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Absolute extrema of two variables functions

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1. Abstract and objectives

In this paper we introduce the concept of absolute extrema of a function of two variables in a compact region. For that, we explain what a compact plain region is and how to calculate the maximum and minumum value of a function of two variables in this region. These extreme value are not necessarily the free extreme values of the function, it depends on the function and the region considered. This fact represents a difference between calculating free or absolute extreme values of a two variables function.

Once studied this paper the student will be able to determine the absolute extrema of a function of two variables in a compact plane region, that is, the maximum and minimum value of the function in this region.

2. Introduction

For many problems it is interesting to know in which points a function reaches the biggest or smallest value in a determined region. For example, it can be useful to know in which point the temperature is smallest or biggest....

First of all, we recall that for a function the concept of maxima and minima values consists on those points for which the value of the function is bigger or smaller than any other point near them. But may be those points are not in the region we are considering or even the value of the function at a point in the boundary of the region can be bigger or smaller too. Then the idea of maxima and minima of the function in the region considered is different.

When we talk about absolute extrema in a compact region D we are talking about those points for which the value of the function is bigger or smaller than for any other point of D. So, our problem consists on determine the maximum and minimum value of a function in a given plane region D. That is to obtain the **absolute extrema** of $f(x, y)$ in D. But this kind of extrema does not always exist, the existence is related to the function and the region considered. What we can assure is that these extrema exist if the region is closed and bounded and the function is continuous in it. So, we are going to work in this situation and to study how to obtain the extrema in this case.

3. Absolute extrema of a function of two variables in a compact region

First of all, we recall that for a function the concept of maxima and minima values consist on those points for which the value of the function is bigger or smaller than any other point near them. That is,

Definition 1 *A real function $f(x, y)$ has*

- *a* **relative maximum** *at the point (x_0, y_0) if there exists $\delta > 0$ such that $\forall (x, y) \in \mathbb{R}^2$ with $|(x, y) - (x_0, y_0)| < \delta$ it is satisfied that*

$$f(x, y) \leq f(x_0, y_0)$$

- *a* **relative minimum** *at the point (x_0, y_0) if there exists $\delta > 0$ such that $\forall (x, y) \in \mathbb{R}^2$ with $|(x, y) - (x_0, y_0)| < \delta$ it is satisfied that*

$$f(x, y) \geq f(x_0, y_0)$$

These extrema are also called free or local extrema of the function. But when we consider a region $D \in \mathbb{R}^2$ and study the maximum and minimum value of the function in this region, we are talking about absolute or global extrema. That is,

Definition 2 *Given a plane region D, a function $f(x, y)$ defined on D has*

- *an* **absolute or global maximum** *at the point $(x_0, y_0) \in D$ if*

$$f(x, y) \leq f(x_0, y_0)$$

  for any point $(x, y) \in D$.

- *an* **absolute or global minimum** *at the point $(x_0, y_0) \in D$ if*

$$f(x, y) \geq f(x_0, y_0)$$

  for any point $(x, y) \in D$.

These definitions are the same concepts of maximum and minumim value for a one variable function in $\mathbb{R}$, both relative and absolute extrema. As we know, with respect to absolute extrema, a one variable function $f(x)$ reaches a maximum and a minimum value in $[a, b]$ if $f(x)$ is continuous in the interval. What happens for two varaible functions?

For a function of two variable, these absolute extrema do not always exist, it depends on the region and the function considered. To guarantee the existence of absolute extrema we need to generalize the idea of closed interval in $\mathbb{R}$ for regions in $\mathbb{R}^2$. In fact, we need that the plane region D be bounded and closed. What is the meaning of these concepts?

A domain D in $\mathbb{R}^2$ is bounded if there exist $M > 0$ such that D is a subset of the disk centered in the origin and with radius M. This means that any point of D is at a distance to the origin smaller than M. Then, one has that a point P is

- an **interior point** of D if there exists an open ball centered at P which is completely contained in D.
- a **boundary point** of D if every ball centered at P has points of D and points from the outside of P.

