

## **Randomness in topological models**

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### **Abstract**

There are two aspects of randomness in topological models. In the first one, topological idealization of random patterns found in the Nature can be regarded as planar representations of three-dimensional lattices and thus reconstructed in the space. Another aspect of randomness is related to graphs in which some properties are determined in a random way. For example, combinatorial properties of graphs: number of vertices, number of edges, and connections between them can be regarded as events in the defined probability space. Random-graph theory deals with a question: at what connection probability a particular property reveals. Combination of probabilistic description of planar graphs and their spatial reconstruction creates new opportunities in structural form-finding, especially in the inceptive, the most creative, stage.

**Keywords:** topological models, random graphs, structural forms

### **1. Introduction**

Significant level of effectiveness and sophistication has been reached by spatial lattice structures during their hundred-year development. Moreover, these structures perfectly corresponded with aesthetical views originated in the “first machine age” and futurists movement. Clear transmission of loads and structural efficiency of elements conformed to traditional meaning of the logic of structural systems. Architectural form stayed in close symbiosis with these systems. Some recent trends in architectural design have changed this point of view. Nowadays, visual impression of the building is a predominant requirement. Unconstrained creation of forms became a new aesthetical paradigm.

Free-form architectural design brought to the fore some new techniques of structural modeling. These techniques that include, among other things, so-called new geometries and adaptation of forms observed in natural objects, are now well established. However, some mathematical concepts developed primarily for use in completely different areas, open new possibilities in creation of structural models.

## 2. Topological models

Space-filling compounds of polyhedra became, in a very natural way, the primary choice for the geometrical models of spatial lattice structures and many of them have been constructed on this basis. Figure 1 gives just a simple example of such a double-layer grid generated from octahedral basic modular elements.

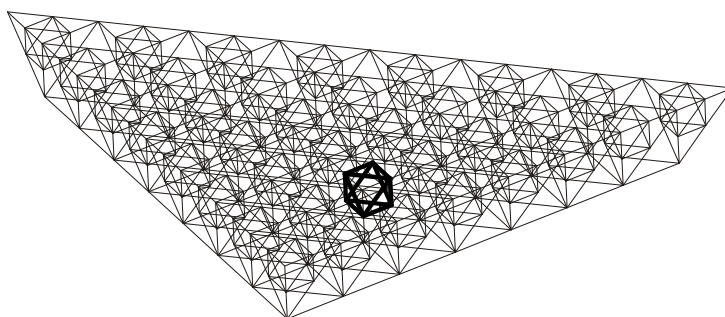


Figure 1: Example of double-layer grid structure

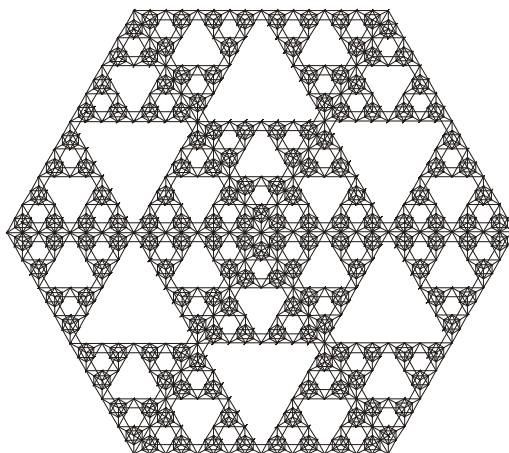


Figure 2: Fractal shaped structure B{T-T}A with Sierpinski's triangles

Some simple modifications of space-filling lattices allowed shaping much more sophisticated configurations. Fractal shaped structure B{T-T}A with Sierpinski's triangles, Fig. 2, exemplifies easiness of inserting various size openings.

Topological models for complex structures can be derived through the Steinitz's theorem. From this theorem we know, that every polyhedron realized in 3D space can be represented by a graph which is planar and 3-connected with edges in every vertex. (Grünbaum [6]), Fig. 3.a. Techniques of conversion from spatial net to planar graphs include Schlegel diagrams and other constructions. Topological representation is on the lowest level, regarding the number of characteristics necessarily needed for description of objects. Graphs preserve geometrical relations of structural components in their most general outlines. They deal not with exact shape of the objects but with their interrelations, described as topological characteristics (e.g. valency).

