Gamma}_+) \neq \emptyset$. If $k \in \Delta$ and we write $k = k_1 + \cdots + k_p$, where $k_i \in \widetilde{\Gamma}_+(F_i)$, for all $i = 1, \dots, p$, then we have $k_i \in \Delta(w, \widetilde{\Gamma}_+(F_i))$, for all $i = 1, \dots, p$ and all $w \in J$, as a consequence of Lemma 2.3(i). In particular $\cap_{w\in J}\Delta(w, \widetilde{\Gamma}_+(F_i)) \neq \emptyset$, for all $i = 1, \dots, p$, and then $\cap_{w\in J}\Delta(w, \widetilde{\Gamma}_+(F_i)) = \Delta(\sum_{w\in J} w, \widetilde{\Gamma}_+(F_i))$, for all $i = 1, \dots, p$. Let us observe that $\Delta(w, F_i) = \Delta(w, \widetilde{\Gamma}_+(F_i))$, for all $i = 1, \dots, p$ and all $w \in J$, since each polynomial F_i is convenient. Then we obtain the equality of polynomials $p_J(F_i) = p_v(F_i)$, for all $i = 1, \dots, p$, where $v = \sum_{w\in J} w$. Thus condition $(\mathrm{C}_{F,J})$ is equivalent to condition $(\mathrm{C}_{F,w})$ and the result follows.

Let us see (ii) $\Rightarrow$ (i). Let $v \in \mathbb{R}^n \smallsetminus \{0\}$. Then there exists some $J \subseteq \mathcal{F}(\widetilde{\Gamma}_+)$ such that $\Delta(v, \widetilde{\Gamma}_+) = \cap_{w\in J}\Delta(w, \widetilde{\Gamma}_+) \neq \emptyset$ and $J \in \mathcal{J}(\Delta(v, \widetilde{\Gamma}_+))$. Then, similarly to the proof of the other implication, we deduce that $p_v(F_i) = p_J(F_i)$, for all $i = 1, \dots, p$, and hence the result follows. $\square$

4. Special monomials with respect to polynomial maps

We say that a given condition holds for all $\|x\| \gg 1$ when there exists a constant $M > 0$ such that the said condition holds for all $x \in \mathbb{K}^n$ for which $\|x\| \geqslant M$.

Definition 4.1. Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. We say that an element $h \in \mathbb{K}[x_1, \dots, x_n]$ is *special with respect to* F when there exists some constant $C > 0$ such that

$$\|h(x)\| \leqslant C\|F(x)\|$$

for all $\|x\| \gg 1$.

In view of the results of Lejeune-Teissier [18] we can consider the previous definition as a kind of global or polynomial version of the notion of integral element over an ideal in a local ring. Let us fix coordinates $x_1, \dots, x_n$ in $\mathbb{K}^n$. Then we define the set:

$$S(F) = \left\{k \in \mathbb{Z}^n_{\geqslant 0} : x^k \text{ is special with respect to } F\right\}.$$

If $S(F) \smallsetminus \{0\} \neq 0$, then it is obvious that there exists some $M > 0$ such that

$$F^{-1}(0) \cap \{x \in \mathbb{K}^n : \|x\| \geqslant M\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}.$$

Proposition 4.2. *Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. Then $S(F) \subseteq \widetilde{\Gamma}_+(F)$.*

Proof. Let us suppose that $S(F) \neq \emptyset$. Let $w = (w_1, \dots, w_n) \in \mathbb{R}^n_0$ and let us consider the meromorphic curve $\varphi_w : \mathbb{K} \smallsetminus \{0\} \to \mathbb{K}^n$ given by $\varphi_w(t) = (t^{w_1}, \dots, t^{w_n})$. If $k \in S(F)$, then there exists a constant $C > 0$ such that

$$|x^k| \leqslant C\|F(x)\| \tag{14}$$

for all $\|x\| \gg 1$. Since $w_0 < 0$, then $\lim_{t\to 0} \|\varphi_w(t)\| = \infty$. In particular, if we compose with $\varphi_w(t)$ both sides of inequality (14) then we obtain that the limit $\lim_{t\to 0} |t^{\langle k,w\rangle}|/\|F(\varphi_w(t))\|$ exists, which is equivalent to saying that the order of $t^{\langle k,w\rangle}$ is bigger than or equal to the order of $\|F(\varphi_w(t))\|$. That is, $\langle k,w\rangle \geqslant \min\{\ell(w,F_1),\dots,\ell(w,F_p)\} \geqslant \ell(w,\widetilde{\Gamma}_+(F))$. Therefore $\langle k,w\rangle \geqslant \ell(w,\widetilde{\Gamma}_+(F))$, for all $w\in\mathbb{R}^n_0$, which means that $k\in\widetilde{\Gamma}_+(F)$, by (4). $\square$

We remark that when $S(F)\neq\emptyset$, then it is easy to check that $S(F)$ is convex. That is, if $k,k'\in S(F)$ then $\lambda k+(1-\lambda)k'\in S(F)$, for all $\lambda\in[0,1]$ such that $\lambda k+(1-\lambda)k'\in\mathbb{Z}^n_{\geqslant 0}$.

In the remaining section we denote by $\widetilde{\Gamma}_+$ a convenient global Newton polyhedron in $\mathbb{R}^n$. Let us recall that $\mathcal{F}(\widetilde{\Gamma}_+)=\mathcal{F}_0(\widetilde{\Gamma}_+)\cup\{e_1,\dots,e_n\}$, where $e_1,\dots,e_n$ denotes the canonical basis of $\mathbb{R}^n$ and $\mathcal{F}_0(\widetilde{\Gamma}_+)$ are the primitive vectors supporting some face of $\widetilde{\Gamma}_+$ of dimension $n-1$ not passing through the origin.

The next two results are tools that allow to give approximations to the set $S(F)$.

Theorem 4.3. *Let $F=(F_1,\dots,F_p):\mathbb{K}^n\to\mathbb{K}^p$ be a polynomial map. Let us suppose that F is adapted to $\widetilde{\Gamma}_+$. Let $k\in\mathbb{Z}^n_{\geqslant 0}$ such that*

$$\langle k,w\rangle \geqslant \max\{\ell(w,F_1),\dots,\ell(w,F_p)\}, \tag{15}$$

for all $w\in\mathcal{F}(\widetilde{\Gamma}_+)$. Then $k\in S(F)$.

We will see the proof of the previous result in Section 5. Let us remark that inequality (15) is assumed for any $w\in\mathcal{F}(\widetilde{\Gamma}_+)$ in Theorem 4.3. We will see that the same conclusion holds if we assume (15) only for the vectors $w\in\mathcal{F}(\widetilde{\Gamma}_+)$ such that $0\notin\Delta(w,\widetilde{\Gamma}_+)$ and F is strongly adapted to $\widetilde{\Gamma}_+$. This fact is shown in the following result, which is independent from Theorem 4.3.

Theorem 4.4. *Let $F=(F_1,\dots,F_p):\mathbb{K}^n\to\mathbb{K}^p$ be a polynomial map. Let us suppose that F is strongly adapted to $\widetilde{\Gamma}_+$. Let $k\in\mathbb{Z}^n_{\geqslant 0}$ such that*

$$\langle k,w\rangle \geqslant \max\{\ell(w,F_1),\dots,\ell(w,F_p)\},$$

for all $w\in\mathcal{F}_0(\widetilde{\Gamma}_+)$. Then $k\in S(F)$.

In Section 5 we give first the proof of Theorem 4.4. As we will see, the proof of Theorem 4.3 will follow a similar argument. We remark that in Sections 6, 7 and 8 we derive some consequences of the argument of the proof of Theorem 4.4.

5. Proof of Theorem 4.4

We need to introduce some definitions and results before proving Theorem 4.4.

Let $a^1,\dots,a^r\in\mathbb{R}^n$ such that $a^i\neq 0$, for all $i=1,\dots,r$. The set $\sigma=\mathbb{R}_{\geqslant 0}a^1+\cdots+\mathbb{R}_{\geqslant 0}a^r$ is called the *cone* spanned, or generated, by $a^1,\dots,a^r$. This is also known as the *positive hull* of $a^1,\dots,a^r$. If σ is minimally generated by $a^1,\dots,a^r$ and a^i is a primitive vector of $\mathbb{Z}^n$, for all $i=1,\dots,r$, then we will say that $a^1,\dots,a^r$ are the *primitive generators* of σ. The intersection of σ with a supporting hyperplane of σ is called a *face* of σ.