On the other hand, a domain D in $\mathbb{R}^2$ is closed if it contains all its interior and boundary points.

As we have mentioned above, these concepts are similar to the concepts of closed and bounded interval in $\mathbb{R}$, $[a, b]$, and the idea of maximum and minumim value of a one variable function in $\mathbb{R}$. A one variable function $f(x)$ has a maximum and a minimum value in $[a, b]$ if $f(x)$ is continuous in the interval. Moreover these extreme values are reached at the critical points of $f(x)$ that are in $]a, b[$ or at the extrema of the interval. Analogously, for a two variables function we have the following result:

Theorem 1 *Let $f(x, y)$ be a continuous function in a closed and bounded plane region D. Then,*

(a) $f(x, y)$ has a maximum and a minimum in D.

(b) The absolute extrema must occur at critical points inside D or at boundary points of D.

Using this result, the method to calculate the absolute extrema of $f(x, y)$ on D is:

Step 1: Obtain the critical points of $f(x, y)$ and select those that are in D.

Step 2: Obtain the constrained extrema of $f(x, y)$ under the condition given by the boundary of the region D.

- If the boundary is given by $g(x, y) = 0$, then we apply the Lagrange multipliers method.
- If the boundary is a polygonal, then we choose the vertex and the critical points at each line.

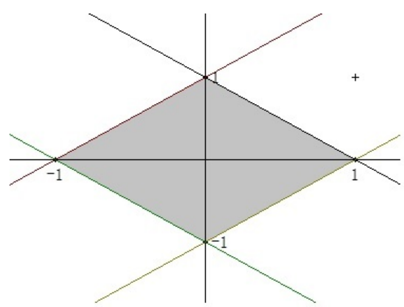
Step 3: Choose the maximum and the minimum between all the points obtained in Steps 1 and 2.

Example 1 *Find the absolute extrema of the function*

$$z = f(x, y) = 2x^2 + y - 3xy$$

in the plane region D bounded by the lines $y = 1 - x$, $y = 1 + x$, $y = -1 - x$ and $y = -1 + x$.

Solution: *The region D represents a quadrilateral*



Step 1: Determine the critical points of z in D.

Critical points of a function $f(x, y)$ are those points where the first partial derivatives are zero or do not exist. In this case, we have that the system giving the nullity of the first partial derivatives of z is

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x}(x, y) = 4x - 3y = 0 \\ \dfrac{\partial f}{\partial y}(x, y) = 1 - 3x = 0 \end{array}\right\} \Rightarrow x = \frac{1}{3} \quad \text{and} \quad y = \frac{4x}{3} = \frac{4}{9}.$$

So, the point $\left(\frac{1}{3}, \frac{4}{9}\right)$ *is a critical point of* z*. Clearly, this point is inside the region* D*.*

Step 2: Analize the boundary points of D.

The vertices of the figure are the points $(1,0)$, $(0,1)$, $(-1,0)$ *and* $(0,-1)$*. So, they must be considered as possible maxima or minima.*

Moreover, one has to consider also the constraint extrema of z *at each line. For the first line* $y = 1 - x$*, one has:*

$$g(x) = f(x, 1-x) = 2x^2 + 1 - x - 3x(1-x) = 5x^2 - 4x + 1 \Rightarrow g'(x) = 10x - 4$$

Then, $g'(x) = 0$ *implies* $x = \frac{2}{5}$ *and* $y = 1 - x = \frac{3}{5}$*. That is, the point* $\left(\frac{2}{5}, \frac{3}{5}\right)$ *is a point to be considered as a possible extrema. / For the other lines, we make a similar reasoning. For* $y = 1 + x$*, one has*

$$g(x) = f(x, 1+x) = -x^2 - 2x + 1 \Rightarrow g'(x) = -2x - 2 = 0 \Rightarrow x = -1 \Rightarrow (-1, 0)$$

For $y = -1 - x$*, it results*

$$g(x) = f(x, -1-x) = 5x^2 + 2x - 1 \Rightarrow g'(x) = 10x + 2 = 0 \Rightarrow x = -\frac{1}{5} \Rightarrow$$

$$\Rightarrow \left(-\frac{1}{5}, -\frac{4}{5}\right)$$

Finally, for $y = -1 + x$*, the point is obtained by*

$$g(x) = f(x, -1+x) = -x^2 + 4x - 1 \Rightarrow g'(x) = -2x + 4 = 0 \Rightarrow x = 2 \Rightarrow (2, 1).$$

Step 3: Choose the maximum and minimum values.