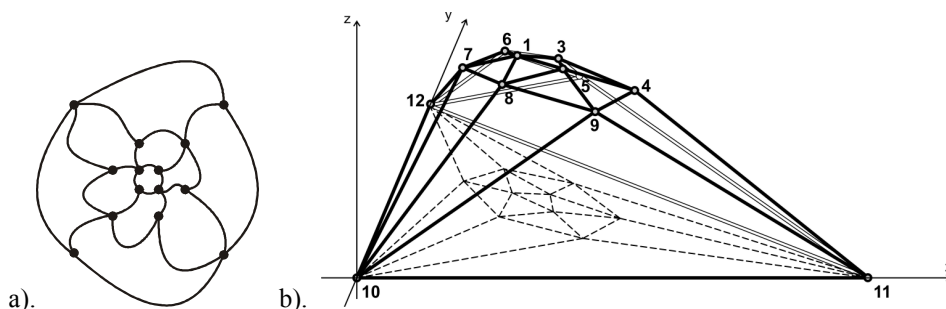


Figure 3: Planar, 3-connected graph (a) and M-T-C method of spatial reconstruction (b)

Topological models are subject of some constrains, from which the Euler's formula is the best known. Some other results in graph theory, e.g. Eberhard's formula, give important information about spatial realizability of graphs of arbitrary configuration. There is a variety of transformations of planar graphs that preserve their realizability. The most important are deleting, contraction and opposite operations (Tarczewski and Bober [12]). Spatial reconstruction of graphs is possible by means of techniques such as Koebe-Andreev-Thurston method (based on circle packing theorem), Lawrence extension, Gale diagrams and Maxwell-Tutte-Cremona method (Tarczewski and Bober [13]). The general idea of the latter one is presented on Fig. 3.b.

### 3. Natural prototypes of structural forms

Sir Frederick Charles Frank has noted that: "In thinking about structures, inspiration can be drawn from surprising sources". Indeed, people have followed structural forms widely appearing in nature, since the very beginning of their conscious structural activity. We can perceive these forms in numerous traditional structural solutions. Moreover, topological relations concerning geometrical entities and mathematical theories such as graph theory,

symmetry groups, crystallography and others, can be related to these forms (Bober and Tarczewski [3]). However, advanced methods in mathematics, allow us to go far beyond simple imitation of natural prototypes and design much more sophisticated objects.

Infinite Platonic polyhedra discovered in 1937 by Coxeter, followed by infinite semiregular polyhedra (Wachman *et al.* [14]) have formed a basis for the most promising concept in description of natural structures – sponge surfaces configurations (Burt [5]).

Figure 4 presents the octahedra-tetrahedra net and its dual. A multidirectional infinite polyhedra can be generated from this pair, by filling cycles of edges with faces (Wachman *et al.* [14]). When cycles of edges are not coplanar, then corresponded faces are spatially curved and form a sponge. Two examples of natural structures and related spongeous nets, shown on Fig. 5 and Fig. 6, bring closer importance of this idea. Surface of coral and internal structure of elephant skull resemble periodic sponge surface based on a uniform trivalent lattice and subdivision of space by two dual complementary networks (Burt [5]).

