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MODELLING 1-MONTH EURIBOR INTEREST RATE BY USING DIFFERENTIAL EQUATIONS WITH UNCERTAINTY

J. C. CORTÉS, J. V. ROMERO, A. SÁNCHEZ-SÁNCHEZ and R. J. VILLANUEVA

Instituto Universitario de Matemática Multidisciplinar
Universitat Politècnica de València, Spain
E-mail: jccortes@imm.upv.es

Abstract

This paper deals with modelling interest rate using continuous models with uncertainty based on Itô-type stochastic differential equations. It is provided an analysis of theoretical aspects that involves the so-called Vasicek's model as well as their practical application. The latter includes model parameter fitting and measurement of goodness-of-fit of the model. The theoretical results are applied to modelling 1-month Euribor interest rate.

1. Introduction

Interest rates' changes have an impact on the stock markets, and as a consequence, they also affect numerous economic activities which influence direct and/or indirectly on companies and domestic economies. When interest rates have longer rises over time, the share prices drop off and, then, the Exchange also does. The relationship between stock cycle and business cycle generates indirect effects that produces changes in interest rates which influence directly in economic activity.

Due to high interest rates, both companies and families have to cope with higher charges and, as a consequence, their benefits and consumption levels reduce, respectively. At the same time, this situation leads to a decreasing in sharing dividends. On the other hand, the rise in interest rates generates

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higher profitability of fixed income investments (bonds, etc.) attracting the investor and the flow of money, against investments equities (shares), leading to falling prices down.

The above comments motivate the search for reliable quantitative methods to predict interest rates. The Ito-type stochastic differential equations (s.d.e.'s) constitute powerful tools for modelling many economic phenomena and, in particular, they have been successfully used to model short term interest rates [1, 2].

In this paper, we revisit the Vasicek model and we show how it can be applied to model 1-month Euribor interest rate. Section 2 is devoted to motivate the stochastic Vasicek model from its deterministic counterpart as well as to obtain its solution taking advantage of the Ito Lemma. A full probabilistic description of the solution is also provided in Section 2. A method to estimate the model parameters and its application to model the 1-month Euribor interest rate is developed in Section 3. Conclusions are drawn in Section 4.

2. The Stochastic Interest Rate Model

This section is divided in two parts. In the first piece, the Vasicek stochastic model, that will be applied later to predict the 1-month Euribor interest rate, is motivated from its deterministic counterpart. This model is based on an Ito-type s.d.e. In the second part, we will show how to compute the solution stochastic process of the Vasicek model as well as its main statistical functions, namely, the mean and the variance functions.

2.1. Motivating the Vasicek Stochastic Model

Numerous mathematical models proposed to describe the time evolution of interest rates belong to the class of *mean-reverting models* [2]. The common property of these models is that their solutions tend to stabilize on the long-run. This property captures the dynamics of the agents in the financial markets, including the authorities which govern them, who try to make decisions providing stability on the interest rates. Otherwise, numerous tensions are created along with the international markets affecting negatively both companies and individual investors.

The Vasicek model is based on a s.d.e. whose formulation can be motivated from a deterministic model which retains the property that its solution is asymptotically stable. Its deterministic counterpart is formulated as follows

$$\begin{cases} r'(t) = k(\mu - r(t)), & k > 0,\ \mu \in \mathbb{R}, \\ r(0) = r_0, & r_0 > 0, \end{cases} \tag{1}$$

where $r(t)$ denotes the interest rate at the time instant t, $\mu \in \mathbb{R}$ denotes the average long-term interest rate, $k > 0$, represents a constant that measures the velocity of adjustment of interest rate $r(t)$ to μ and, r_0 is the initial interest rate. It is straightforward to check that the solution $r(t)$ of (1) is unconditionally stable since

$$r(t) = (r_0 - \mu)e^{-k(t-t_0)} + \mu \xrightarrow[t \to \infty]{} \mu. \tag{2}$$

Model (1) is completely deterministic. However, it is obvious that, in practice, the description of interest rates depends on a number of different and complex factors that are far from being known with certainty. This motivates the consideration of randomness into the deterministic model (1) but trying to still keeping its mean reversion property. With this aim, the model parameter μ, which represents the long-term behaviour of the model, is going to be considered as a random variable rather than a deterministic value. To this end, this parameter is perturbed as follows

$$\mu \to \mu + \lambda B'(t) \quad \lambda > 0, \tag{3}$$

where $B(t)$ is the Brownian motion or Wiener stochastic process, $B'(t)$ denotes its derivative (in a generalized sense based on distributions theory) which is usually termed white noise process, and $\lambda > 0$ represents the intensity of the perturbation.