We define the *dimension* of the cone $\sigma=\mathbb{R}_{\geqslant 0}a^1+\cdots+\mathbb{R}_{\geqslant 0}a^r$, denoted by $\dim(\sigma)$, as the dimension of the real vector subspace spanned by $a^1,\dots,a^r$. We say that σ is *simplicial* when $\dim(\sigma)=r$.

Along this section we denote by $\widetilde{\Gamma}_+$ a convenient global Newton polyhedron in $\mathbb{R}^n$. Then let us consider the equivalence relation in $\mathbb{R}^n$ defined as follows. If $u, v \in \mathbb{R}^n$, then $u \sim v$ if and only if $\Delta(u, \widetilde{\Gamma}_+) = \Delta(v, \widetilde{\Gamma}_+)$. Obviously the corresponding quotient space $X = \mathbb{R}^n / \sim$ is bijective with the set of faces of $\widetilde{\Gamma}_+$.

If Δ is a face of $\widetilde{\Gamma}_+$, then we denote by $[\Delta]$ the closure, in the euclidian sense, of the set of vectors supporting Δ. Hence $[\Delta]$ is equal to a cone $\mathbb{R}_{\geqslant 0}a^1 + \cdots + \mathbb{R}_{\geqslant 0}a^r$, for some primitive vectors $a^1, \dots, a^r \in \mathbb{Z}^n$. In particular, if Δ has dimension $n-1$ and $\Delta = \Delta(w, \widetilde{\Gamma}_+)$, for some $w \in \mathbb{Z}^n \smallsetminus \{0\}$, then $[\Delta] = \mathbb{R}_{\geqslant 0} w$. It is immediate to see that $\dim[\Delta] = n - \dim \Delta$, for each face Δ of $\widetilde{\Gamma}_+$.

Given a cone $\sigma = \mathbb{R}_{\geqslant 0}a^1 + \cdots + \mathbb{R}_{\geqslant 0}a^r \subseteq \mathbb{R}^n$, by Caratheodory's theorem (see [12, p. 139]), we have that σ can be expressed as the union of cones $\{\sigma_1, \dots, \sigma_m\}$ of $\mathbb{R}_n$ such that

(i) $\sigma_i \cap \sigma_j$ is a face of σ_i and of σ_j, for all $i, j \in \{1, \dots, m\}$,
(ii) each cone σ_i is written as $\mathbb{R}_{\geqslant 0}a^{i_1} + \cdots + \mathbb{R}_{\geqslant 0}a^{i_s}$, where $1 \leqslant i_1 < \cdots < i_s \leqslant r$ and $\{a^{i_1}, \dots, a^{i_s}\}$ is linearly independent.

This fact also follows from [14, p. 147, Theorem 1.12]. Then we can decompose σ as the union of simplicial cones with generators contained in $\{a^1, \dots, a^r\}$. We call such a decomposition a *simplicial subdivision* of σ. Let us fix a simplicial subdivision of each n-dimensional cone $[\Delta]$, where Δ denotes a vertex of $\widetilde{\Gamma}_+$. Then we denote by $\Sigma^{(n)}$ the set of simplicial cones of dimension n arising from the fixed simplicial subdivisions of $[\Delta]$, for any vertex Δ of $\widetilde{\Gamma}_+$.

For each $\sigma \in \Sigma^{(n)}$ let us consider a copy of $\mathbb{K}^n$ which we will denote by $\mathbb{K}^n(\sigma)$. We will write the elements of $\mathbb{K}^n(\sigma)$ as $y_\sigma = (y_{\sigma,1}, \dots, y_{\sigma,n})$. Let $W_\sigma = \{y_\sigma \in \mathbb{K}^n(\sigma) : 0 < |y_{\sigma,j}| \leqslant 1, \text{for all } j = 1, \dots, n\}$, for all $\sigma \in \Sigma^{(n)}$, and let $V = \{x \in (\mathbb{K} \smallsetminus \{0\})^n : \max_i |x_i| \geqslant 1\}$.

Let us consider, for each cone $\sigma \in \Sigma^{(n)}$, the monomial map $\pi_\sigma : W_\sigma \to \mathbb{K}^n$ given by

$$\pi_\sigma(y_{\sigma,1}, \dots, y_{\sigma,n}) = (y_{\sigma,1}^{a_1^1(\sigma)} \cdots y_{\sigma,n}^{a_1^n(\sigma)}, \dots, y_{\sigma,1}^{a_n^1(\sigma)} \cdots y_{\sigma,n}^{a_n^n(\sigma)}),$$

where we suppose that $a^1(\sigma), \dots, a^n(\sigma)$ are the primitive generators of σ and each vector $a^i(\sigma)$ is written as $a^i(\sigma) = (a^i_1(\sigma), \dots, a^i_n(\sigma))$, for all $i = 1, \dots, n$.

Lemma 5.1. *Let W denote the union of all sets W_σ, where $\sigma \in \Sigma^{(n)}$. Let $\pi : W \to (\mathbb{K} \smallsetminus \{0\})^n$ be the map defined by $\pi(y_\sigma) = \pi_\sigma(y_\sigma)$, for all $y_\sigma \in W_\sigma$, $\sigma \in \Sigma^{(n)}$. Then the restriction $\pi_{|_{\pi^{-1}(V)}} : \pi^{-1}(V) \to V$ is surjective.*

Proof. We will develop the proof in the case $\mathbb{K} = \mathbb{C}$. The case $\mathbb{K} = \mathbb{R}$ is analogous. Let $x = (x_1, \dots, x_n) \in V$. Let us write x_j as $x_j = r_j e^{2\pi\alpha_j \mathbf{i}}$, where $r_j \in [0, +\infty[$, $\alpha_j \in [0, 1[$, for all $j = 1, \dots, n$, and $\mathbf{i} = \sqrt{-1}$. Let us define the vector $v(x) = (-\log(r_1), \dots, -\log(r_n)) \in \mathbb{R}^n$. There exists a cone $\sigma \in \Sigma^{(n)}$ such that $v(x) \in \sigma$. Let $\{a^1(\sigma), \dots, a^n(\sigma)\}$ be a linearly independent set of vectors of $\mathbb{R}^n$ such that $\sigma = \mathbb{R}_{\geqslant 0}a^1(\sigma) + \cdots + \mathbb{R}_{\geqslant 0}a^n(\sigma)$. Thus there exist $\beta_1, \dots, \beta_n \geqslant 0$ such that $v(x) = \beta_1 a^1(\sigma) + \cdots + \beta_n a^n(\sigma)$.

Let $r_{\sigma,j} = e^{-\beta_j}$, for all $j = 1, \dots, n$. We observe that

$$r_{\sigma,1}^{a_j^1(\sigma)} \cdots r_{\sigma,n}^{a_j^n(\sigma)} = r_j. \tag{16}$$

Let $y_{\sigma,j} = r_{\sigma,j} e^{2\pi\theta_{\sigma,j}\mathbf{i}}$, where $\theta_{\sigma,j} \in [0,1[$, and let $y_\sigma = (y_{\sigma,1},\dots,y_{\sigma,n})$. Using (16) we observe that $\pi_\sigma(y_\sigma) = x$ if and only if the vector $\theta_\sigma = (\theta_{\sigma,1},\dots,\theta_{\sigma,n})$ verifies that

$$a_j^1(\sigma)\theta_{\sigma,1} + \cdots + a_j^n(\sigma)\theta_{\sigma,n} \equiv \alpha_j \mod \mathbb{Z},$$

for all $j = 1,\dots,n$. We can find such a vector θ_σ, since $\{a^1(\sigma),\dots,a^n(\sigma)\}$ is linearly independent. Moreover, we observe that $0 < r_{\sigma,j} \leqslant 1$, for all $j = 1,\dots,n$, then $y_\sigma \in W_\sigma$ and hence π is surjective. □

In order to simplify the notation, if $w \in \mathcal{F}(\widetilde{\Gamma}_+)$ then in this section we will denote the number $\ell(w,\widetilde{\Gamma}_+)$ only by $\ell(w)$. Let us fix a cone $\sigma \in \Sigma^{(n)}$ and let $a^1(\sigma),\dots,a^n(\sigma)$ be the primitive generators of σ. Since $\{a^1(\sigma),\dots,a^n(\sigma)\} \subseteq \mathcal{F}(\widetilde{\Gamma}_+)$, some vector $a^j(\sigma)$ can coincide with some vector of the canonical basis. Then, we can assume that

$$\{j : \ell(a^j(\sigma)) < 0\} = \{1,\dots,r\} \tag{17}$$

$$\{j : \ell(a^j(\sigma)) = 0\} = \{r+1,\dots,r+s\} \tag{18}$$

for some integers $r,s \geqslant 0$ such that $r+s = n$. Hence, if $s \geqslant 1$, there exist indices $1 \leqslant i_1 < \cdots < i_s \leqslant n$ such that $a^{r+j}(\sigma) = e_{i_j}$, for all $j = 1,\dots,s$.

