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Kinematic and static analysis of plane reciprocal frames

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Abstract
Reciprocal frames and reciprocal structures have already been considered of great interest in the past, as witnessed by the work of Leonardo da Vinci and of other important scientists and architects. Recent researches show that both the shape and the mechanical behaviour of such structures are really complexes, due to the concept of reciprocity. The position of each structural element in the space, indeed, as well as the way they transfer loads, depends on the position and on the role of adjacent elements, so that the structure must be studied as a whole and it can hardly be decomposed in simpler substructures. In this paper we focus on the statical and mechanical determinacy of a specific subset of this kind of structures, i.e. the multiple plane reciprocal frames. In this structures the shape can be easily defined in plane coordinates, so that only the mechanical behaviour needs to be studied. The presence of mechanisms or the possibility of self-stress states is largely related to the kind of internal constraints joining each bar to the others. The study is developed starting from the basic works on pinned bars assemblies [5], [8], in which the statical and kinematical determinacy are discussed by means of a matrix formulation of the equilibrium and compatibility equations. The extension of such formulation from pinned bars assemblies to plane reciprocal frames is the main goal of the research.

Keywords: Plane Reciprocal Frames, Repetitive structure, Kinematics

1. Introduction
Reciprocal frames (RF) and multiple reciprocal frames (MRF) are structures composed by mutually supported elements, arranged to form, respectively, one or more closed circuits, (Di Carlo [3]). They differ from better known truss assemblies because bars, in RF and MRF, join to each other not only at the ends but even at intermediate points. There are many examples of structures conceived following the reciprocity principle, starting from the technique adopted in Japan by the monk Chogen (1121-1206), for the construction of temples. In Europe, during Middle Age, a plane reciprocal frame (PRF) have been proposed

for building the floor of big rooms using short beams, as described by the medieval architect Villard de Honnecourt (1225-1250). The wider and most interesting studies were made by Leonardo da Vinci in the "Codex Atlanticus": he explored various patterns of beams grillages, and studied three dimensional arch structures for domes and bridges, in which short elements are supported by each others. Starting from Honnencourt's work, Sebastiano Serlio in 1537 first introduced a multiple plane reciprocal frame (MPRF) as a general solution for the construction of large floor by means of short elements simply supported to each other. A complete historical overview on ancient and contemporary realisations can be found in Rizzuto *et al.* [7].

Many different configurations are possible for MPRF and they can be classified on the basis of their geometrical properties, as proposed by Popovic [6]. In the present work we will focus mainly on the distinction between regular frames (Figure 1.a), obtained by means of the repetition of unit cells with the same shape and dimensions, and non-regular frames. Furthermore, non-regular frames can have regular topology (Figure 1.b), when only the length of bars and the position of joints change, while the topological scheme is the same, or they can be totally non-regular (Figure 1.c).

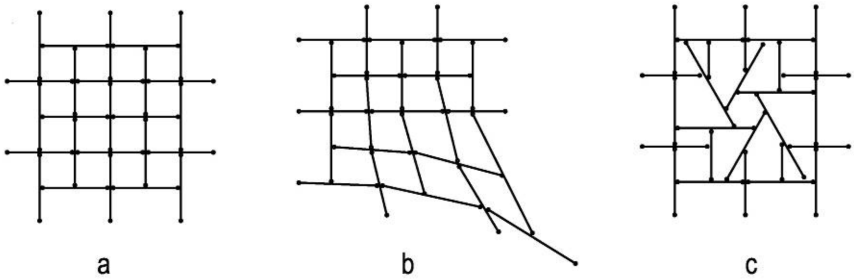

Figure 1

In contemporary architecture, as it has already been underlined by Popovic [6], regular configurations are more frequent than non-regular: many interesting realisations reflect this trend, as the Japanese architectures designed by Ishii, Kijima and Kan. Nevertheless it is clear that, besides structures composed by regular patterns, many other shapes are possible, even completely non-regular. These structures could be of great interest for architects and designers, but the complexity of the geometry makes difficult the analysis of the mechanical behaviour. A general analytical approach become then necessary for the study of MRF and for the assessment of their applicability to architecture.

