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A Static Analysis of the Brunelleschi's Dome in Florence

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Abstract

The paper comments on the quite recent discovery that the Brunelleschi dome in Florence, in opposition to its appearance, is a rotational and not a segmental dome. The edges at the eight corners, that we can see outside, don't reproduce, in fact, the internal structure of the masonry: the bricks' beds, called "corda blanda", were put in place following circles around the dome. These circles are geometrically obtained by intersection of the eight dome shells with ideal reverse cones, coaxially disposed within the dome itself. [2.7]. In the framework of the Heyman masonry material [6], the paper points out the consequences following from this new rotational structure of the dome. The membrane state of stress, that, as a rule, initially takes place in the dome, cannot produce stress discontinuities at the octagon corners [1,9]. Consequently, a smooth rotational dome is thus effectively present inside the thick octangular structure. The dome weight, not collected anymore just at the corners, is continuously conveyed by meridional compression stresses to the drum. In the framework of the Heyman masonry material, the minimum thrust of the dome is thus evaluated.

The particular crack pattern of the dome is then discussed: meridional cracks run in fact only through the middle of the four shells placed on the four large piers while the other four shells, placed on the arches, are uncracked [5]. The paper then analyzes the evolution of the stress state in the dome which occurred during its construction, by arranging bricks in the " fishbone" pattern, ring by ring and without scaffoldings. Stress evaluation at the piers base, ends the paper.

Keywords: masonry domes, cracking of domes, historical monuments.

1. Introduction

The construction of the Church of S. Maria del Fiore in Florence started in the year 1296 by Arnolfo di Cambio. After the death of Arnolfo, in the 1302, works slowed down and had various new starts and stops during the time. At the year 1418 the octagonal drum was finished and only the dome was not yet built. A competition was banned to build the dome. The project was very ambitious: the height of the dome would be of about 105m and the external diameter of the octagonal basis was of about 54m. The Brunelleschi proposal to build the dome without scaffolding wined.

Special cranes, planned by the same Brunelleschi were used to raise the huge quantities of bricks and mortar required to build the dome. (Di Pasquale 2002). The construction of the dome was completed in the year 1434; the lantern was built in the 1461, after the death of Brunelleschi occurred in 1446. Figg. 36, 37 e 38 show the plan and the transversal and longitudinal sections of the cathedral. (B.S. Sgrilli, 1733).

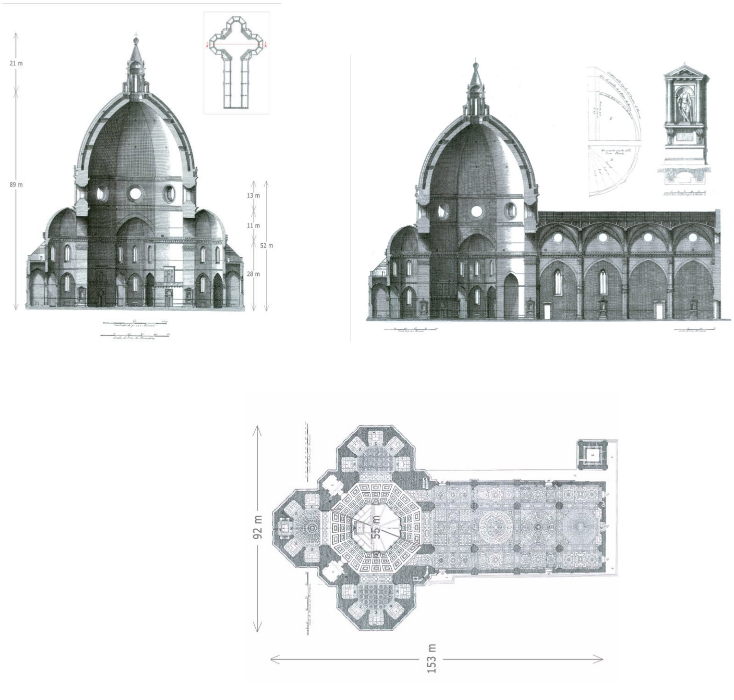

Figure 1: transversal and longitudinal sections, plan of the church [4]

The top section of the four large piers sustaining the drum reaches the height of 28 m. from the ground level. In turn the octagonal drum, reaches the height of 52 m, the level of the spring of the dome. The piers have foundations over thick banks of gravel. The drum, that sustains the weight of the dome rises from 28 m, the height top sections of the piers and the height of 52,5 m, at the dome springing. The drum is a large masonry ring beam, high about 24,5 m and of width of 4,65 m. (Fig. 3). Each side of the octagonal drum has length of 20,74m.