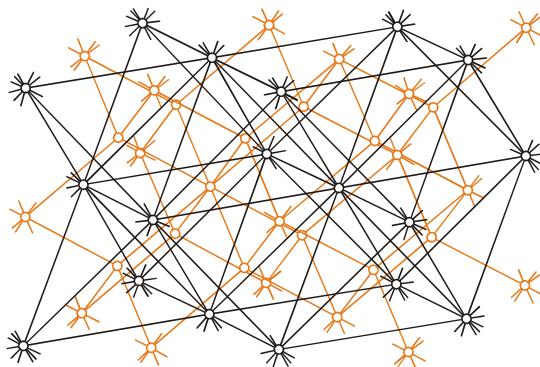


Figure 4: The octahedra-tetrahedra net and its dual

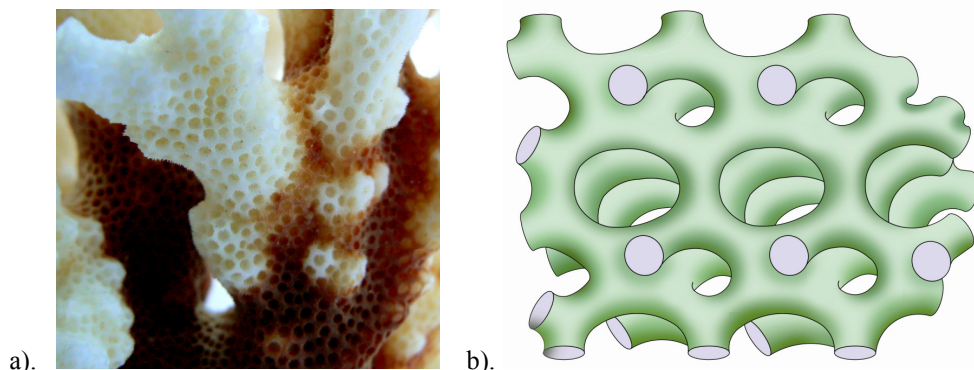


Figure 5: Surface of coral (a) and periodic uniform trivalent lattice (b)

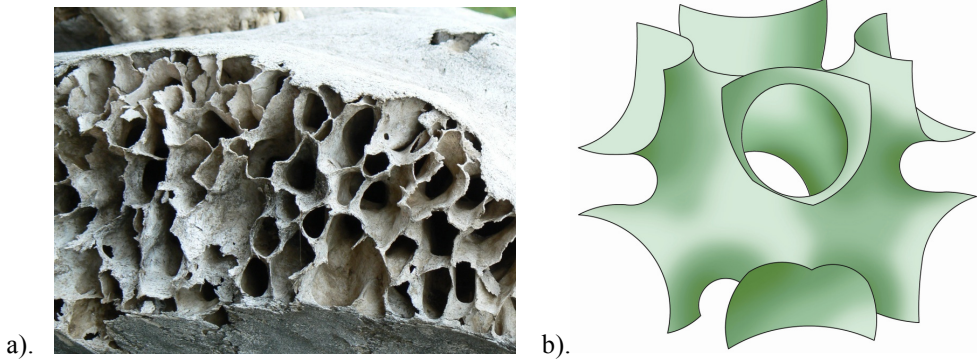


Figure 6: Internal structure of elephant skull (a) and subdivision of space by two dual complementary networks (b)

Existence of miscellaneous polyhedra and sponges as well as some of their properties can be predicted from the periodic table, which is based on properties of statistical symmetry (Burt [4]). In this approach, average values of topological characteristics of the net are considered, instead of exact ones (Loeb [8]).

Two examples of natural structures presented above, are quite regular, so idealization by perfect geometrical construction is acceptable. It is easy to see however, many irregularities and random imperfections. On the other hand, following quoted thought of Ch. Frank, one can take into consideration random patterns apparently with no structural properties. Figure 7 presents a photo of cracked icecap and a graph drawing derived from this photo.