Let $F = (F_1,\dots,F_p) : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. Let us fix an index $i \in \{1,\dots,n\}$. Let us suppose that F_i is written as $F_i = \sum a_k x^k$. Let us define

$$Z = \Big\{k \in \operatorname{supp}(F_i) : \langle k, a^j(\sigma)\rangle = \ell^*(a^j(\sigma),F_i),\ \text{for some } j \in \{1,\dots,n\}\Big\} \tag{19}$$

$$Z_J = \Big\{k \in \operatorname{supp}(F_i) : \langle k, a^j(\sigma)\rangle = \ell^*(a^j(\sigma),F_i),\ \text{if and only if } j \in J\Big\} \tag{20}$$

for all $J \subseteq \{1,\dots,n\}$. Therefore Z is the disjoint union $\cup_J Z_J$, where J varies in the set of non-empty subsets of $\{1,\dots,n\}$. If $k \in \mathbb{Z}^n_{\geqslant 0}$, then $x^k \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\langle k,a^1(\sigma)\rangle} \cdots y_{\sigma,n}^{\langle k,a^n(\sigma)\rangle}$. Hence

$$\begin{aligned}
F_i \circ \pi_\sigma(y_\sigma) &= \left(\sum_{k\in Z} a_k x^k + \sum_{k\notin Z} a_k x^k\right) \circ \pi_\sigma(y_\sigma)\\
&= \sum_{J\subseteq\{1,\dots,n\}} \sum_{k\in Z_J} a_k y_{\sigma,1}^{\ell(a^1(\sigma),F_i)} \cdots y_{\sigma,r}^{\ell(a^r(\sigma),F_i)} \Bigg(\prod_{\substack{j\notin J\\ \ell(a^j(\sigma))<0}} y_{\sigma,j}^{\langle k,a^j(\sigma)\rangle - \ell(a^j(\sigma),F_i)}\Bigg)\Bigg(\prod_{\substack{j\notin J\\ \ell(a^j(\sigma))=0}} y_{\sigma,j}^{k_{i_{j-r}}}\Bigg)\\
&\quad + \sum_{k\notin Z} a_k y_{\sigma,1}^{\langle k,a^1(\sigma)\rangle} \cdots y_{\sigma,r}^{\langle k,a^r(\sigma)\rangle} y_{\sigma,r+1}^{k_{i_1}} \cdots y_{\sigma,n}^{k_{i_s}}
\end{aligned}$$

$$= y_{\sigma,1}^{\ell(a^1(\sigma),F_i)} \cdots y_{\sigma,r}^{\ell(a^r(\sigma),F_i)} \Bigg(\sum_{J\subseteq\{1,\dots,n\}} \sum_{k\in Z_J} a_k \Bigg(\prod_{\substack{j\notin J\\ \ell(a^j(\sigma))<0}} y_{\sigma,j}^{\langle k,a^j(\sigma)\rangle - \ell(a^j(\sigma),F_i)}\Bigg)\Bigg(\prod_{\substack{j\notin J\\ \ell(a^j(\sigma))=0}} y_{\sigma,j}^{k_{i_{j-r}}}\Bigg) \tag{21}$$

$$+ \sum_{k\notin Z} a_k y_{\sigma,1}^{\langle k,a^1(\sigma)\rangle - \ell(a^1(\sigma),F_i)} \cdots y_{\sigma,r}^{\langle k,a^r(\sigma)\rangle - \ell(a^r(\sigma),F_i)} y_{\sigma,r+1}^{k_{i_1}} \cdots y_{\sigma,n}^{k_{i_s}}\Bigg). \tag{22}$$

We denote by $F^*_{\sigma,i}$ the polynomial such that

$$F_i \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\ell(a^1(\sigma),F_i)} \cdots y_{\sigma,r}^{\ell(a^r(\sigma),F_i)} \cdot F^*_{\sigma,i}(y_\sigma) \tag{23}$$

for all $y_\sigma \in W_\sigma$. That is, $F^*_{\sigma,i}$ is the polynomial in the variables $y_{\sigma,1}, \dots, y_{\sigma,n}$ given by the expression that appears between (21) and (22) in parentheses.

If $M > 0$ then we denote by V_M the set $\{x \in (\mathbb{K} \setminus \{0\})^n : \|x\| \geqslant M\}$.

Proposition 5.2. *Let us suppose that F is strongly adapted to $\widetilde{\Gamma}_+$. Then for all $\sigma \in \Sigma^{(n)}$ there exists a constant $M_\sigma > 0$ such that*

$$\inf_{y_\sigma \in \pi_\sigma^{-1}(V_{M_\sigma})} \sup_i |F^*_{\sigma,i}(y_\sigma)| \neq 0.$$

Proof. Let us assume the opposite. That is, let $\sigma \in \Sigma^{(n)}$ such that

$$\inf_{y_\sigma \in \pi_\sigma^{-1}(V_M)} \sup_i |F^*_{\sigma,i}(y_\sigma)| = 0$$

for all $M > 0$. Then there exists a sequence $\{y_m\}_{m\geqslant 1} \subseteq W_\sigma$ such that $\{\pi_\sigma(y_m)\}_{m\geqslant 1} \to \infty$ and $\{F^*_{\sigma,i}(y_m)\}_{m\geqslant 1} \to 0$ as $m \to \infty$, for all $i = 1, \dots, p$.

Let $\overline{W}_\sigma$ denote the closure of W_σ, that is $\overline{W}_\sigma = \{y_\sigma \in \mathbb{K}^n(\sigma) : \|y_\sigma\| \leqslant 1\}$. Let $\mathbf{y} = (\mathbf{y}_1, \dots, \mathbf{y}_n) \in \overline{W}_\sigma$ be a limit point of the sequence $\{y_m\}_{m\geqslant 1}$. Let $J_0 = \{j : \mathbf{y}_j = 0\}$. We have that $J_0 \neq \emptyset$, since $\{\pi_\sigma(y_m)\}_{m\geqslant 1} \to \infty$. Moreover $F^*_{\sigma,i}(\mathbf{y}) = 0$, for all $i = 1, \dots, p$.

Let $a^1(\sigma), \dots, a^n(\sigma)$ be the primitive generators of σ. The condition $\{\pi_\sigma(y_m)\}_{m\geqslant 1} \to \infty$ implies that there exists some $j \in J_0$ such that $a^j(\sigma)$ has some negative component. In particular $\ell(a^j(\sigma)) < 0$, since $\widetilde{\Gamma}_+$ is convenient. Therefore $0 \notin \Delta(a^j(\sigma), \widetilde{\Gamma}_+)$ for some $j \in J_0$, which is to say that the face $\cap_{j\in J_0}\Delta(a^j(\sigma), \widetilde{\Gamma}_+)$ does not contain the origin.