Figure 2: The outer side of the dome: (model of the Opera del Duomo, Di Pasquale 2002)

Figure 3: the drum

2. The dome

To an outside observer a segmental ogival dome covers the space bounded by the drum. The dome in fact adapts itself to the octagonal geometry of the drum and is composed by eight sectors springing from the sides of the octagon. Shell sectors cross one another along the sharp edges each within four different vertical planes.

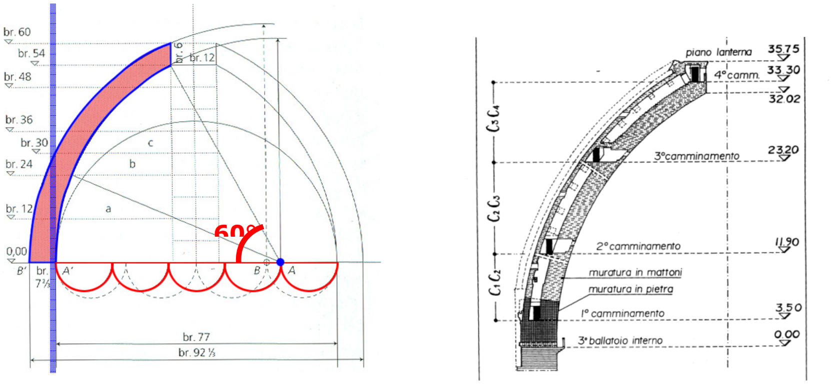

Figure 4: geometry and composition of the dome section

The octagon is inscribed into an annulus having external and internal radii of length of 27, 10 m and 22,62m. [7]. These measures hardly correct those quoted in Fig. 1. Each vertical plane passing through the dome axis cuts the intrados of the dome into an arch of circle whose radius is equal to the 4/5 of the diameter of the internal circle inscribed into the octagon. (Fig. 4). The internal structure of the dome is shown in Fig. 4.

Two different devices were used by Brunelleschi to build the masonry dome. Firstly, he used the herringbone pattern disposing the bricks in order to avoid their sliding during the construction, as shown in Fig. 5.

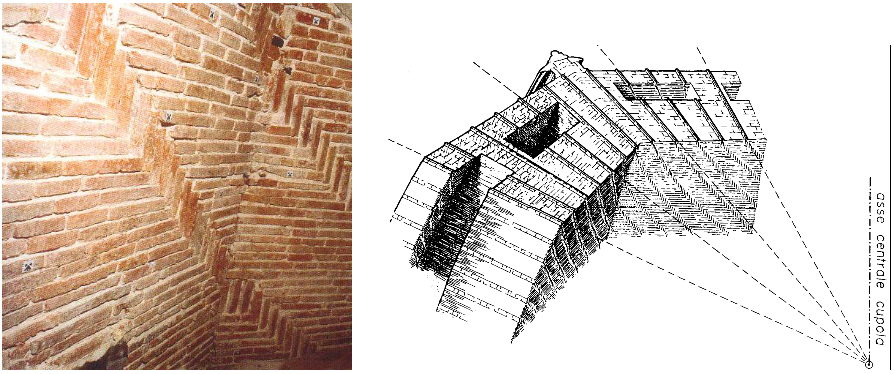

Figure 5: the herringbone producing convergence of the corresponding radial bricks rows

The second device, called " corda blanda" was used to really build a hidden rotational dome inside the segmental one. The rows of bricks run along a reverse conical section as shown in Fig.6. At the external hip the conical section reaches the dome at a position higher than the intercept of the dome at the middle of the octagon sides. The bricks rows show thus a cusp at corners and bend slightly, as shown in Fig. 6.