2. Aim of the research

In spatial MRF the position of each element can be defined only in relation with all the others, making the geometrical representation of the structure a very hard task. While very simple configurations, as the ones derived by regular polyhedrons, the geometrical description can be easily obtained, in case of more complex spatial configurations the final geometry can only be obtained by means of numerical tools, as the genetic algorithm proposed by Baverel *et al.* [1]. MPRF, on the other hand, are plane configurations that can be easily defined 'a priori' through their plane coordinates, without the need of a numerical configuration processing, so that the attention can be focused mainly on the mechanical behaviour of the assembly.

The present work then focuses on the kinematic analysis of MPRF. These structures show different mechanical behaviour when loaded by in-plane and out-of-plane forces and their global stability is largely influenced by the topology and the type, number and position of internal and external constraints. Starting from these considerations, the aim of the research is the study of the statical and mechanical determinacy of regular and non-regular plane reciprocal frames through a matrix kinematical description. The presence of finite or infinitesimal inextensional mechanisms (kinematic indeterminacy) or the possibility of self-stress states (statical indeterminacy) is largely related to the kind of internal constraints joining each bar to the others. The study is developed starting from the basic works on pinned bars assemblies (Crapo [2], Guest and Hutchinson [4], Pellegrino and Calladine [5], Vassart *et al.* [8]), in which the static and kinematical determinacy are discussed by means of a matrix formulation of the equilibrium and compatibility equations. The extension of such formulation from pinned bars assemblies to MPRF is the main goal of the research.

3. Kinematics and statics of MPRF

When dealing with an assembly of bars, the concept of kinematic (compatibility) equations refers to the description of the mutual constraints between the assembly elements. On he other hand the static (equilibrium) equations express the fact that forces present inside the assembly elements must be equilibrated. These two sets of equations involve four groups of variables, accordingly with Pellegrino and Calladine [5]. In kinematic equations of truss assemblies the variables are the displacements of nodes and the elongations of bars. If bars are rigid and subjected only to forces, elongations are null, so that bars can be regarded as constraints between the nodes. Bars can be subjected to imposed elongations, as thermal strains, acting as a boundary condition in the kinematic problem. In equilibrium equations, on the other hand, variables are the internal stresses in bars and the external forces applied on the nodes. External forces are null when the structure is stress free or when it is subjected only to a self stresses state. Following this approach, the compatibility equations are written for the elements of the structures, while equilibrium equations are written for the nodes.

In an alternative approach, the displacements of bars and the 'dislocations' of joints can be assumed as the kinematic variables of the problem. As for the elongations, the joints 'dislocations' are null if joints are rigid, and prescribed non null joints dislocations can be imposed as a kinematic boundary condition to the structure. The dual static variables are in this case the external forces applied to the assembly elements, assuming the same local origin adopted to evaluate the elements displacements, and the forces mutually transferred by joints. Even in this case a self stress state can be present in the assembly, without external forces applied. The compatibility is expressed for the mutual reciprocal constraints, while the equilibrium is written for each bar.

The presence of a infinitesimal mechanism corresponds then to the presence of non trivial solutions of the homogeneous kinematic system, as well as the presence of a self stress state corresponds to a non trivial solution of the homogeneous static system.

4. Geometry and topology of regular repetitive MPRF

In order to introduce a general matrix formulation for regular repetitive MPRF we first analyse the planar configuration obtained by indefinitely repeating a three bars cell (or 'fan', following the denomination proposed by Baverel *et. al.* [1]), as shown in Figure 2. The unit cell is repeated in two directions, namely ***i*** direction and ***j*** direction, forming a hexagonal grid, so that each cell is connected to the six adjacent. In this way the relation between the nodes of the configuration, i.e. the points in which bars are joined to each other, and the bars can be written iteratively, starting from the description of the unit cell. In non-regular frames, on the other hand, the relation between nodes and bars can only be expressed through a topology matrix, or incidence matrix, that must be assigned when the configuration is defined. In the configuration of Figure 2 each bar has three nodal points and in each nodal point three bars are joined. For sake of generality each nodal point is considered composed by two coincident nodes, and the corresponding displacements are considered separately. The kinematic matrix of the frame can then be defined in a repetitive way, independently from the number of cells composing the frame, through the assembling of the local kinematical equations of each unit cell.