Introducing in the model (1) the differential notation for the derivative, i.e., $r'(t) = \dfrac{dr(t)}{dt}$ and considering the randomness in the model parameter μ according to (3), after formal algebraic manipulations one gets the following Itô-type s.d.e.

$$\begin{cases} dr(t) = \alpha(r_e - r(t))dt + \sigma dB(t), \\ r(0) = r_0, \end{cases} \tag{4}$$

where the following identification among the parameters has been considered for the sake of clarity: $\alpha = k$, $r_e = \mu$ and $\sigma = k\lambda > 0$. This model is usually referred to as in the financial literature the Vasicek or Ornstein-Uhlenbeck's model [2, 3].

A part from Vasicek's model, there are alternative interest rates models based on Ito s.d.e.'s that keep the mean reversion property. All of them adapt to the following pattern

$$\begin{cases} dr(t) = \alpha(r_e - r(t))dt + \sigma(t)dB(t), \\ r(0) = r_0. \end{cases} \tag{5}$$

It has been long recognized in the finance literature that one of the most important features of (5) for derivative security pricing is the specification of the function $\sigma(t)$. As a consequence, every model has tried to specify $\sigma(t)$ correctly. Here, we account for the most important ones used in practice. First, notice that in the Vasicek's model, the instantaneous volatility of the spot rate $r(t)$ is assumed to be constant, $\sigma(t) = \sigma$. The so-called CIR's model by Cox-Ingersoll-Ross (see [4]) assumes that $\sigma(t) = \sigma\sqrt{t}$, being $\sigma > 0$. Hence, this model considers time-dependent local volatility. The advantage of CIR's model against the Vasicek's model is that, under certain conditions affecting their parameters, one can guarantee the positivity of the solution. Currently, this theoretical advantage does not seem to be critical since trading values of Euribor interest rates are closer to zero. In [5], one considers $\sigma(t) = \sigma r(t)^{\lambda}$, $\lambda > 0$ as a mathematical generalization of CIR's model that provides more degrees of freedom in practical situations. In the outstanding contribution [6], one accounts for different interest short-term rate models. Moreover, one proposes a new model that overcomes some drawbacks encountered in real applications. A comprehensive presentation of interest rate models in theory and practice can be found in the book [9].

2.2. Solving the Stochastic Model and Computing its Main Statistical Properties

This section is addressed to obtain the solution of model (4). First, notice that it is based on an Ito s.d.e. of the form:

$$\begin{cases} dX(t) = f(t, X(t))dt + g(t, X(t))dB(t), \\ X(0) = x_0, \end{cases} \tag{6}$$

where $X(t) = r(t)$, $f(t, X(t)) = \alpha(r_e - r(t))$, $g(t, X(t)) = \sigma$ and $x_0 = r_0$.

A key result that has demonstrate to be very useful to calculate the solution of an Ito s.d.e. of the form (5), for specific forms of the coefficients $f(t, X(t))$ and $g(t, X(t))$, is the so-called Itô Lemma. Numerous versions and adaptations to different stochastic processes are available regarding the Ito Lemma. In this paper, we will take advantage of the differential version adapted to the Brownian motion established in Theorem 3.12 of [1]. Under mild conditions, this result assures that if $X = X(t)$ is a solution of the s.d.e. (5) and $F(t, x)$ is a deterministic mapping such that the following partial derivatives $\frac{\partial F(t, x)}{\partial t}$, $\frac{\partial F(t, x)}{\partial x}$, $\frac{\partial^2 F(t, x)}{\partial t^2}$, $\frac{\partial^2 F(t, x)}{\partial x^2}$ and $\frac{\partial^2 F(t, x)}{\partial t \partial x}$ are continuous, then $F(t, x)$ satisfies the following Itô s.d.e.

$$dF(t, X(t)) = \left(\frac{\partial F(t, x)}{\partial t} + f(t, X)\frac{\partial F(t, x)}{\partial x} + \frac{1}{2}(g(t, X))^2 \frac{\partial^2 F(t, x)}{\partial x^2} \right) dt + g(t, X)\frac{\partial F(t, x)}{\partial x} dB(t). \tag{7}$$