We observe that

$$F^*_{\sigma,i}(\mathbf{y}) = \sum_{\substack{J \subseteq \{1,\dots,n\} \\ J_0 \subseteq J}} \sum_{k \in Z_J} a_k \Bigg(\prod_{\substack{j \notin J \\ \ell(a^j(\sigma))<0}} \mathbf{y}_j^{\langle k, a^j(\sigma)\rangle - \ell(a^j(\sigma,F_i))} \Bigg) \Bigg(\prod_{\substack{j \notin J \\ \ell(a^j(\sigma))=0}} \mathbf{y}_j^{k_{i_{j-r}}} \Bigg). \tag{24}$$

On the other hand, given any $z_\sigma = (z_{\sigma,1}, \dots, z_{\sigma,n}) \in W_\sigma$, we have

$$\mathrm{p}^*_{J_0}(F_i) \circ \pi_\sigma(z_\sigma) = \sum_{\substack{\langle k, a^j(\sigma)\rangle = \ell^*(a^j(\sigma),F_i) \\ \text{for all } j \in J_0}} a_k z_{\sigma,1}^{\langle k, a^1(\sigma)\rangle} \cdots z_{\sigma,n}^{\langle k, a^n(\sigma)\rangle} \tag{25}$$

$$= z_{\sigma,1}^{\ell(a^1(\sigma),F_i)} \cdots z_{\sigma,r}^{\ell(a^r(\sigma),F_i)} \sum_{\substack{J \subseteq \{1,\dots,n\} \\ J_0 \subseteq J}} \sum_{k \in Z_J} a_k \Bigg(\prod_{\substack{j \notin J \\ \ell(a^j(\sigma))<0}} z_{\sigma,j}^{\langle k, a^j(\sigma)\rangle - \ell(a^j(\sigma,F_i))} \Bigg) \Bigg(\prod_{\substack{j \notin J \\ \ell(a^j(\sigma))=0}} z_{\sigma,j}^{k_{i_{j-r}}} \Bigg) \tag{26}$$

Let us consider the point $\widetilde{\mathbf{y}} = (\widetilde{\mathbf{y}}_1, \dots, \widetilde{\mathbf{y}}_n)$ defined by

$$\widetilde{\mathbf{y}}_j = \begin{cases} \mathbf{y}_j, & \text{if } j \notin J_0 \\ 1, & \text{if } j \in J_0. \end{cases}$$

Comparing (24) and (26) we obtain

$$\mathrm{p}^*_{J_0}(F_i)\circ\pi_\sigma(\widetilde{\mathbf{y}}) = \prod_{j\notin J_0} \mathbf{y}_{\sigma,j}^{\ell(a^j(\sigma),F_i)} F^*_{\sigma,i}(\mathbf{y}) = 0 \tag{27}$$

for all $i=1,\dots,p$. Since $\cap_{j\in J_0}\Delta(a^j(\sigma),\widetilde{\Gamma}_+)$ is a face of $\widetilde{\Gamma}_+$ not containing the origin, relation (27) gives a contradiction and then the result follows. □

Proof of Theorem 4.4. Let us fix a cone $\sigma\in\Sigma^{(n)}$. By Proposition 5.2 there exist positive constants D_σ and M_σ such that

$$D_\sigma < \sup_i |F^*_{\sigma,i}(y_\sigma)|$$

for all $y_\sigma\in\pi_\sigma^{-1}(V_{M_\sigma})$. Then, for any $y_\sigma\in\pi_\sigma^{-1}(V_{M_\sigma})$ we have the following chain of inequalities:

$$\sup_i |F_i(x)|\circ\pi_\sigma(y_\sigma) = \sup_i |y_{\sigma,1}^{\ell(a^1(\sigma),F_i)}\cdots y_{\sigma,r}^{\ell(a^r(\sigma),F_i)}\cdot F_i^*(y_\sigma)| \tag{28}$$

$$\geqslant |y_{\sigma,1}|^{\max_i \ell(a^1(\sigma),F_i)}\cdots|y_{\sigma,r}|^{\max_i \ell(a^r(\sigma),F_i)}\sup_i|F^*_{\sigma,i}(y_\sigma)| \tag{29}$$

$$\geqslant |y_{\sigma,1}|^{\langle k,a^1(\sigma)\rangle}\cdots|y_{\sigma,r}|^{\langle k,a^r(\sigma)\rangle}\cdot D_\sigma \tag{30}$$

$$\geqslant |y_{\sigma,1}|^{\langle k,a^1(\sigma)\rangle}\cdots|y_{\sigma,r}|^{\langle k,a^r(\sigma)\rangle}|y_{\sigma,r+1}|^{k_{i_1}}\cdots|y_{\sigma,n}|^{k_{i_s}}\cdot D_\sigma \tag{31}$$

$$= D_\sigma\|x^k\|\circ\pi_\sigma(y_\sigma). \tag{32}$$

Let $M=\max_{\sigma\in\Sigma^{(n)}}M_\sigma$. We can assume $\sqrt{n}\leqslant M$. Then V contains V_M and, by Lemma 5.1 we have that the set $\{\pi_\sigma(y_\sigma) : y_\sigma\in\pi_\sigma^{-1}(V_M),\ \sigma\in\Sigma^{(n)}\} = V_M$. In particular, if $C=(\min_{\sigma\in\Sigma^{(n)}}D_\sigma)^{-1}$ we conclude that

$$\|x^k\|\leqslant C\sup_i|F_i(x)| \tag{33}$$

for all $x\in(\mathbb{K}\smallsetminus\{0\})^n$ such that $\|x\|\geqslant M$. By the continuity of the functions of both sides of the previous inequality, we obtain that (33) holds for all $x\in\mathbb{K}^n$ such that $\|x\|\geqslant M$. Thus $k\in S(F)$. □

Proof of Theorem 4.3. Let us modify the definitions of Z and of Z_J, in (19) and (20) respectively, by replacing $\ell^*(a^j(\sigma),F_i)$ by $\ell(a^j(\sigma),F_i)$. Then we obtain, as in (23), a polynomial $F'_{\sigma,i}\in\mathbb{K}[y_{\sigma,1},\dots,y_{\sigma,n}]$ such that

$$F_i\circ\pi_\sigma(y_\sigma) = y_{\sigma,1}^{\ell(a^1(\sigma),F_i)}\cdots y_{\sigma,n}^{\ell(a^n(\sigma),F_i)}\cdot F'_{\sigma,i}(y_\sigma)$$

for all $y_\sigma\in W_\sigma$ and all $i=1,\dots,p$. Following the proof Proposition 5.2, we obtain that for each $\sigma\in\Sigma^{(n)}$ there exists a constant $M_\sigma>0$ such that

$$\inf_{y_\sigma\in\pi_\sigma^{-1}(V_{M_\sigma})}\ \sup_i|F'_{\sigma,i}(y_\sigma)|\neq 0.$$

Hence we can reproduce the argument of the proof of Theorem 4.4 to obtain that $k\in S(F)$. □

6. Consequences of the main result

Let us fix along this section a convenient global Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n$ and a polynomial map $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$. Let us define $\mathbf{L}(w, F) = \max\{\ell(w, F_1), \dots, \ell(w, F_p)\}$, for any vector $w \in \mathbb{R}^n$.

Corollary 6.1. *Let us suppose that F is strongly adapted to $\widetilde{\Gamma}_+$. Let $k \in \mathbb{Z}^n_{\geqslant 0}$ and $\theta \geqslant 0$ such that $\theta\langle k, w\rangle \geqslant \mathbf{L}(w, F)$, for all $w \in \mathcal{F}_0(\widetilde{\Gamma}_+)$. Then there exist positive constants C and M such that*

$$\|x^k\|^\theta \leqslant C\|F(x)\|$$

for all $x \in \mathbb{K}^n$ such that $|x| \geqslant M$.