Now, we have to compare the value of the function at each point obtained in Steps 1 and 2:

Critical point (x, y)	$f(x, y) = 2x^2 + y - 3xy$
$\left(\frac{1}{3}, \frac{4}{9}\right)$	$\frac{2}{9}$
$(1, 0)$	2
$(0, 1)$	1
$(-1, 0)$	2
$(0, -1)$	-1
$\left(\frac{2}{5}, \frac{3}{5}\right)$	$\frac{1}{5}$
$\left(-\frac{1}{5}, -\frac{4}{5}\right)$	$-\frac{30}{25}$
$(2, 1)$	3

Then, the absolute maximum of $f(x, y)$ *is* 3 *and occurs at the point* $(2, 1)$. *And the absolute minimum of* $f(x, y)$ *is* $-\frac{30}{25}$ *and occurs at the point* $\left(-\frac{1}{5}, -\frac{4}{5}\right)$.

Example 2 *Find the absolute extrema of the function*

$$f(x, y) = x^2 + 3y^2$$

in the circle $D = \{(x, y) \in \mathbb{R}^2 \ / \ (x-1)^2 + y^2 \leq 4\}$.

Solution: *The region* D *represents a circle centered at the point* $(1, 0)$ *and with radius* 2.

Step 1: Determine the critical points of z in D.

The system given by the nullity of the first partial derivatives of z *is*

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x}(x, y) = 2x = 0 \\ \dfrac{\partial f}{\partial y}(x, y) = 6y = 0 \end{array}\right\} \Rightarrow x = 0 \quad \text{and} \quad y = 0$$

So, the point $(0, 0)$ *is a critical point of* z. *Clearly, this point is in* D.

Step 2: Analize the boundary points of D.

In this case we do not have vertices. Then, we only analize the constraint extrema of $f(x, y)$ *at the circumference* $(x-1)^2 + y^2 = 4$, *which can be calculated using Lagrange multiplyers. The lagrangian function is*

$$L(x, y, \lambda) = x^2 + 3y^2 + \lambda((x-1)^2 + y^2 - 4)$$

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and its critical points are given by the system

$$\begin{cases} \dfrac{\partial L}{\partial x}(x,y,\lambda) = 2x + 2\lambda(x-1) = 0 \\ \dfrac{\partial f}{\partial y}(x,y,\lambda) = 6y + 2\lambda y = 0 \\ \dfrac{\partial f}{\partial \lambda}(x,y,\lambda) = (x-1)^2 + y^2 - 4 = 0. \end{cases}$$

From the second equation one gets that $y = 0$ *or* $\lambda = -3$. *If* $y = 0$ *then the third equation gives* $x = 3$ *or* $x = -1$; *and if* $\lambda = -3$, *the first equation gives* $x = \dfrac{3}{2}$ *and then, the third equation allows to obtain* $y = \pm\dfrac{\sqrt{15}}{2}$.

Then, the critical points on the boundary are $(3,0)$, $(-1,0)$, $\left(\dfrac{3}{2}, \dfrac{\sqrt{15}}{2}\right)$ *and* $\left(\dfrac{3}{2}, -\dfrac{\sqrt{15}}{2}\right)$.

Step 3: Choose the maximum and minimum values.