In order to obtain the solution of (4), let us introduce the following change of variables:

$$X(t) = r(t) - r_e \Rightarrow \begin{cases} r(t) = X(t) + r_e, \\ dX(t) = dr(t). \end{cases} \tag{8}$$

This permits to rewrite (4) as follows

$$\left. \begin{aligned} dX(t) &= -\alpha X(t)dt + \sigma dB(t), \\ X(0) &= r_0 - r_e. \end{aligned} \right\}. \tag{9}$$

Now, denoting by $X = X(t)$ and considering the deterministic mapping $F(t, X) = e^{\alpha t}X$, we apply the Ito Lemma taking into account the following identification between (9) and the general pattern (6)

$$f(t, X) = -\alpha X,\ g(t, X) = \sigma. \tag{10}$$

This leads to $d(e^{\alpha t}X) = \sigma e^{\alpha t}dB(t)$ or, equivalently, in its integral form

$$\int_0^t d(e^{\alpha s}X) = \sigma\int_0^t e^{\alpha s}dB(s) \Leftrightarrow e^{\alpha t}X(t) - X(0) = \sigma\int_0^t e^{\alpha s}dB(s). \tag{11}$$

As a consequence, isolating $X(t)$ and expressing the result in terms of the original variable $r(t)$ one gets

$$r(t) = r_e + (r_0 - r_e)e^{-\alpha t} + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dB(s). \tag{12}$$

This stochastic process is the solution of the Vasicek model (4). Notice that it is expressed in terms of an Ito-type integral. Although this integral cannot be explicitly calculated, based on properties of the Ito integral for deterministic integrands [1, Ch.3], a full probabilistic description of the solution (12) of the Vasicek model can be obtained as follows

$$r(t) \sim N(\mu[r(t)];\ \sigma[r(t)]),\ \begin{cases} \mu[r(t)] = r_e + (r_0 - r_e)e^{-\alpha t} \\ \sigma^2[r(t)] = \dfrac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}), \end{cases} \tag{13}$$

which means that for every t, the random variable $r(t)$ is gaussian with mean $r_e + (r_0 - r_e)e^{-\alpha t}$ and variance $\sigma^2(1 - e^{-2\alpha t})/(2\alpha)$. Moreover, as $\alpha > 0$, one gets

$$\lim_{t\to\infty} r(t) \sim N\left(r_e;\ \frac{\sigma^2}{2\alpha}\right). \tag{14}$$

This is in agreement with the interpretation of parameter r_e as a generalization in the stochastic context of parameter μ that appears in the deterministic model (see (1)-(2)).

3. Modelling the 1-Month Euribor Interest Rate

From a practical standpoint, once a sample of data corresponding to interest rate is available, the model parameters r_0, r_e and σ must be fitted. This section describes the process followed to do that including its application for modelling the 1-month Euribor interest rate.

At this point, it is important to motivate the convenience of using Vasicek's model to describe the dynamics of 1-month Euribor interest rate. In the financial market, there are fifteen types of Euribor rates available depending on their maturity. All these values range from one week to one year (1, 2 and 3 weeks; 1, 2 ..., 12 months). Here, we consider 1-month Euribor because this type of interest rate belongs to the class of short-term interest rate where local volatility remains approximately constant. This feature is in agreement with the formulation of the Vasicek's model by an Ito s.d.e. (see expression (4)), where the coefficient σ of the stochastic term $dB(t)$ is constant.

3.1. Model Parameter Estimation

Taking advantage that $r(t)$ has a gaussian distribution (see (13)), model parameters will be estimate by Maximum Likelihood Method. Let $\mathcal{S} = (r_0, r_1, \ldots, r_N)$ be a sample data corresponding to interest rate at equal times instant $t_0, t_0 + \Delta t, \ldots, t_0 + N\Delta t, \Delta t > 0.$

Let us denote by $L(\alpha, r_e, \sigma; r_0, r_1, \ldots, r_N)$ the likelihood function associated to the sample $\mathcal{S}$. On the one hand, by the Total Probability Law one gets

$$L(\alpha, r_e, \sigma; r_0, r_1, \ldots, r_N) = L(\alpha, r_e, \sigma; r_0) L(\alpha, r_e, \sigma; r_1 | r_0) \ldots L(\alpha, r_e, \sigma; r_2 | r_0, r_1)$$

$$\times L(\alpha, r_e, \sigma; r_N | r_0, r_1, \ldots, r_{N-1}).$$

On the other hand, the solution stochastic process of an Itô s.d.e. of the form (6) is Markovian (see [7, Th. 5.2.5.]), in particular, this holds true for the Vasicek's model (4). This permits writing the likelihood function as follows

$$L(\alpha, r_e, \sigma; r_0, r_1, \ldots, r_N) = L(\alpha, r_e, \sigma; r_0) L(\alpha, r_e, \sigma; r_1 | r_0) \ldots L(\alpha, r_e, \sigma; r_N | r_{N-1}). \quad (15)$$