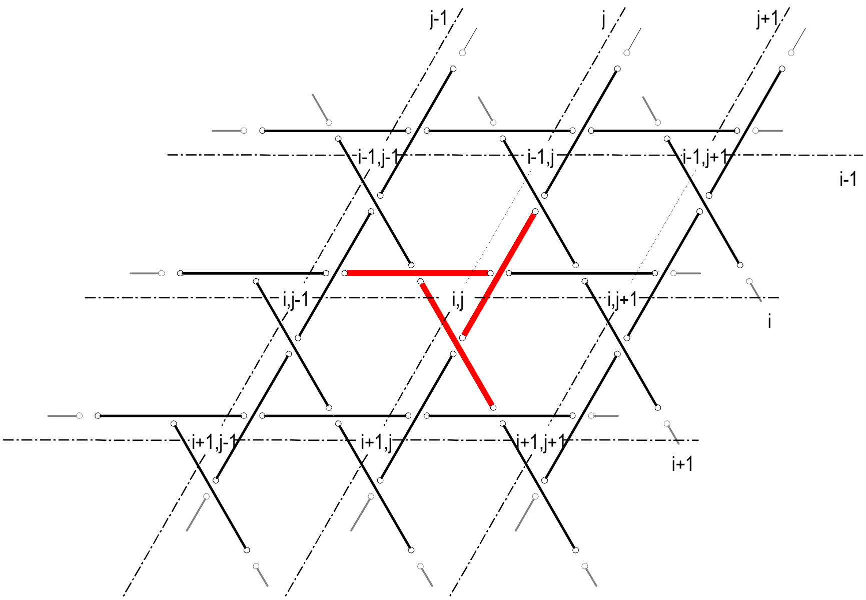

Figure 2

5. Geometry of the unit cell

In each cell the three bars are ordered counterclockwise starting from the top and labeled with capital letters. The three nodes of each bar are numbered from the inner to the outer, as shown in Figure 3 and their plane coordinates are defined as follows (1):

$$\begin{array}{ccc} A_1 & x_{A1} & y_{A1} \\ A_2 & x_{A2} & y_{A2} \\ A_3 & x_{A3} & y_{A3} \\ B_1 & x_{B1} & y_{B1} \\ B_2 & x_{B2} & y_{B2} \\ B_3 & x_{B3} & y_{B3} \\ C_1 & x_{C1} & y_{C1} \\ C_2 & x_{C2} & y_{C2} \\ C_3 & x_{C3} & y_{C3} \end{array} \qquad (1)$$

All the nine nodes of the unit cell, and the corresponding coordinates, are defined separately: the coincidence of the first node of one bar and the second node of the adjacent must be expressed explicitly by giving the same values to the corresponding coordinates. Such redundancy in the geometric input data will make possible to deal with irregular and more complex configurations.

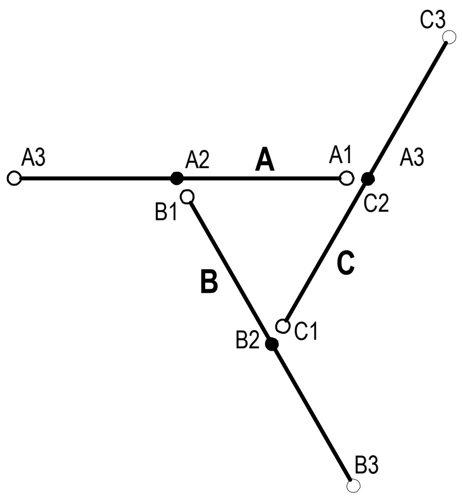

Figure 3

6. Linear kinematics of bars

Given a rigid body (bar) free to move on a plane, the displacement of the nodes can be defined through the three kinematic parameters, $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{\varphi}$: representing the displacements of the reference point $\boldsymbol{O}$, accordingly with Figure 4:

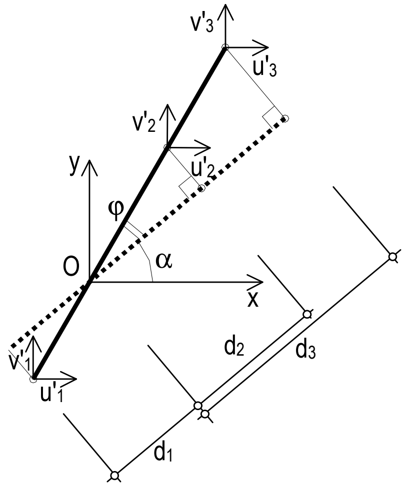

Figure 4

The linear kinematic relations are expressed by the following six equations:

$$\begin{aligned} u_k &= u + u'_k \\ v_k &= v + v'_k \end{aligned} \qquad \begin{aligned} u_k &= u - \varphi d_k \sin\alpha \\ v_k &= v + \varphi d_k \cos\alpha \end{aligned} \qquad \begin{bmatrix} 1 & 0 & -d_k \sin\alpha \\ 0 & 1 & d_k \cos\alpha \end{bmatrix} \begin{bmatrix} u \\ v \\ \varphi \end{bmatrix} = \begin{bmatrix} u_k \\ v_k \end{bmatrix} \tag{2}$$

and in matrix form:

$$[A_k]_{2x3} \{D\}_3 = \{d_k\}_2 \tag{3}$$

with k = 1 to 3 in the number of the joint in the bar.

For sake of convenience in the application to the reciprocal frame the reference point for the each bar of the assembly is assumed to be coincident with the first node of the bar.

7. Kinematics of the repetitive frame

The kinematic equations of the indefinite assembly represent the internal and external constraint conditions due to the presence of joints. If the assembly is considered indefinte, of course, external constraints lose their significance, because the assembly does not have boundaries. The kinematic variables are the generalysed plane displacements of the bars, $\boldsymbol{u}$,

v, ***φ***, and the 'dislocations' of joints. If joints are rigid, the dislocations will be null. Accordingly with Pellegrino and Calladine [5], the corresponding dual static variables are the external forces ***H, V, M***, applied to bars in the origin of the local axes systems, and the internal forces reciprocally traded in joints. It can be easily shown that these forces are univocally related to the internal state of stress of each bar.

The indefinite frame is obtained by means of the repetition of a unit cell: hence the kinematic description of constraints can consider separately the joints belonging to only one unit cell from the joints that connect cells between each other. This distiction is useful for the construction of the global kinemati matrix.

If we consider just one cell, its bars are mutually connected accordingly with this rule: provided that the bars are named counerclockwise as ***A***, ***B***, and ***C*** and the nodes are numbered as indicated in par. 4, then node 2 of each bar is linked with node 1 of the next bar. The kinematic equations of a single unit cell can then be written as follows, assembling the coefficients defined in par. 5:

$$\begin{bmatrix} A_{A1} & 0 & A_{C2} \\ -A_{A2} & A_{B1} & 0 \\ 0 & -A_{B2} & A_{C1} \end{bmatrix}_{6x9} \begin{bmatrix} D_A \\ D_B \\ D_C \end{bmatrix}_9 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}_6 \tag{4}$$

In order to build the whole kinematic matrix of the indefinite frame we start from a finite rectangular ***m*** rows, ***n*** columns frame. The bar displacements vectors of cells must be ordered into a unique vector. Grouping the displacements of bars belonging to the same ***i***, ***j*** cell:

$$\left\{D^{i,j}\right\}_9 = \begin{bmatrix} D_A^{i,j} \\ D_B^{i,j} \\ D_C^{i,j} \end{bmatrix} \tag{5}$$

the whole vector can be obtained ordering the vectors of unit cells by column first, then by rows, accordingly with the following scheme:

$$\{D\}_{9nm\,x\,1} = \left[D^{1,1} \cdots D^{1,j} \cdots D^{1,n} \cdots\cdots D^{i,1} \cdots D^{i,j} \cdots D^{i,n} \cdots\cdots D^{m,1} \cdots D^{m,j} \cdots D^{m,n}\right]^T \tag{6}$$

Each unit cell is then connected to the six adjacent as shown in Figure 5.

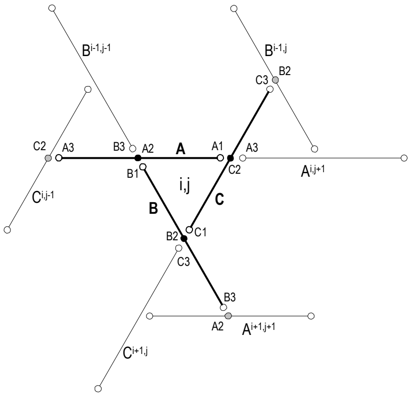

Figure 5

Reordering the unit cells into a one dimensional array, coherently with the vector of generalised displacements (17) the coefficients related to each unit cell can be reorganised in submatrices as shown in Figure 6.