Consequently the brick rows run without interruption around the dome and the stress state is smooth, without presenting the strong discontinuities occurring in the segmental dome.

A further confirmation of this behavior of the dome is obtained by examining the crack pattern within.

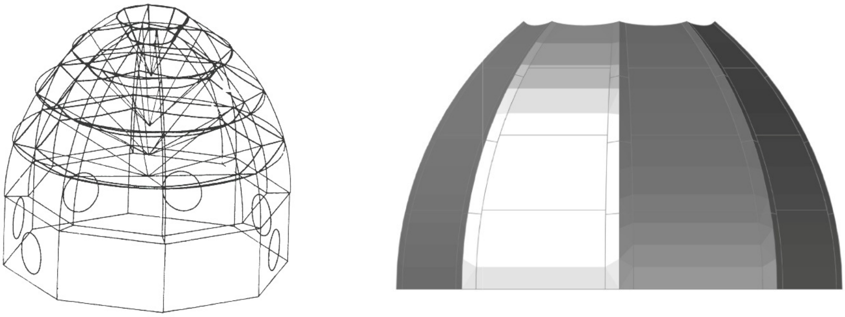

Figure 6: Intersections of groins with the reverse cones and the curved bricks rows [9]

Figure 6: Curved bricks rows emerging under the horizontal tiles rows[7]

3. Crack patterns analysis and the consequent definition of the rotational dome behavior

Crack analysis can give useful information on the static behavior of the dome and we will examine if the detected crack pattern corresponds to that of a rotational or of a segmental dome.

Cracks reproduce the initial membrane stresses in the vault. A rotational dome exhibits a meridian crack pattern, produced by tensile stresses in the rings acting in the lower band of the dome, of easy determination. More complex is the evaluation of the membrane stresses in a cylindrical segmental dome. For sake of simplicity, we can make reference to a hip roof with square plan.

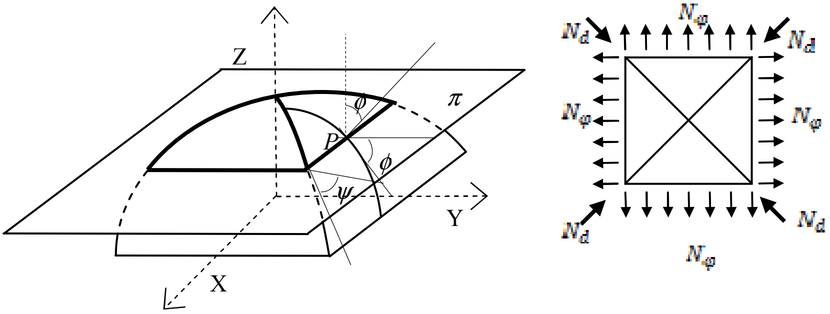

Figure 8: Vertical equilibrium of a portion of a segmental dome with square plan

Equilibrium equations of the membrane stresses are well known

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{\phi x}}{\partial s}=0 \qquad \frac{\partial N_\phi}{\partial s}+\frac{\partial N_{x\phi}}{\partial x}+g\sin\phi=0 \qquad N_\phi=-gR\cos\phi \tag{1}$$

where stresses N_x are directed along the direction of the rectilinear generatrices. Thus we get

$$\frac{\partial N_{x\phi}}{\partial x}=-2g\sin\phi \tag{2}$$

and by integration

$$N_{x\phi}=-2gx\sin\phi+C_1(\phi) \tag{3}$$

At $x=0$, for symmetry, we have $N_{x\varphi}=0$ and $C_1(\phi)=0$. Shearing stresses $N_{x\phi}$ thus are

$$N_{x\phi}=-2gx\sin\phi \tag{4}$$

From the second equilibrium equation thus we have

$$\frac{\partial N_x}{\partial x}=-\frac{1}{R}\frac{\partial}{\partial\phi}(-2gx\sin\phi)=\frac{1}{R}2gx\cos\phi \tag{6}$$

and, by integration we get

$$N_x=\frac{1}{R}gx^2\cos\phi+C_2(\phi) \tag{7}$$

Membrane stresses acting in the single sector sum up along the diagonals and give rise to the compression efforts N_d. These efforts increase getting down towards the drum and take increasing shares of the weight of the dome. Efforts N_φ are in fact unable to transfer vertical loads at the base.