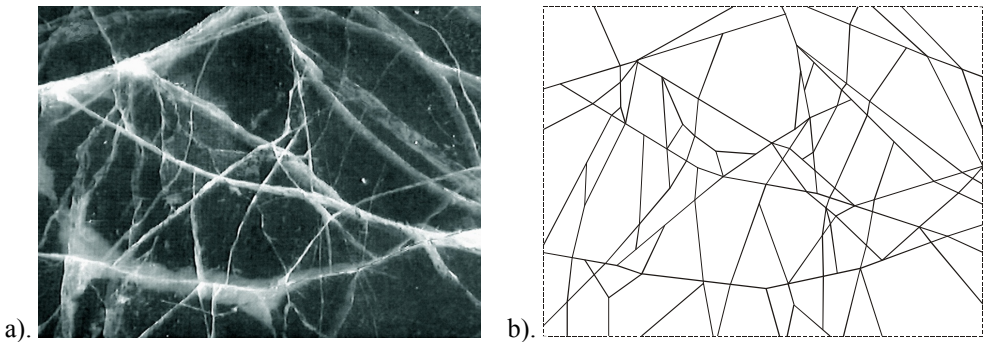


Figure 7: Cracked icecap and corresponding graph

These two phenomena: irregularities or imperfections in generally ordered structures and structural connotations of various patterns are generators of randomness in topological models of structures related to natural prototypes. Also some man-made forms, especially those “freely” designed, can introduce random patterns of nodes, random distribution of supporting points etc. When the number of elements increases, a probabilistic approach to description of all these structures becomes reasonable.

#### 4. Random graphs and random networks

Let’s imagine a set of vertices with slots for all available edges. For each slot we can flip a coin. Slot receives an edge if the coin shows heads, but disappears when the coin shows tails. Thus each edge receives its realization with probability  $p = 1/2$ . This simple example, adopted from (Palmer [8]), demonstrates general idea of random graphs. More formal definition states that graph  $G = G(n, p)$  is a random graph with  $n$  vertices, if its edges are chosen independently with probability  $p = p(n)$ . In other words, for each pair of vertices  $(v_i, v_j)$  exists edge  $e_{ij}$  with probability  $p$ . Expected number of edges  $q$ , of  $G$ , is equal (Janson *et al.* [7]):

$$q = \binom{n}{2} \cdot p \tag{1}$$

Figure 8 visualizes this example: slots for edges are marked by thin red lines, while realized edges (by means of numerical experiment) – by black lines. Eight vertices and twenty eight slots for edges allow 251 548 592 graphs of order 8 to be drawn.

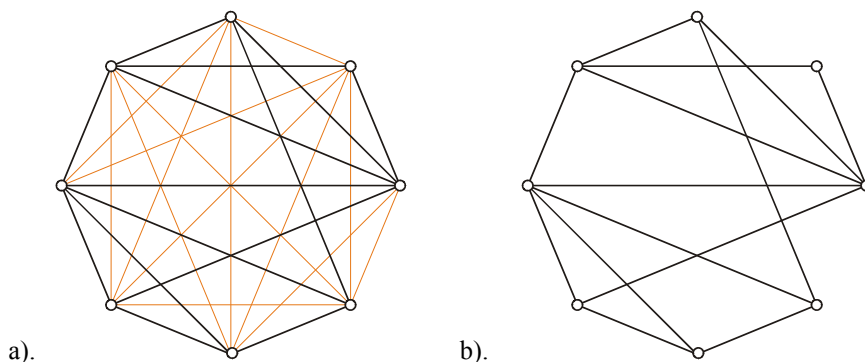


Figure 8: Random realization of edges for a given set of vertices

The above example refers to the classic model of random graphs, originated in works of P. Erdős and A. Rényi some fifty years ago. Actually there are three submodels: A, B and C, which vary with some properties. Only A-model is discussed here.

For description of complex structures (both regular and totally unsymmetrical) one makes use of average values of topological characteristics. Most important of these is the average vertex valency  $\bar{r}$  (edge valency) and average face valency  $\bar{q}$  (edge valency) :

$$\bar{r} = \frac{\sum_{r=2}^{\infty} V_r}{n} \quad (2)$$

$$\bar{q} = \frac{\sum_{q=2}^{\infty} V_q}{n} \quad (3)$$

where  $V_r$  is a number of  $r$ -valent vertices and  $V_q$  is a number of  $q$ -valent faces (Loeb [8]).