In order to obtain the expression of likelihood function of r_i given r_{i-1}, for $i = 1, 2, \ldots, N$, it is important to observe that, from (13), one gets

$$r_i \mid r_{i-1} \sim N\left(r_e + (r_{i-1} - r_e)e^{-\alpha\Delta t};\ \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha\Delta t})\right),\ i = 1, 2, \ldots, N. \quad (16)$$

This expression is obtained by repeating the development previously exhibited to establish (13) on the interval $[0, t]$, but now on the interval $[t_{i-1}, t_i] = [t_{i-1}, t_{i-1} + \Delta t]$. Notice that the approximation $r_i \mid r_{i-1}$ can be constructed using, for example, the Euler-Maruyama scheme for the s.d.e. (4) (see [8])

$$r_i \approx r_{i-1} + \alpha(r_e - r_{i-1})\Delta t + \sigma(B(t_{i-1}) + \Delta t) - B(t_{i-1})),\ i = 1, 2, \ldots, N. \quad (17)$$

Hence, the probability density function of the r.v. r_i fixed r_{i-1} is given by

$$L(\alpha, r_e, \sigma; r_i \mid r_{i-1}) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t}}\, e^{-\frac{1}{2}\frac{(r_i - (r_e + (r_{i-1} - r_e)e^{-\alpha\Delta t}))^2}{\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha\Delta t})}},\ i = 1, 2, \ldots, N.$$

Substituting this expression in (15) and simplifying one gets the following representation for the log-likelihood function associated to sample $\mathcal{S}$.

$$\ln(L(\alpha, r_e, \sigma; r_0, \ldots, r_N)) = -\frac{N}{2}\left(\ln(2\pi) + \ln\left(\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha\Delta t})\right)\right)$$
$$-\frac{1}{2}\sum_{i=1}^{N}\left(\frac{r_i - (r_e + (r_{i-1} - r_e)e^{-\alpha\Delta t})}{\sqrt{\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha\Delta t})}}\right)^2. \quad (18)$$

Since r_0 is a deterministic value fixed by the initial condition, the term $L(\alpha, r_e, \sigma; r_0)$ can be assumed to be 1. This fact has been used in (18).

From expression (18), we can obtain the α, r_e, σ parameters that maximize the function $\ln(L(\alpha, r_e, \sigma; r_0, \ldots, r_N))$. This process has been carried out using the *stats* 4 and *sde* packages in R software [10]. Figure 1 shows the developed code.

We have considered a sample of 134 values corresponding to the 1-month Euribor interest rate from January 2, 2012 until July 11, 2013 [11]. They are plotted in Figure 2. The maximum likelihood estimates for each model parameter are collected in Table 1. In this manner, the model (4) is completely specified.