Proof. It follows by the same argument of the proof of Theorem 4.4 by replacing inequalities (28)-(32) by the following inequalities:

$$\sup_i |F_i(x)| \circ \pi_\sigma(y_\sigma) = \sup_i |y_{\sigma,1}^{\ell(a^1(\sigma),F_i)} \cdots y_{\sigma,r}^{\ell(a^r(\sigma),F_i)} \cdot F_i^*(y_\sigma)| \tag{34}$$

$$\geqslant |y_{\sigma,1}|^{\mathbf{L}(a^1(\sigma),F)} \cdots |y_{\sigma,r}|^{\mathbf{L}(a^r(\sigma),F)} \sup_i |F^*_{\sigma,i}(y_\sigma)| \tag{35}$$

$$\geqslant |y_{\sigma,1}|^{\theta\langle k,a^1(\sigma)\rangle} \cdots |y_{\sigma,r}|^{\theta\langle k,a^r(\sigma)\rangle} \cdot D_\sigma \tag{36}$$

$$\geqslant |y_{\sigma,1}|^{\theta\langle k,a^1(\sigma)\rangle} \cdots |y_{\sigma,r}|^{\theta\langle k,a^r(\sigma)\rangle}|y_{\sigma,r+1}|^{\theta k_{i_1}} \cdots |y_{\sigma,n}|^{\theta k_{i_s}} \cdot D_\sigma \tag{37}$$

$$= D_\sigma\|x^k\|^\theta \circ \pi_\sigma(y_\sigma). \tag{38}$$

Let us remark that the condition $\theta \geqslant 0$ is used to obtain (37), since we assume $0 < |y_{\sigma,i}| \leqslant 1$, for all $i = 1, \dots, n$ and all $\sigma \in \Sigma^{(n)}$. □

Let us fix an index $i \in \{1, \dots, n\}$. Then we define

$$E_i(F, \widetilde{\Gamma}_+) = \left\{\theta \geqslant 0 : \theta w_i \geqslant \mathbf{L}(w, F), \text{ for all } w \in \mathcal{F}_0(\widetilde{\Gamma}_+)\right\}. \tag{39}$$

Let us decompose $\mathcal{F}_0(\widetilde{\Gamma}_+)$ as $\mathcal{F}_0(\widetilde{\Gamma}_+) = A_{i,-} \cup A_{i,0} \cup A_{i,+}$ where $A_{i,-} = \{w \in \mathcal{F}_0(\widetilde{\Gamma}_+) : w_i < 0\}$, $A_{i,0} = \{w \in \mathcal{F}_0(\widetilde{\Gamma}_+) : w_i = 0\}$ and $A_{i,+} = \{w \in \mathcal{F}_0(\widetilde{\Gamma}_+) : w_i > 0\}$. We define

$$a_i(F, \widetilde{\Gamma}_+) = \max_{w\in A_{i,+}} \frac{\mathbf{L}(w, F)}{w_i} \qquad\qquad b_i(F, \widetilde{\Gamma}_+) = \min_{w\in A_{i,-}} \frac{\mathbf{L}(w, F)}{w_i}. \tag{40}$$

In the remaining section we also denote the numbers defined above by a_i and b_i, respectively, for all $i = 1, \dots, n$. Let us remark that $E_i(F, \widetilde{\Gamma}_+) = [0, b_i]$, whenever $E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$.

Lemma 6.2. *The following conditions are equivalent:*

(i) *$E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$*
(ii) *$\ell(w, F_j) \leqslant 0$, for all $w \in \mathcal{F}_0(\widetilde{\Gamma}_+)$ and all $j = 1, \dots, p$.*

Proof. Let us fix an index $i \in \{1, \dots, n\}$. By (39) and (40) we obtain that $E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$ if and only if

$$a_i \leqslant b_i,\ 0 \leqslant b_i \text{ and } \ell(w, F_j) \leqslant 0, \text{ for all } j = 1, \dots, p \text{ and all } w \in A_{i,0}. \tag{41}$$

Let us assume (i). If $w \in \mathcal{F}_0(\widetilde{\Gamma}_+)$, then $w_0 < 0$. In particular $w_0 = w_i < 0$, for some $i \in \{1, \dots, n\}$. This implies $w \in A_{i,-}$. Since $b_i \geqslant 0$, we conclude $\mathbf{L}(w, F) \leqslant 0$, and then (ii) follows. The converse is obvious using (41). □

In particular, if F_j is convenient, for all $j = 1, \dots, p$, then $E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$.

Corollary 6.3. *Let us suppose that F is strongly adapted to $\widetilde{\Gamma}_+$ and $E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$. Then*

$$\min \left\{ \frac{\mathbf{L}(w, F)}{w_0} : w \in \mathcal{F}_0(\widetilde{\Gamma}_+) \right\} \leqslant \mathcal{L}_\infty(F). \tag{42}$$

Proof. Since F is strongly adapted to $\widetilde{\Gamma}_+$ and $E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$, we can apply Corollary 6.1 to each monomial x_i, $i = 1, \dots, n$, to obtain that there exist constants $C, M > 0$ such that

$$|x_i|^{b_i} \leqslant C \|F(x)\| \tag{43}$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. Let $b_0 = \min\{b_1, \dots, b_n\}$. We can assume $M \geqslant \sqrt{n}$. If $\|x\| \geqslant M$ then $\sqrt{n} \leqslant M \leqslant \|x\| \leqslant \sqrt{n} \max_i |x_i|$. In particular $1 \leqslant \max_i |x_i|$ and then (43) implies that

$$\max |x_i|^{b_0} \leqslant C \|F(x)\|$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. Therefore $b_0 \leqslant \mathcal{L}_\infty(F)$. Let us observe that

$$b_0 = \min_i \min_{w \in A_{i,-}} \frac{\mathbf{L}(w, F)}{w_i} = \min_{w \in \mathcal{F}_0(\widetilde{\Gamma}_+)} \frac{\mathbf{L}(w, F)}{w_0}.$$

Hence (42) follows. □

In particular, by Lemma 6.2, the conclusion of the above result follows if we replace the condition $E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$, by the condition that F_s is convenient, for all $s = 1, \dots, p$. Let us denote the left hand side of (42) by $b(F, \widetilde{\Gamma}_+)$. Let us remark that $b(F, \widetilde{\Gamma}_+) = \min_i b_i(F, \widetilde{\Gamma}_+)$. We remark that Corollary 6.3 applies both to real and complex maps and is analogous to [2, Theorem 5.9], which applies only to real polynomial maps.

If $h \in \mathbb{K}[x_1, \dots, x_n]$ and h is convenient, then we denote by $r_i(h)$ the number $r_i(\widetilde{\Gamma}_+(h))$, as defined in (5), for all $i = 1, \dots, n$.

Corollary 6.4. *Let us suppose that F is non-degenerate and F_j is convenient, for all $j = 1, \dots, p$. Then*

$$\min_{i,j} r_i(F_j) \leqslant \mathcal{L}_\infty(F) \leqslant r_0(\widetilde{\Gamma}_+(F)). \tag{44}$$

Proof. The right hand side of (44) follows by [2, Lemma 3.3] (the proof of this result is given for $\mathbb{K} = \mathbb{R}$ but it also applies to the case $\mathbb{K} = \mathbb{C}$). By Proposition 3.9, the map F is strongly adapted to the polyhedron $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(F_1) + \cdots + \widetilde{\Gamma}_+(F_p)$. By Lemma 2.5 we have

$$r_i(F_j) = \min_{w \in \mathbb{R}^n_0(i)} \frac{\ell(w, F_j)}{w_i}$$

for all $j = 1, \dots, p$, $i = 1, \dots, n$. Then

$$\min\big\{r_1(F_s), \dots, r_n(F_s)\big\} = \min_{\substack{w \in \mathbb{R}^n_0 \\ j=1,\dots,p}} \frac{\ell(w, F_j)}{w_0} \leqslant \min\left\{\frac{\mathbf{L}(w, F)}{w_0} : w \in \mathcal{F}_0(\widetilde{\Gamma}_+)\right\} \leqslant \mathcal{L}_\infty(F)$$

where the last inequality comes from Corollary 6.3. □

Corollary 6.5. *Let $F : \mathbb{K}^n \to \mathbb{K}^n$ be a polynomial map such that F is a local homeomorphism. Let us suppose that F is strongly adapted to $\widetilde{\Gamma}_+$ and $E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$. Then F is a homeomorphism.*

Proof. The hypothesis imply that $\mathcal{L}_\infty(F)$ exists and $\mathcal{L}_\infty(F) > 0$. Then F is a proper map and therefore, by Hadamard's theorem [26, p. 240], F is a homeomorphism. □

Remark 6.6. Let us observe that, in the previous result, we do not assume that each component function of F is convenient. Corollary 6.4 is proven in [2, Theorem 3.8] only for the case $\mathbb{K} = \mathbb{R}$ with a specific technique developed to study real polynomial maps. We remark that, under the conditions of Corollary 6.4, if we assume that F is a local homeomorphism, then the same proof of the previous result works to deduce that F is a global homeomorphism (see also [6, Theorem 1.4]).

7. Pre-weighted homogeneous maps

In this section we expose some results concerning a wide class of polynomial maps $\mathbb{K}^n \to \mathbb{K}^p$, which is part of the motivation of our study.