To evaluate the diagonal efforts N_d we can image to cut the dome with a horizontal plane at the generic height Z (Fig. 9). With some elaborations thus we get

$$N_d = 2gR^2 \frac{\sqrt{2-\sin^2\phi}}{\sin\phi}[\cos\phi(1+\sin^2\phi)-1] \tag{8}$$

where we have

$$\lim_{\phi\to 0} N_d = 0 \qquad\qquad \lim_{\phi\to\pi/2} N_d = -2gR^2 \tag{9}$$

To obtain now the unknown function $C_2(\varphi)$, according to [1,9], we can analyze all the forces acting on the corner band of Fig.9. From the equilibrium of this band in the diagonal direction, with some developments, we can obtain the function $C_2(\varphi)$. Thus we have

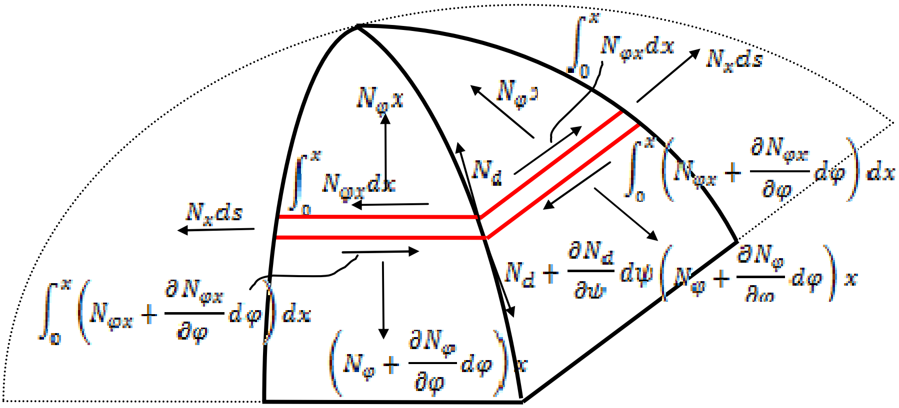

Figure 9: Forces acting on the corner band of the vault

$$C_2(\phi) = gR\cos\phi[3\cos(2\phi)-2] - N_d\frac{\sin\psi}{R\sin^2\phi} \tag{10}$$

and the efforts N_x are given by

$$N_x(\phi) = gR[\frac{x^2}{R^2} + 3\cos(2\phi)-2]\cos\phi - N_d\frac{\sin\psi}{R\sin^2\phi} \tag{11}$$

A strong hooping action is produced by stresses N_x at the base of the segmental dome. Fig. 10 points out the tensile forces acting at the corners, balanced by diagonal compressions. Strong discontinuities with long diagonal cracks thus take place at corners. Similar cracks occurring in a segmental dome are sketched in Fig. 11.

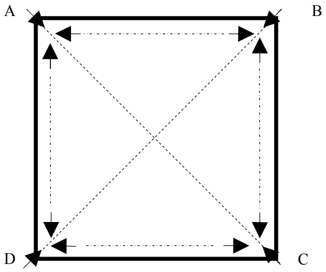

Figure 10: Equilibrium of the segmental dome at corners

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Figure 11: typical cracks in a segmental vault, Tomasoni, 2006)

Reports of first cracking occurring in the S. Maria del Fiore dome go out to the year 1639. Some meridian cracks were detected at the intrados of the dome. They propagated from the central eye as far as the drum, crossing the Vasari fresco and, in some cases, following a toothed pattern of the bricks. This cracking worsened gradually in the time which then permanently stabilized.