For random graphs of order  $n$  (with  $n$  vertices), probability  $P(G)$ , that the graph  $G$  has  $q$  edges, is given by (Palmer [8]):

$$P(G) = p^q (1-p)^{\binom{n}{2}-q} \quad (4)$$

and the valency  $r_i$  of a node  $i$  follows a binomial distribution with parameters  $n-1$  and  $p$  (Barabási and Albert [1]):

$$P(r_i = r) = C_{n-1}^r \cdot p^r (1-p)^{n-1-r} \quad (5)$$

where  $C_{n-1}^r$  is a number of equivalent ways of selecting the  $r$  end points for edges that start in vertex  $i$ . Expected value of average vertex valency is

$$\bar{r} = p(n-1) \quad (6)$$

Distribution of the number of vertices with valency  $r_i$  approaches a Poisson distribution, which indicates, that most vertices have approximately the same value of valency, close to the average valency  $\bar{r}$ .

If we return to the graph of cracked icecap from Fig. 7, and draw the diagram of appearances of all  $r$ -valent vertices, Fig. 9.a, this distribution is noticeable. The same is for appearances of all  $q$ -valent faces, Fig. 9.b. Average values for this graph are:  $\bar{r} = 3.963$  and  $\bar{q} = 4.041$ .

For large systems, average valencies must satisfy (Loeb [8])

$$\frac{1}{r} + \frac{1}{q} = \frac{1}{2} \quad (7)$$

For example of cracked icecap graph we have:  $1 / 3.963 + 1 / 4.041 = 0.4998 \cong 1/2$ .

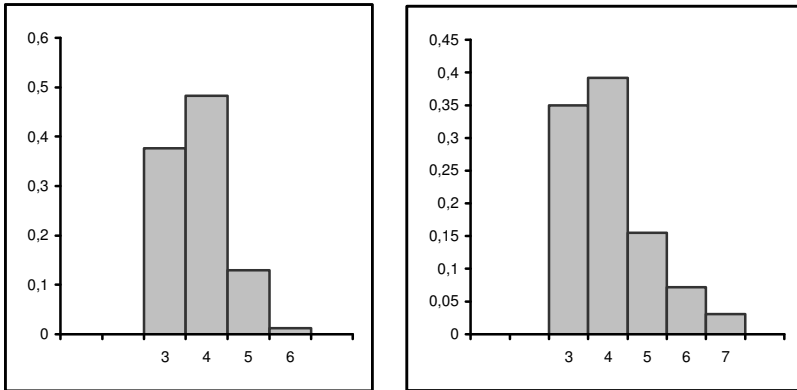


Figure 9: Distribution of  $r$ -valent vertices (a) and  $q$ -valent faces (b) in graph from Fig. 7.b.

Some aspect of effectiveness in lattice systems can be characterized by average path length  $\bar{l}$ . Distance from one vertex  $v_i$  to vertex  $v_j$  is measured by number of edges that are necessary to pass through between these two vertices. The shortest path is the distance with the smallest number of edges (in directed graph distance from  $v_i$  to  $v_j$  is not necessarily the same as from  $v_j$  to  $v_i$ ). The average path length  $\bar{l}$  is the average value of shortest paths for all pairs of vertices in graph. In random graphs the average path length is proportional to the logarithm of the order of graph:  $\bar{l} \propto \log n$  (Barabási and Oltvai [2]). It was observed, that even for very large graphs  $\bar{l}$  is relatively small. This is known as a “small-world effect”.

The Erdős-Rényi model assumed “ideal” randomness, i.e. every edge has the same probability of existence. This model can be “improved” by construction of so called configuration model. In this model graph has a specific distribution of vertex valencies,  $p_k$ , instead of constant value of probability.