Definition 7.1. Let $h \in \mathbb{K}[x_1, \dots, x_n]$ and let $v = (v_1, \dots, v_n) \in \mathbb{Z}^n_{\geqslant 1}$. Let us write h as $h = \sum_k a_k x^k$. Then we denote by $d_v(h)$ the maximum of the scalar products $\langle v, k\rangle$ such that $a_k \neq 0$. We call $d_v(h)$ the *degree of h with respect to v*. We denote by $\mathrm{q}_v(h)$ the sum of those terms $a_k x^k$ such that $\langle v, k\rangle = d_v(h)$. Let us remark that $\mathrm{q}_v(h) = p_w(h)$, where $w = -v$ (see Definition 3.1). We say that h is *weighted homogeneous with respect to v* when $\mathrm{q}_v(h) = h$.

Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. Then we define $\mathrm{q}_v(F) = (\mathrm{q}_v(F_1), \dots, \mathrm{q}_v(F_p))$ and $d_v(F) = (d_v(F_1), \dots, d_v(F_p))$. If $q_v(F) = F$, then we say that F is *weighted homogeneous with respect to v*. The map F is said to be *pre-weighted homogeneous with respect to v* when $\mathrm{q}_v(F)^{-1}(0) = \{0\}$.

Let $v = (v_1, \dots, v_n) \in \mathbb{Z}^n_{\geqslant 1}$. Then we denote by $\widetilde{\Gamma}^v_+$ the global Newton polyhedron given by the convex hull of $\{\frac{v_1 \cdots v_n}{v_1} e_1, \dots, \frac{v_1 \cdots v_n}{v_n} e_n\} \cup \{0\}$. Then $\widetilde{\Gamma}^v_+$ has a unique face Δ of dimension $n-1$, which is contained in the hyperplane $v_1 x_1 + \cdots + v_n x_n = v_1 \cdots v_n$ and hence $\mathcal{F}(\widetilde{\Gamma}^v_+) = \{-v, e_1, \dots, e_n\}$ and $\mathcal{F}_0(\widetilde{\Gamma}^v_+) = \{-v\}$. Let us recall that, if $\mathtt{I}$ is a non-empty subset of $\{1, \dots, n\}$, then $\pi_{\mathtt{I}} : \mathbb{R}^n \to \mathbb{R}^n_{\mathtt{I}}$ denotes the natural projection.

Corollary 7.2. *Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map and let $v \in \mathbb{R}^n_{\geqslant 1}$. Then the following conditions are equivalent:*

(i) *F is pre-weighted homogeneous with respect to v;*

(ii) *F is strongly adapted to $\widetilde{\Gamma}^v_+$.*

Proof. Let us see (i) $\Rightarrow$ (ii). We will use Proposition 3.6. Let $w = -v$. Let Δ be a face of $\widetilde{\Gamma}^v_+$ such that $0 \notin \Delta$. If Δ is not contained in any coordinate subspace $\mathbb{R}^n_{\mathtt{I}}$, for some proper subset $\mathtt{I} \subsetneq \{1, \dots, n\}$, then $\Delta = \Delta(w, \widetilde{\Gamma}^v_+)$ and $\mathcal{J}(\Delta) = \{\{w\}\}$. Hence condition $(\mathrm{C}_{\{w\},F})$ follows, since $p_w(F) = q_v(F)$ and $q_v(F)^{-1}(0) = \{0\}$.

If $\Delta \subseteq \mathbb{R}^n_{\mathtt{I}}$, for some proper subset $\mathtt{I} \subsetneq \{1, \dots, n\}$ such that $\dim \Delta = |\mathtt{I}| - 1$, then $\mathcal{J}(\Delta) = \{J\}$, where $J = \{w, e_{i_1}, \dots, e_{i_s}\}$ and $\{i_1, \dots, i_s\} = \{1, \dots, n\} \smallsetminus \mathtt{I}$, $s = n - |\mathtt{I}|$.

Then we observe that $\mathrm{p}^*_J(F) = \mathrm{p}_{\pi_{\mathtt{I}}(w)}(F^{\mathtt{I}}) = \mathrm{q}_{\pi_{\mathtt{I}}(v)}(F^{\mathtt{I}})$. The map $\mathrm{q}_{\pi_{\mathtt{I}}(v)}(F^{\mathtt{I}})$ is weighted homogeneous with respect to $\pi_{\mathtt{I}}(v)$. Therefore the condition $\mathrm{q}_v(F)^{-1}(0) = \{0\}$ implies $\mathrm{q}_{\pi_{\mathtt{I}}(v)}(F^{\mathtt{I}})^{-1}(0) = \{0\}$ and hence $\mathrm{p}^*_J(F)^{-1}(0) = \{0\}$. Thus $(\mathrm{C}^*_{F,J})$ holds and (ii) follows, by Proposition 3.6.

Let us see (ii) $\Rightarrow$ (i). Since $\mathcal{F}_0(\widetilde{\Gamma}^v_+) = \{-v\}$ it is clear that $E_i(\mathrm{q}_v(F), \widetilde{\Gamma}^v_+) \neq \emptyset$, for all $i = 1, \dots, n$ (see Lemma 6.2). By Definition 3.3 we observe that F is strongly adapted to $\widetilde{\Gamma}^v_+$ if and only if $\mathrm{q}_w(F)$ is strongly adapted to $\widetilde{\Gamma}^v_+$. Then we can apply Corollary 6.3 to $\mathrm{q}_v(F)$ to deduce that there exists constants $\alpha, C, M > 0$ such that

$$\|x\|^\alpha \leqslant C\|\mathrm{q}_v(F)(x)\|$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. In particular $\mathrm{q}_v(F)^{-1}(0)$ is contained in the open ball $B(0; M)$ centered at 0 and of radius M. But this implies $\mathrm{q}_v(F)^{-1}(0) = \{0\}$ since $\mathrm{q}_v(F)$ is weighted homogeneous with respect to v. Thus F is pre-weighted homogeneous. □

If $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, then we define $A(v) = \{j : v_j = \max_i v_i\}$.

Proposition 7.3. *Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. Let $v \in \mathbb{Z}^n_{\geqslant 1}$ such that F is pre-weighted homogeneous with respect to v. Then*

$$\frac{\min\{d_v(F_1), \dots, d_v(F_p)\}}{\max\{v_1, \dots, v_n\}} \leqslant \mathcal{L}_\infty(F). \tag{45}$$

Let us assume that F is weighted homogeneous with respect to v and $F^{-1}(0) = \{0\}$. Let $i_0 \in \{1, \dots, p\}$ such that $\min\{d_v(F_1), \dots, d_v(F_p)\} = d_v(F_{i_0})$. If

$$\big\{x \in \mathbb{K}^n : F_i(x) = 0, \text{ for all } i \neq i_0\big\} \nsubseteq \{x \in \mathbb{K}^n : x_j = 0, \textit{ for all } j \in A(v)\}, \tag{46}$$

then equality holds in (45).

Proof. By Corollary 7.2, the map F is strongly adapted to $\widetilde{\Gamma}^v_+$. Then (45) follows from Corollary 6.3, since $\mathcal{F}_0(\widetilde{\Gamma}^v_+) = \{-v\}$.

Let us see the second part. In order to simplify the notation, let us assume $i_0 = 1$. Then the quotient on the left hand side of (45) is equal to $d_v(F_1)/\max_i v_i$. By (46) and the hypothesis $F^{-1}(0) = \{0\}$, there exists a point $a = (a_1, \dots, a_n) \in \mathbb{K}^n$ such that $F_2(a) = \cdots = F_p(a) = 0$, $F_1(a) \neq 0$ and $a_j \neq 0$, for some $j \in A(v)$.