To end this section, we want to point out that apart from maximum likelihood method to estimate model parameters, alternative methods can be also used such as the interesting techniques presented in [12, 13].

```
# The following code takes care of maximum likelihood estimation of the Vasicek model.

# Loading the library: Simulation and Inference for Stochastic Differential Equations.
library(stats4)
library(sde)

# W defined as BM() function. The function BM returns a trajectory of the translated Brownian motion B(t); t >= 0,B(t0) = x; i.e., x + B(t t0) for t >= t0.
# The standard Brownian motion is obtained choosing x=0 and t0=0 (the default values BM(x=0, t0=0, T=1, N=100)).
W<-BM()

# t defines time [0,1] by 0.01.
t<-time(W)

# Length of the vector t.
N<-length(t)

# N is defined by numeric vector and called by X.
X<-numeric(N)

# The first component of X is x.
X[1]<-x

# Obtaining X.
ito.sum<-c(0,sapply(2:N,function(x){exp(-theta*(t[x]-t[x-1]))*(W[x]-W[x-1]) } ))
X<-sapply(1:N,function(x){X[1]*exp(-theta*t[x])+sum(ito.sum[1:x])})

# Considering X as time-series objets. The function ts is used to create time-series objects.
X<-ts(X,start=start(W),deltat=deltat(W))

# Definition of the conditional density of the process by the function dcOU.
dcOU<-function(x,t,x0,theta,log=FALSE){
        Ex<-theta[1]/theta[2]+(x0-theta[1]/theta[2])*exp(-theta[2]*t)
        Vx<-theta[3]^2*(1-exp(-2*theta[2]*t))/2*theta[2]
        dnorm(x,mean=Ex, sd=sqrt(Vx),log=log) }

#The function OU.lik needs as input the 3 parameters and assumes that sample observations of the process are available in the current R workspace in the time-series object.
OU.lik<-function(theta1,theta2,theta3){
        n<-length(X)
        dt<-deltat(X)
        -sum(dcOU(X[2:n],dt,X[1:(n-1)],c(theta1,theta2,theta3),log=TRUE)) }

# set.seed() declares the seed for the random generator. If we use this command before a random number generating statement, we are able to retain the same number each time we provide the same seed.
set.seed(123)

# Simulation of stochastic differential equation.
X<-sde.sim(model="OU",theta=c(3,1,2),N=1000,delta=1)

# Estimate parameters by the method of maximum likelihood considering as a initial values of the parameters theta1=1,theta2=0.5,theta3=1.
mle(OU.lik,start=list(theta1=1,theta2=0.5,theta3=1),method="L-BFGS-B",lower=c(-Inf,0,0))->fit

# Summary of the estimated parameters.
summary(fit)

#Returns the variance-covariance matrix of the main parameters of a fitted model object.
vcov(fit)

#Computes confidence intervals for one or more parameters in a fitted model.
confint(fit)
```

Figure 1. *R*-Code developed to estimate the Vasicek model parameters.

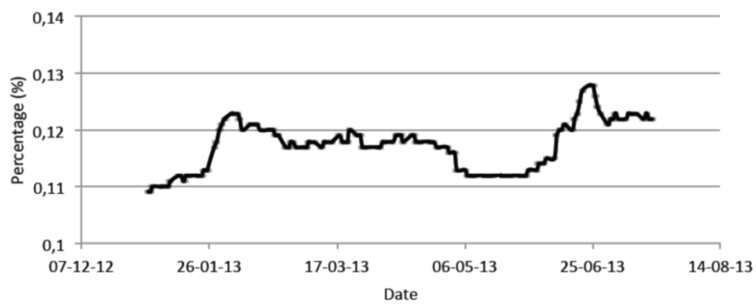

Figure 2. 1-month Euribor interest rates data (black line) from January 2, 2012 until July 11, 2013 used to put forward model (4).

Table 1. Estimated parameters in the model (4) considering 1-month Euribor interest rate data from January 2, 2012 until July 11, 2013.

Parameter	Maximum Likelihood Method
α	8.25510
r_e	0.12038
σ	0.01552

3.2. Validation of the model

Once the parameters have been estimated, the next step is to validate the model. For that, we will use the historical time series (Figure 2) and we will calculate, as measures of goodness-of-fit, MAPE (Mean Absolute Percentage Error) and RMSE (Root Mean Square Error). Moreover, we have built confidence intervals centred about the average taking advantage that the solution $r(t)$ has a gaussian distribution (see (13)).

The computation of MAPE and RMSE has been carried out in two different ways to provide robustness:

- Considering as predictions the exact values of the theoretical expectation given by (13).

- Considering as predictions the averaged value from predictions obtained by Monte Carlo sampling at every time instant generated by normal distribution according to (13). The number of simulations has been set in order to guarantee that the daily volatility, say $\delta(t)$, of the 134 sample data at each time instant $t \in \mathcal{T} = \{0, 1, 2, \ldots, 133\}$ lies inside the 95% confidence interval

$$CI[r(t)] = \left[\mu[r(t)] - 1.96 \frac{\sigma[r(t)]}{\sqrt{M(t)}},\ \mu[r(t)] + 1.96 \frac{\sigma[r(t)]}{\sqrt{M(t)}} \right], \tag{19}$$

for each $t \in \mathcal{T}$, we impose that

$$2 \times 1.96 \times \frac{\sigma[r(t)]}{\sqrt{M(t)}} \leq \delta(t),\ \forall t \in \mathcal{T}. \tag{20}$$

Hence, taking

$$M = \max_{t \in \mathcal{T}} \{M(t)\},\ \text{where}\ M(t) \geq \left(2 \times 1.96 \times \frac{\sigma[r(t)]}{\delta(t)} \right)^2. \tag{21}$$

In our case, we have obtained $M = 550$.