A different approach is specific for scale-free networks, such as the Barabási-Albert network (Barabási and Albert [1]). In this model, graph is constructed in a recursive procedure, with preferential attachment principle. Every new vertex with  $m$  edges is added to the graph. The new edges connect to the existing vertices and the likelihood of connecting to a vertex depends on its valency:

$$\Pi_i = \frac{r_i}{\sum_j r_j} \quad (8)$$

where  $r_i$  is the valency of vertex  $i$ , and  $j$  denotes summation over graph vertices. This means, that new vertex connects first to the vertex with highest valency. Thus, the graph contains a (relatively small) number of vertices of high valency. These vertices are called



hubs. The probability that vertex  $i$  has valency  $r_i$  follows power-law distribution (Barabási and Oltvai [2]):

$$P(r_i) \propto r_i^{-\gamma} \quad (9)$$

where  $\gamma$  is a degree exponent. In Barabási-Albert model  $\gamma=3$ , while in graphs of this type, that are observed in nature  $2 < \gamma < 3$ . Average path length in these graphs follows  $\bar{l} \propto \log \log n$ . Through the power-law distribution of vertex valency, system manifests its self-organization into a scale-free state. Figure 10 gives a comparison of a random graph (Erdős-Rényi model) and scale-free network (Barabási-Albert model).

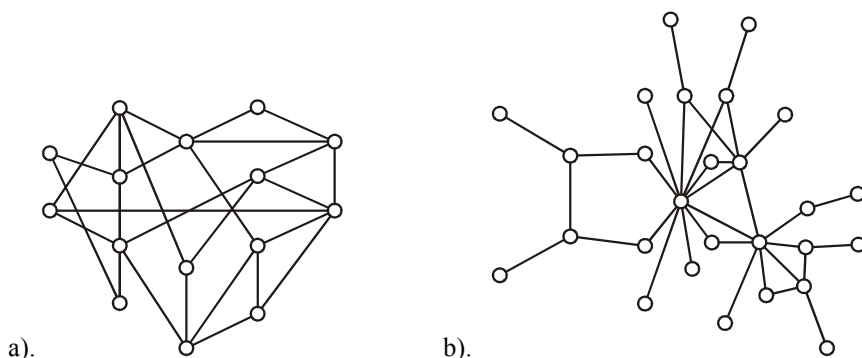


Figure 10: Example of random graph (a) and scale-free network (b)

The construction of a random graph is often described as an evolution. Process starts with a set of  $n$  vertices and continues with successive addition of randomly chosen edges. At the each successive stage of this process, the graph represents increasing probability  $p$ . If  $p \rightarrow 1$ , graph becomes fully connected.

Surprising feature of random graphs is that many of their important properties appear suddenly. This means that, for a given value of probability  $p$ , almost all graphs has some property or almost no graph has it. The change is rather rapid and there often exists a critical probability  $p_c(n)$ . It's a topic of the random graph theory to determine its value for a particular property (Barabási and Albert [1]).

## 5. Random graphs in structural world

Techniques of spatial reconstruction of planar graphs, mentioned above, allow “structurization” of various random patterns found in nature and made by man. Lifting of these patterns materializes their intuitively perceptible properties allowing formation of interesting lattices. These techniques, which are discussed in (Tarczewski and Bober [13]) can be combined with various method of generation of random patterns. An example is

covering a sphere with spherical caps (Sugimoto and Tanemura [10]) and transformation of this pattern by means of circle packing theorem (Koebe-Andreev-Thurston method).

Structural applications of random graphs are not limited, however, only to lattices. Many beautiful structures have been designed according to the flow pattern of forces. Works of P.L. Nervi are well-known examples, Fig. 11.

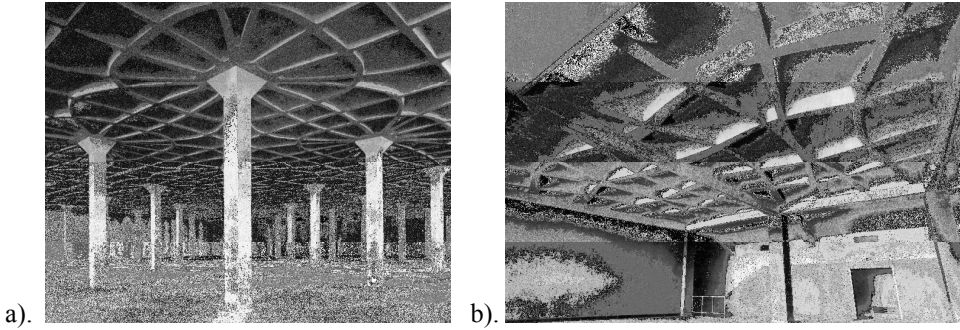


Figure 11: Examples of P.L. Nervi floors: Gatti factory (a) and Palazzo dello Sport (b)