In particular the curve $\gamma : \mathbb{K} \smallsetminus \{0\} \to \mathbb{K}$ defined by $\gamma(t) = (a_1 t^{-v_1}, \dots, a_n t^{-v_n})$ is not the zero curve. We observe that $\mathrm{ord}(\gamma) = -(\max_i v_i)$. Moreover, since we assume that F is weighted homogeneous with respect to v, we have

$$F(\gamma(t)) = (t^{-d_v(F_1)} F_1(a), t^{-d_v(F_2)} F_2(a), \dots, t^{-d_v(F_p)} F_p(a)) = (t^{-d_v(F_1)} F_1(a), 0, \dots, 0).$$

This shows that $F \circ \gamma$ is not the zero curve. Let $\beta > d_v(F_1)/\max_i v_i$. Then we have

$$\lim_{t\to 0} \frac{\|F(\gamma(t))\|}{\|\gamma(t)\|^\beta} = \lim_{t\to 0} \frac{\|(t^{-d_v(F_1)} F_1(a), 0, \dots, 0)\|}{\|\gamma(t)\|^\beta} = 0, \tag{47}$$

where the last equality follows from

$$\mathrm{ord}(\|\gamma(t)\|^\beta) = -(\max_i v_i)\beta < -d_v(F_1) = \mathrm{ord}(F(\gamma(t))).$$

In particular $\mathcal{L}_\infty(F) < \beta$ (otherwise the limit (47) would be greater than or equal to some positive constant). Therefore $\mathcal{L}_\infty(F) \leqslant d_v(F_1)/\max_i v_i$ and the result follows. □

8. Index of polynomial maps

In this section we show a result concerning the index of real polynomial vector fields. This result will follow as a consequence of the argument of the proof of Theorem 4.4. If $F : \mathbb{R}^n \to \mathbb{R}^n$ is real a polynomial map such that $F^{-1}(0)$ is finite, then we denote by $\mathrm{ind}(F)$ the *index* of F, that is

$$\mathrm{ind}(F) = \sum_{x \in F^{-1}(0)} \mathrm{ind}_x(F)$$

where $\mathrm{ind}_x(F)$ denotes the topological index of F at x (see for instance [21], [22] or [23]). We recall that if $f : U \to \mathbb{R}^n$ denotes a continuous map defined in an open set $U \subseteq \mathbb{R}^n$ and $x \in U$ is an isolated zero of f, then the *index of* f *at* x is defined as follows. Let D be a ball centered at x, $D \subseteq U$, such that $f^{-1}(0) \cap D = \{x\}$ and let us consider the map $\partial D \to S^{n-1}$ given by $z \mapsto \frac{f(z)}{\|f(z)\|}$, for all $z \in \partial D$, where ∂D denotes the boundary of D. Then $\mathrm{ind}_x(f)$ is the degree of this map between spheres of dimension $n-1$.

Theorem 8.1. *Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ be polynomial maps such that F and $F + G$ have a finite number of zeros. Let us assume that*

(i) *F is strongly adapted to $\widetilde{\Gamma}_+$;*
(ii) *$E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$;*
(iii) *$\ell(w, G_i) > \ell(w, F_i)$, for all $w \in \mathcal{F}_0(\widetilde{\Gamma}_+)$, $i = 1, \dots, n$.*

Then

$$\mathrm{ind}(F) = \mathrm{ind}(F + G).$$

Proof. Let us consider the homotopy $H : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $H(t, x) = F(x) + tG(x)$. Let $H_t = (H_{t,1}, \dots, H_{t,n}) : \mathbb{R}^n \to \mathbb{R}^n$ be given by $H_t(x) = H(t, x)$, for all $x \in \mathbb{R}^n$ and all

$t \in [0,1]$. We claim that there exists a uniform Łojasiewicz inequality at infinity for the family of maps $\{H_t\}_{t\in[0,1]}$. That is, there exist some constants $M, \alpha > 0$ such that

$$\|x\|^\alpha \leqslant C\|H_t(x)\| \tag{48}$$

for all pair $(t,x) \in [0,1] \times \mathbb{R}^n$ such that $\|x\| \geqslant M$. As a consequence we would obtain that $H_t^{-1}(0)$ is contained in the open ball $B(0;M)$, for all $t \in [0,1]$. In particular $H(t,x) \neq 0$ for all $(t,x) \in [0,1] \times \partial B(0;M)$, where $\partial B(0;M)$ denotes the boundary of $B(0;M)$. This fact implies $\mathrm{ind}(F) = \mathrm{ind}(F+G)$, as a consequence of a known result about the invariance of the index by homotopies (see for instance [21, Theorem 2.2.4] or [7]).

From $\widetilde{\Gamma}_+$ we can construct a subdivision of $\mathbb{R}^n$ into simplicial cones as explained in Section 5. Let us keep the notation introduced in Section 5, before Lemma 5.1. Let us fix a cone $\sigma \in \Sigma^{(n)}$. Let $a^1(\sigma), \dots, a^n(\sigma)$ be the primitive generators of σ. Let us consider the decomposition of $\{a^1(\sigma), \dots, a^n(\sigma)\}$ as in (17) and (18).

Analogous to (23) we can consider, for each $i = 1, \dots, n$ and each $t \in [0,1]$, the polynomial $H^*_{\sigma,t,i} \in \mathbb{K}[y_{\sigma,1}, \dots, y_{\sigma,n}]$ such that

$$H_{t,i} \circ \pi_\sigma(y_\sigma) = y_{\sigma,1}^{\ell(a^1(\sigma),H_{t,i})} \cdots y_{\sigma,r}^{\ell(a^r(\sigma),H_{t,i})} \cdot H^*_{\sigma,t,i}(y_\sigma),$$

for all $y_\sigma \in W_\sigma$.

By hypothesis we have $\ell(w, G_i) > \ell(w, F_i)$, for all $w \in \mathcal{F}_0(\widetilde{\Gamma}_+)$, $i = 1, \dots, n$. This implies $\ell(a^j(\sigma), H_{t,i}) = \ell(a^j(\sigma), F_i)$, for all $i = 1, \dots, n$, $j = 1, \dots, r$. Hence the principal parts of F_i and of $H_{t,i}$ with respect to any subset of $\mathcal{F}_0(\widetilde{\Gamma}_+)$ coincide, for all $i = 1, \dots, n$ (see Definition 3.1) and

$$E_i(H_t, \widetilde{\Gamma}_+) = E_i(F, \widetilde{\Gamma}_+) \neq \emptyset \tag{49}$$

for all $i = 1, \dots, n$ and all $t \in [0,1]$ (the sets $E_i(F, \widetilde{\Gamma}_+)$ are non-empty by hypothesis).

Let us see that there exists some constant $M > 0$ such that

$$\inf_{\substack{y_\sigma \in \pi_\sigma^{-1}(V_M)\\ t\in[0,1]}} \sup_i |H^*_{\sigma,t,i}(y_\sigma)| \neq 0 \tag{50}$$

for all $\sigma \in \Sigma^{(n)}$. If we assume the opposite then there exists a cone $\sigma \in \Sigma^{(n)}$ and a sequence $\{(t_m, y_m)\}_{m\geqslant 1} \subseteq [0,1] \times W_\sigma$ verifying that $\{\pi_\sigma(y_m)\}_{m\geqslant 1} \to \infty$ and $\{H^*_{\sigma,t_m,i}(y_m)\}_{m\geqslant 1} \to 0$, for all $i = 1, \dots, n$. Analogous to the proof of Proposition 5.2, let us consider a limit point $(\mathbf{t}, \mathbf{y}) = (\mathbf{t}, \mathbf{y}_1, \dots, \mathbf{y}_n) \in [0,1] \times \overline{W_\sigma}$ of $\{(t_m, y_m)\}_{m\geqslant 1}$. By continuity we have $H^*_{\sigma,\mathbf{t},i}(\mathbf{y}) = 0$, for all $i = 1, \dots, n$. Moreover, since $\{\pi_\sigma(y_m)\}_{m\geqslant 1} \to \infty$, we have that the set $J_0 = \{j : \mathbf{y}_j = 0\}$ is non-empty.