The values for the MAPE and RMSE using both approaches are given in Table 2. Notice that both methods provide similar accuracy being these values very acceptable. This permits to validate the model. To carry out computations time-step Δt has been taken as $1/252$, which represents the averaged number of days per year the 1-month Euribor interest rate is traded.

Table 2. Validation of the model (4) by RMSE and MAPE using both approaches, the exact solution (13) and Monte Carlo simulations.

Parameter	RMSE	MAPE
Vasicek model	0.00418	2.64%
Monte Carlo	0.00423	2.70%

Figure 3 shows the model validation. In there, we have plotted the theoretical 95% confidence intervals and the mean as well as the sample data. Very similar results are obtained when 95% confidence intervals based on 2:5 and 97:5 percentiles are built by Monte Carlo simulations.

3.3. Predictions

On account of the previous validation, we are legitimated to apply model (4) in order to obtain predictions of the 1-month Euribor rate. Tables 3 and 4 collect punctual predictions and 95% confidence intervals for the next five days using both the theoretical Vasicek's model and Monte Carlo sampling. These results are also compared against the real data. We observe that both approaches provide very good predictions.

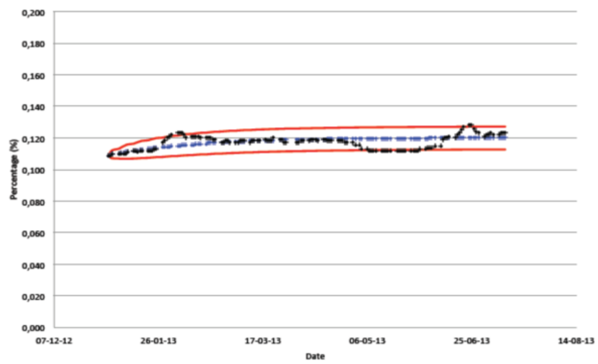

Figure 3. Vasicek model validation 1-month Euribor interest rate data from January 2, 2012 until July 11, 2013. The black line represents the available data of 1-month Euribor and the blue one represents the obtained theoretical mean. The 95% confidence intervals are represented by the red lines.

Table 3. Predictions for the 1-month Euribor rate using the theoretical solution of model (4).

Date	1-month Euribor	Punctual estimation	95% Confidence Interval
12 Jul 2013	0.123	0.1202378	[0.1277231, 0.1127525]
15 Jul 2013	0.122	0.1202423	[0.1277277, 0.1127570]
16 Jul 2013	0.123	0.1202467	[0.1277321, 0.1127613]
17 Jul 2013	0.122	0.1202510	[0.1277364, 0.1127656]
18 Jul 2013	0.122	0.1202551	[0.1277406, 0.1127697]

Table 4. Predictions for the 1-month Euribor rate using Monte Carlo simulations.

Date	1-month Euribor	Punctual estimation	95% Confidence Interval
12 Jul 2013	0.123	0.1199788	[0.1273538, 0.1124896]
15 Jul 2013	0.122	0.1199834	[0.1273584, 0.1124941]
16 Jul 2013	0.123	0.1199878	[0.1273629, 0.1124984]
17 Jul 2013	0.122	0.1199920	[0.1273672, 0.1125027]
18 Jul 2013	0.122	0.1199962	[0.1273713, 0.1127507]

4. Conclusions

In this paper we have presented a stochastic dynamic model, called Vasicek's model, to study the evolution of short term interest rates. The model is based on an Ito-type stochastic differential equation. The model assumes constant local volatility and mean reversion asymptotic behaviour. These hypotheses are reasonable in dealing with short term interest rate data. This model has been successfully applied to study 1-month Euribor interest rate. The obtained results provide very good predictions lying most of real data inside of the constructed 95% confidence intervals. Alternatively to Vasicek's model, it is important to point out that other models are available. Here, we highlight the so-called CIR's model by Cox-Ingersoll-Ross (see [4]). This model generalizes the Vasicek's model. It considers time-dependent local volatility, and it will be very interesting to compare the numerical predictions of the CIR's model against the ones provided by the Vasicek's model in a forthcoming study.

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