The submatrix contains, for the generic ***i***, ***j*** unit cell, the coefficients that represent the constraints between the bars of the cell and between the cell and the six adjacent. In order to allow the final assembling of the kinematic matrix, the rows of submatrices can be grouped in two sets: 18 rows used to store the coefficients and a variable number of rows filled with zeros and used only as spacing. The number of rows ***h*** in submatrices depends on the number of cells in the frame. It can be calculated with this formula:

$$h = 8 + 12\ m \tag{7}$$

where ***m*** is again the number of rows of the rectangular assembly.

$$\begin{array}{c c c}
 & \textbf{i, j} & \\
\textbf{i, j-1} \leftarrow & \left[\begin{array}{ccc}
[0] & [0] & A_{C3} \\
[0] & [0] & [0] \\
-A_{A2} & [0] & [0] \\
\ddots & \ddots & \ddots \\
A_{A1} & [0] & -A_{C2} \\
-A_{A2} & A_{B1} & [0] \\
[0] & -A_{B2} & A_{C1} \\
\ddots & \ddots & \ddots \\
[0] & [0] & -A_{C2} \\
A_{A3} & [0] & [0] \\
[0] & -A_{B2} & [0] \\
[0] & A_{B3} & [0]
\end{array}\right] & \begin{array}{l}
 \\ \\ \\ \\ \\ \\ \\ \\
\leftarrow \textbf{i+1, j} \\
\rightarrow \textbf{i+1, j+1} \\
\leftarrow \textbf{i, j+1} \\
\end{array} \\
\textbf{i-1, j-1} \rightarrow & & \\
\textbf{i-1, j} \leftarrow & &
\end{array}$$

Figure 6

The submatrices can finally be assembled into the global matrix accordingly with the repetitive scheme shown in Figure 7. The horizontal position ***posH*** and vertical position ***posV*** of each submatrix into the global matrix depends on the values of ***i*** and ***j***:

$$\text{posH} = 9\text{m}(\text{i-1})+9(\text{j-1}) \tag{8}$$

$$\text{posV} = 6(\text{n-i})+(\text{h-4})(\text{j-1}) \tag{9}$$

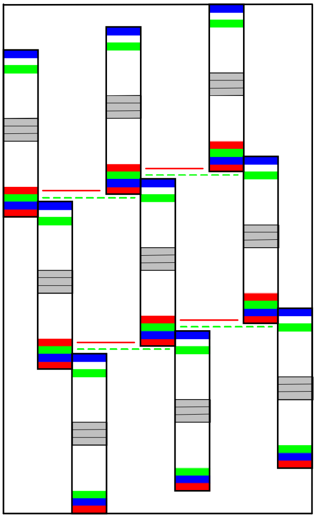

Figure 7

For a finite rectangular ***m*** x ***n*** frame the kinematic equations take then the following form:

$$[A]_{12mn \ x \ 9mn} \{D\}_{9mn} = \{ 0 \}_{12mn} \tag{10}$$

The corresponding dual equilibrium equations can be written by transposing the coefficient matrix:

$$[A]^T_{9mn \ x \ 12mn} \{ Q \}_{12mn} = \{F\}_{9mn} \tag{11}$$

As stated in par. 6, vector {F} represents the external forces applied to the bars, while vector {Q} is the vector of the internal forces mutually exchanged in joints.

8. Concluding remarks

In this paper a general procedure for the construction of the kinematic matrixes of repetitive reciprocal plane frames is presented. The procedure is based on the subdivision of the global configuration into unit cells, topologically homogeneous, and on a repetitive indexing of bar, nodes and DoF's. As it is well known the kinematic matrix is the transpose of the static matrix and its properties are related to the kinematical and statical determinacy of the reciprocal assembly. Future developments of the research will focus on the properties of specific finite and infinite, regular and non-regular configurations, with particular attention to kinematically indefinite assemblies. The presence of inextensional mechanisms is in fact of large interest when these configurations are applied to engineering or architectural problems. The possibility to adapt the structural shape to the surrounding conditions, as well as the application in kinetic structures, can then be regarded as the most interesting features of reciprocal plane frames.

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