Following the same procedure as in the proof of Proposition 5.2 (see (27)) we obtain

$$\mathrm{p}_{J_0}(H_{\mathbf{t},i}) \circ \pi_\sigma(\widetilde{\mathbf{y}}) = \prod_{j\notin J_0} \mathbf{y}_{\sigma,j}^{\ell(a^j(\sigma),H_{\mathbf{t},i})} H^*_{\sigma,\mathbf{t},i}(\mathbf{y}) = 0 \tag{51}$$

for all $i = 1, \dots, n$, where $\widetilde{\mathbf{y}} = (\widetilde{\mathbf{y}}_1, \dots, \widetilde{\mathbf{y}}_n)$ is the point defined by

$$\widetilde{\mathbf{y}}_j = \begin{cases} \mathbf{y}_j, & \text{if } j \notin J_0 \\ 1, & \text{if } j \in J_0. \end{cases}$$

Then we have a contradiction, since $\mathrm{p}_{J_0}(H_{\mathbf{t},i}) = \mathrm{p}_{J_0}(F_i)$, for all $i = 1, \dots, n$, and F is strongly adapted to $\widetilde{\Gamma}_+$. Hence relation (50) holds for some $M > 0$ and all $\sigma \in \Sigma^{(n)}$. As a consequence, since $E_i(H_t, \widetilde{\Gamma}_+) = E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$ and all $t \in [0,1]$, we can reproduce inequalities (28)-(32), by replacing F_i by $H_{t,i}$ and taking $k = e_i$, to obtain that for all $i = 1, \dots, n$, there exists a constant $C_i > 0$ such that

$$|x_i|^{b_i} \leqslant C_i \|F(x) + tG(x)\| \tag{52}$$

for all $x \in \mathbb{R}^n$ such that $\|x\| \geqslant M$ and all $t \in [0,1]$, where $b_i = \sup E_i$, for all $i = 1, \dots, n$ (see (40)).

Let $b_0 = \min\{b_1, \dots, b_n\}$ and $C_0 = \max\{C_1, \dots, C_n\}$. We can assume that $M \geqslant \sqrt{n}$. Hence $\|x\| \geqslant M$ implies $\max_i |x_i| \geqslant 1$. Taking $\max_i$ at both sides of (52), we obtain

$$\|x\|^{b_0} \leqslant (\sqrt{n})^{b_0} \max |x_i|^{b_0} \leqslant (\sqrt{n})^{b_0} C_0 \|F(x) + tG(x)\|$$

for all $t \in [0,1]$ and all $x \in \mathbb{R}^n$ such that $\|x\| \geqslant M$. Then there exists a uniform Łojasiewicz inequality at infinity for the family of maps $\{H_t\}_{t \in [0,1]}$ and the result follows, as explained at the beginning of the proof. $\square$

If $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map, then we denote by $\mathbf{I}(F)$ the ideal of $\mathbb{C}[x_1, \dots, x_n]$ generated by the component functions of F. Moreover, we denote by $F_{\mathbb{R}}$ the map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ obtained from F under the identification $x + \mathbf{i}y \leftrightarrow (x, y)$ between $\mathbb{C}$ and $\mathbb{R}^2$. We remark that the proof of Theorem 8.1 also works to deduce the following result.

Theorem 8.2. *Let $F, G : \mathbb{C}^n \to \mathbb{C}^n$ be polynomial maps such that*

(i) *F is strongly adapted to $\widetilde{\Gamma}_+$;*
(ii) *$E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, \dots, n$;*
(iii) *$\ell(w, G_i) > \ell(w, F_i)$, for all $w \in \mathcal{F}_0(\widetilde{\Gamma}_+)$, $i = 1, \dots, n$.*

Then F and $F + G$ have a finite number of zeros and

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbf{I}(F)} = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbf{I}(F+G)}.$$

Proof. Conditions (i), (ii) and (iii) imply that $F + G$ also satisfy the hypothesis of Corollary 6.3. Then $\mathcal{L}_\infty(F)$ and $\mathcal{L}_\infty(F + G)$ are positive numbers. This implies that $F^{-1}(0)$ and $(F + G)^{-1}(0)$ are compact and hence finite. The same proof of Theorem 8.1 works to obtain that there is a uniform Łojasiewicz inequality for the homotopy $H : [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$ defined by $H(t, z) = F(z) + tG(z)$, for all $(t, z) \in [0,1] \times \mathbb{C}^n$. That is, there exist some constants M, $\alpha > 0$ such that

$$\|z\|^{\alpha} \leqslant C \|F(z) + tG(z)\|$$

for all $z \in \mathbb{C}^n$ such that $\|z\| \geqslant M$ and all $t \in [0,1]$. In particular, this means that $\mathrm{ind}(F_{\mathbb{R}}) = \mathrm{ind}(F_{\mathbb{R}} + G_{\mathbb{R}})$.

It is well known (see for instance [9, p. 150]) that

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x_1,\dots,x_n]}{\mathbf{I}(F)} = \sum_{z\in F^{-1}(0)} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,z}}{\mathbf{I}_z(F)}$$

where $\mathcal{O}_{n,z}$ denotes the germ of analytic function germs $(\mathbb{C}^n, z) \to \mathbb{C}$ and $\mathbf{I}_z(F)$ is the ideal of $\mathcal{O}_{n,z}$ generated by the germs of the component functions of F at z, for any $z \in \mathbb{C}^n$. It is also known that, if $z = x + \mathbf{i}y \in F^{-1}(0)$, then $\dim_{\mathbb{C}} \mathcal{O}_{n,z}/\mathbf{I}_z(F) = \mathrm{ind}_{(x,y)}(F_{\mathbb{R}})$ (see for instance [4, p. 146] or [13, p. 15]). Then the result follows. □

If $F : \mathbb{R}^n \to \mathbb{R}^n$ denotes a real polynomial map, then we denote by $F_{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^n$ the map obtained from F by complexifying the variables.

Remark 8.3. Under the hypothesis of Theorem 8.1, if we assume that $F_{\mathbb{C}}$ is strongly adapted to $\widetilde{\Gamma}_+$, then $\mathcal{L}_\infty(F_{\mathbb{C}}) \geqslant 0$, by Corollary 6.3, and consequently the zero set of the maps F and $F + G$ are finite.

Example 8.4. Let us consider the polynomial map $F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$, where

$$\begin{aligned} F_1(x,y,z) &= x^{a_1} + x^{a_1}y^{b_1} + x^{a_1}z^{c_1} + \alpha x^{a_1}y^{b_1}z^{c_1} \\ F_2(x,y,z) &= y^{b_2} + x^{a_2}y^{b_2} + y^{b_2}z^{c_2} + \beta x^{a_2}y^{b_2}z^{c_2} \\ F_3(x,y,z) &= z^{c_3} + x^{a_3}z^{c_3} + y^{b_3}z^{c_3} + \gamma x^{a_3}y^{b_3}z^{c_3} \end{aligned}$$

where α, β, γ are mutually different non-zero real numbers and the supports of the above polynomials are contained in $\mathbb{Z}^3_{\geqslant 1}$. Let g be the function defined by $g(x,y,z) = x + y + z + xy + xz + yz + xyz$ and let $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(g) \subseteq \mathbb{R}^3$. Then we observe that $\widetilde{\Gamma}_+$ is convenient and $\mathcal{F}_0(\widetilde{\Gamma}_+) = \{-e_1, -e_2, -e_3\}$. It is straightforward to check that $F_{\mathbb{C}}$ is strongly adapted to $\widetilde{\Gamma}_+$ and $E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$, for $i = 1, 2, 3$. In particular $0 < \min\{a_i, b_i, c_i : i = 1, 2, 3\} \leqslant \mathcal{L}_\infty(F)$, by Corollary 6.3. Therefore $F^{-1}(0)$ is finite. Let $G = (G_1, G_2, G_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a polynomial map such that $\mathrm{supp}(G_i)$ is contained in the cube $\{(k_1, k_2, k_3) \in \mathbb{Z}^3_{\geqslant 0} : k_1 < a_i, k_2 < b_i, k_3 < c_i\}$, for $i = 1, 2, 3$. Then $(F + G)^{-1}(0)$ is also finite and $\mathrm{ind}(F) = \mathrm{ind}(F + G)$, by Theorem 8.1.

Example 8.5. Let $a, b, c \in \mathbb{Z}_{\geqslant 1}$ and let us consider the map $F = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$F(x,y,z) = (x^a + y^b + x^a y^b, x^a y^b + z^c, z^c).$$

Let $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(F)$. Then it is immediate to see that $F_{\mathbb{C}}$ is strongly adapted to $\widetilde{\Gamma}_+$ and $E_i(F, \widetilde{\Gamma}_+) \neq \emptyset$, for all $i = 1, 2, 3$. Hence $F^{-1}(0)$ is finite. Then, by Theorem 8.1, any polynomial map $G : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathrm{supp}(G_i)$ is contained in the interior of $\widetilde{\Gamma}_+$ verifies that $(F + G)^{-1}(0)$ is finite and $\mathrm{ind}(F) = \mathrm{ind}(F + G)$.

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