Trajectories of internal forces in these regularly supported floor slabs remind typical drawings of graphs. In this case “graphs” are derived from the continuous structure. But we can consider a reverse approach. Let’s imagine a random set of vertices that present the only acceptable supporting points in a freely designed form. For given points we can generate a set of random Voronoi cells on the surface. It is remarkable, that statistical properties of these cells follow Poisson distribution (Tanemura [11]), Fig. 12.a.

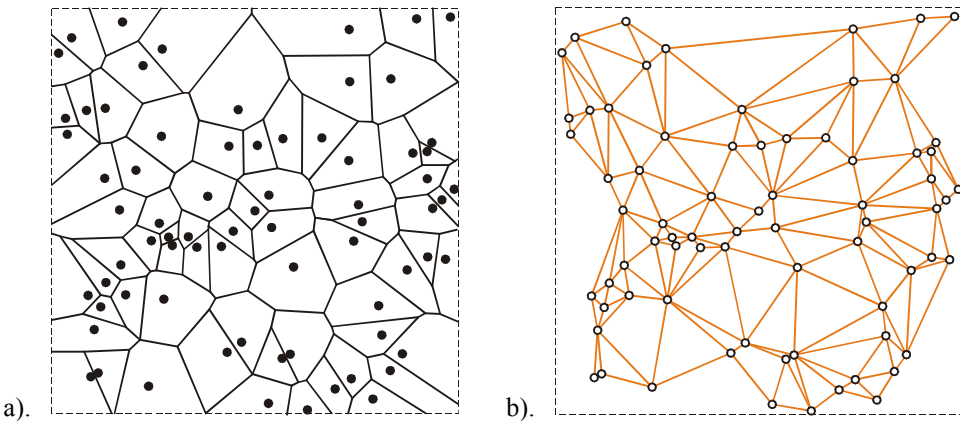


Figure 12: Random Voronoi cells (a) and corresponding Delaunay triangulation (b)

Then, construction of Delaunay triangles is possible, which results in new pattern of lines. Fig. 12.b. Now, we can regard triangulation lines as trajectories of internal forces and search for the corresponding structure. Applying this approach to deep beams, one can take advantage from the strut-and-tie method (STM). In this case, some edges are assumed to be compression struts, while other – tensile ties, with a requirement that equilibrium of the nodes must be maintained. Moreover, some properties can be assigned to the edges, by means of minimax theorems in graph theory. By filling neutral space between strut and ties, a continuous structure is obtained. One recognizes that several possible solutions may exist for problem of reverse construction of loaded structure, for which flow of stresses conforms considered pattern. Thus, this method seems to be a highly flexible and conceptual tool for shaping structures.

## 6. Conclusions

Graph theory usually modeled complex structures as regular objects. Topology of these models was a subject of many important results within the theory. Large class of networks observed in Nature has rather complex topology. Some of these networks reveal more or less ordered structure, which can be modeled as spatial lattices or sponges. Other natural structures appear as completely random with unknown organizing principles. For modeling these structures random graph theory may be used.

Topological model for evolutionary formed natural structures relies on scale-free networks, which are formed in the recursive procedure, with some initial assumptions. In regular networks final topology is determined by a global optimization process. In this process edges are distributed to minimize or maximize, in the whole network, some predefined quantity, e.g. minimize average path length.

In evolving networks the global optimization is absent, as the decision about where to connect is taken at the node level (preferential attachment principle). However, due to eq. (8), every new vertex has information about the degree of all vertices in the network, thus this decision is not entirely local. This makes the system consistent. It is still not clear, to what degree the topology of random natural structures is shaped by global optimization, or the local processes observed in scale-free networks.

Disregarding method of generation, random structural models are useful in topological analyze. They can be modified, optimized and reconstructed in space as standard graphs.

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