We observe that the occurrence of cracks two centuries later, after the completion of the dome was due, as a rule, to the fictitious tensile strength of the masonry due to friction.(Fig. 12). When, during the time, humidity penetrates inside the masonry, friction strength decreases and meridian cracks take place.

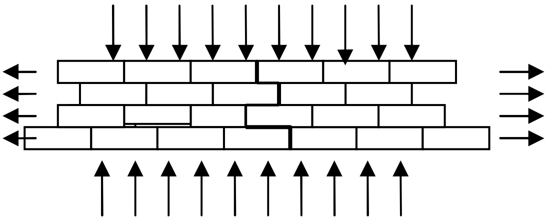

Figure 12: the fictitious tensile strength of the masonry of the dome due to friction[9]

According to the survey [5], the meridian cracks are localized only in the middle of the four groins placed over the four large piers. The width of the cracks ranges between 6 – 3 cm. No cracks, on the contrary, are present in the groins springing over the large arches between the piers.

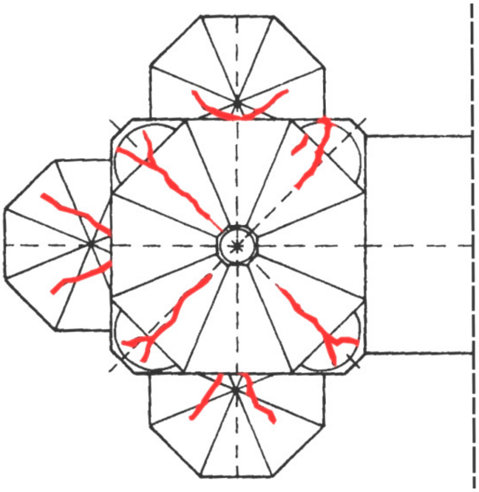

Figure 12: Crack pattern in the dome of S. Maria del Fiore [5]

Cracks detected at the intrados of the S:Maria del Fiore dome have the typical pattern of the cracking occurring in a rotational dome. The fact that cracks are not present in the groins placed over the arches spanning among the piers is due to the compression forces acting at the keys of these arches.

We can conclude that the behavior of the dome is that typical of a rotational dome. This result has thus shown that Brunelleschi decided to build a rotational dome over an octagonal drum by using a special building device able to transforms the segmental dome into a rotational one, with a smoother static behavior.

4. Minimum thrust of the dome

The occurrence of small radial deformations at the springing of the dome is very likely and justifies the presence of the minimum thrust in the dome. The pressure line has been traced by a funicular polygon technique taking into account the various loads acting on a slice of angular width equal to 1/8 of that of the whole dome.

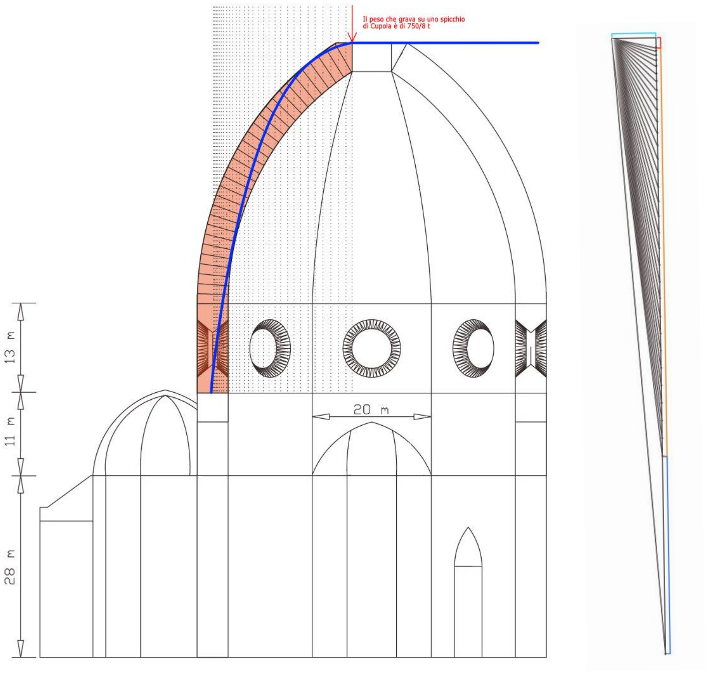

Figure 13: The pressure line in the slice of the dome at the minimum thrust state

The pressure line starts from the extrados, at the attack of the dome with the lantern and, running inside the slice, is tangent at the intrados of the dome (Fig. 13). The total weight of the dome is about 29.198 *t* and the thrust acting along the circle at the springing of the dome equals about 20,50 t/ml.