Compare the value of the function at each point obtained in Steps 1 and 2:

Critical point (x,y)	$f(x,y) = x^2 + 3y^2$
$(0,0)$	0
$(3,0)$	9
$(-1,0)$	1
$\left(\dfrac{3}{2}, \dfrac{\sqrt{15}}{2}\right)$	$\dfrac{27}{2}$
$\left(\dfrac{3}{2}, -\dfrac{\sqrt{15}}{2}\right)$	$\dfrac{27}{2}$

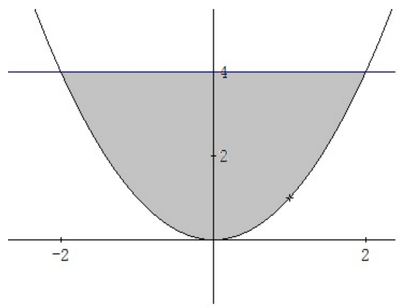
Then, the absolute maximum of $f(x,y)$ *is* $\dfrac{27}{2}$ *and occurs at the points* $\left(\dfrac{3}{2}, \pm\dfrac{\sqrt{15}}{2}\right)$. *And the absolute minimum of* $f(x,y)$ *is* 0 *and occurs at the point* $(0,0)$.

Example 3 *Find the absolute extrema of the function*

$$z = f(x, y) = 4x^2 + y^2 - 4yx$$

in the plane region bounded by the curves $y = x^2$ *and* $y = 4$.

Solution: *The region* D *represents the intersection between a parabola and a line:*



Step 1: Determine the critical points of z in D.

The system given by the nullity of the first partial derivatives of z *is*

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x}(x, y) = 8x - 4y = 0 \\ \dfrac{\partial f}{\partial y}(x, y) = 2y - 4x = 0 \end{array}\right\} \Rightarrow y = 2x$$

So, every point of the form $(x, 2x)$*, with* $x \in \mathbb{R}$ *is a critical point of* $f(x, y)$*. But only the points with* $x \in [0, 2]$ *are in* D.

Step 2: Analize the boundary points of D.

In this case we have a combination of a polygonal and a curve defined by an equation. We have to consider the vertices of the figure, that is, the points $(-2, 4)$ *and* $(2, 4)$.

Moreover, we have to consider the constraint extrema of z *at each curve too. For the first curve* $y = x^2$*, one has:*

$$g(x) = f(x, x^2) = 4x^2 + x^4 - 4x^3 \Rightarrow g'(x) = 8x + 4x^3 - 12x^2 = 4x(2 + x^2 - 3x)$$

Then, $g'(x) = 0$ *implies* $x = 0$*,* $x = 2$ *or* $x = 1$*. From* $y = x^2$*, the points* $(0, 0)$*,* $(2, 4)$ *and* $(1, 1)$ *are points to be considered as possible extrema.*

For the other line, $y = 4$, one has

$$g(x) = f(x, 4) = 4x^2 + 16 - 16x \Rightarrow g'(x) = 8x - 16 = 0 \Rightarrow x = 2 \Rightarrow (2, 4).$$

Step 3: Choose the maximum and minimum values.
Compare the value of the function at each point obtained in Steps 1 and 2:

Critical point (x, y)	$f(x, y) = 4x^2 + y^2 - 4yx$
$(x, 2x)$ *with* $x \in [0, 2]$	0
$(-2, 4)$	64
$(1, 1)$	1

Note that the point $(2, 4)$ *is included in the poins* $(x, 2x)$ *for* $x = 2$.

Then, the absolute maximum of $f(x, y)$ *is* 64 *and occurs at the point* $(-2, 4)$. *And the absolute minimum of* $f(x, y)$ *is* 0 *and occurs at all the points of the form* $(x, 2x)$ *with* $x \in [0, 2]$.

4. Closing

We have studied that a function $f(x, y)$ that is continuous in a compact region D reaches a maximum and a minimum values in this region. These extrema values are obtained between the critical points of the function that are inside D and the points of the boundary of D. The method to find these absolute extrema consists on the obtaining of all the points that can be possible extrema and the comparison between the value of the function on them to finally choose the maximum and minimum value of $f(x, y)$ and the points where they are reached.

The examples presented illustrate the method in compact regions with different characteristics: defined by a function ($g(x, y) = 0$), defined by a polygonal and defined by a combination of both. Depending on the case we will have to consider the critical points of the function in the region D, the vertices of the region and the constrained extrema of $f(x, y)$ in the boundary of the region.

5. Bibliography

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