This thrust state is reached some time after the end of the construction of the dome.

During the erection, being carried out without centering and working circle after circle, the stress state was different. Until the reached by the construction was sufficiently contained, each slice of the dome is self- supporting because the weight of the built falls inside the base of the slices. Little by little, while the construction proceeds, this resultant will become more eccentric until it will reach the external boundary of the base section. By assuming a sliced behavior for the dome, it has been evaluated that the critical state will be attained when the construction, circle after circle, reaches the ring at the height of about 26 m from the springing. For heights of constructions larger than this critical value, the slice will tend to turn over inwards and the rings will become compressed.

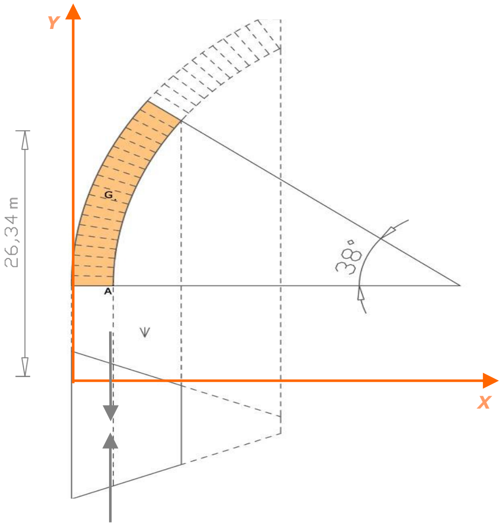

Figure 14: the equilibrium of the slice at the limit of self-supporting

At the end of the construction, with the positioning of the lantern, the dome will reach its complete membrane equilibrium. Then, after some time, meridian cracking occurs and the dome reaches the state of minimum thrust, sketched in the previous Fig. 13.
From the knowledge of the thrust acting at the base of the dome, it has been possible to calculate stresses in the piers. In order to perform this analysis also the thrust of the large arches spanning among the large piers at the base of the dome have been taken into account.
Fig. 14 gives a picture of the base of the dome with the piers and the arches spanning among them. Thrusts are also transmitted by arches to the piers.

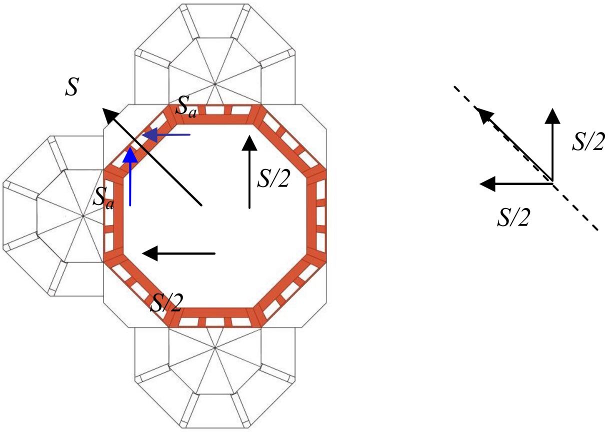

Figure: The various thrusts transmitted to the piers

Summing up all the various contributions the total radial thrust acting at the head of the piers results to be $S_{tot} = S + S/\sqrt{2} + S_2 = 400 + 282,84 + 798,67 = 1481,51t$
The axial load at the head of the pier results to be N = 13.360 *t.* At the base of the pier the axial load reaches the value of 22.360 t taking ino account the pier weight.
The base of the pier is subjected at an eccentric axial load. The maximum compression reaches the value of 2,15 MPa; the minimu is about to 0,3 MPa.

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