A local fixed point theorem for set-valued mappings on partial metric spaces

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ABSTRACT

The purpose of this paper is to study the existence and location of fixed points for pseudo-contractive-type set-valued mappings in the setting of partial metric spaces by using Bianchini-Grundolfi gauge functions.

1. INTRODUCTION

In [24], Matthews introduced the notion of a partial metric space, which is a generalization of usual metric spaces in which the self-distance for any point need not be equal to zero. The partial metric space has wide applications in many branches of mathematics as well as in the fields of computer domain and semantics. Later, many authors studied fixed point theorems for set-valued mapping on partial metric spaces (see, e.g., [1, 4, 5, 11, 22, 23, 32] and references cited therein). A basic result is the Nadler fixed point theorem [5, Theorem 3.2] for contractive set-valued mappings, using the partial Hausdorff metric, which reduces to the Banach contraction mapping theorem for single-valued mappings [24, Theorem 5.3].

In [10], Dontchev and Hager presented a fixed point theorem for set-valued mappings on complete metric space speaks about a location of a fixed point with respect to an initial value of the set-valued mapping. Let (X, d) be a

metric space and let A and B be two nonempty closed subsets of X. Recall that the generalized Hausdorff metric H between A and B is given by

$$H(A,B) = \max\{e(A,B), e(B,A)\}$$

where $e(A,B) = \sup\limits_{a \in A} d(a,B) = \sup\limits_{a \in A} \inf\limits_{b \in B} d(a,b)$ (the excess of A over B). This (extended) metric could take the value $+\infty$; see [7, 18]. We denote by $\overline{B}(x,r)$ the closed ball of radius r centered at x defined by

$$\overline{B}(x,r) = \{y \in X : d(x,y) \leqslant r\}.$$

Then the theorem of Dontchev and Hager [10] reads as follows:

Theorem 1.1. *Let (X,d) be a complete metric space, and consider a point $\overline{x} \in X$, nonnegative scalars $r > 0$ and λ be such that $0 \leqslant \lambda < 1$, and a set-valued mapping ϕ from the closed ball $\overline{B}(\overline{x},r)$ to the closed subset of X and the following conditions hold:*

> **(i):** $d(\overline{x}, \phi(\overline{x})) < r(1-\lambda)$,
> **(ii):** $e(\phi(x_1) \cap \overline{B}(\overline{x},r), \phi(x_2)) \leqslant \lambda d(x_1, x_2) \quad \forall x_1, x_2 \in \overline{B}(\overline{x},r)$,

then ϕ has a fixed point in $\overline{B}(\overline{x},r)$, that is, there exists $x \in \overline{B}(\overline{x},r)$ such that $x \in \phi(x)$. If ϕ is a single-valued mapping, then x is the unique fixed point of ϕ in $\overline{B}(\overline{x},r)$.

This theorem is the main tool to establish the convergence of several iterative methods for variational inclusion problem: find $x \in X$ such that

$$\text{(VI)} \qquad\qquad 0 \in f(x) + F(x)$$

where f be a single-valued mapping acting between two Banach spaces X and Y, F is a set-valued mapping from X into the subsets of Y. See e.g. [9, 14, 15, 16, 20, 29] for further informations on the applications of this theorem.

Recall that the variational inclusions (VI) are an abstract model of a wide variety including systems of nonlinear equations (when $F = \{0\}$), systems of inequalities (when F is the positive orthant in $\mathbb{R}^m$), linear and nonlinear complementary problems, variational inequalities (mixed quasi-variational inequality, Hartman-Stampacchia variational inequality), first-order necessary conditions for nonlinear programming, etc. In particular, they may characterize optimality or equilibrium (traffic network equilibrium, spatial price equilibrium problems, migration equilibrium problems, environmental network problems, etc.) and then have several applications in engineering and economics (analysis of elastoplastic structures, Walrasian equilibrium, Nash equilibrium, financial equilibrium problems, etc.) see e.g [12, 13, 17, 21, 30, 31].

In this paper, we extend Theorem 1.1 on partial metric spaces by using Bianchini-Grandolfi gauge functions and give some related corollaries.

2. Preliminaries

We start by recalling some basic definitions and properties of partial metric spaces.

Definition 2.1. Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}^+$ (where $\mathbb{R}^+$ denotes the set of all nonnegative real numbers) is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:

(P$_1$): $p(x,x) = p(y,y) = p(x,y) \Leftrightarrow x = y$; ($T_0$-separation axiom)
(P$_2$): $p(x,x) \leqslant p(x,y)$; (small self-distance axiom)
(P$_3$): $p(x,y) = p(y,x)$; (symmetry)
(P$_4$): $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$. (modified triangular inequality)

The pair (X, p) is then called a partial metric space. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Other examples of partial metric spaces may be found in [6, 24].

Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where

$$B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$$

for all $x \in X$ and $\epsilon > 0$. The closed p-ball of radius r centered at x is denoted by $\overline{B_p}(x, r)$ where

$$\overline{B_p}(x, r) = \{y \in X : p(x, y) \leqslant p(x, x) + r\}.$$

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric on X.

Let (X, p) be a partial metric space. Then:

- A sequence $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \to +\infty} p(x, x_n)$. This will be denoted by $x_n \to x$, as $n \to +\infty$.
- A sequence $\{x_n\}$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m \to +\infty} p(x_n, x_m)$.
- The partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to +\infty} p(x_n, x_m)$.

Lemma 2.2. *Let (X, p) be a partial metric space.*

(a): *$\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).*

(b): *A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete. Furthermore,*

$$(2.1) \quad \lim_{n \to +\infty} p^s(x_n, x) = 0 \Leftrightarrow p(x, x) = \lim_{n \to +\infty} p(x_n, x) = \lim_{n,m \to +\infty} p(x_n, x_m)$$

where x is a limit of $\{x_n\}$ in (X, p^s).

Lemma 2.3 ([2])**.** *Assume that $x_n \to x$ as $n \to +\infty$ in a partial metric space (X, p) such that $p(x, x) = 0$. Then $\lim_{n \to +\infty} p(x_n, y) = p(x, y)$ for every $y \in X$.*

Let (X, p) be a partial metric space and let $C^p(X)$ be the family of all nonempty and closed subsets of the partial metric space (X, p), induced by the partial metric p. For $x \in X$ and $A, B \in C^p(X)$, we define

$$p(x, A) = \inf\{p(x, a), a \in A\},$$

$$\delta_p(A, B) = \sup\{p(a, B), a \in A\},$$

and

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}.$$

We adopt the convention that $\delta_p(\varnothing, B) = 0$.

Lemma 2.4 ([4])**.** *Let (X, p) be a partial metric space and $A \subset X$. Then*
$$p(a, A) = p(a, a) \Leftrightarrow a \in \overline{A}$$

where $\overline{A}$ denotes the closure of A with respect to the partial metric p. Note that A is closed in (X, p) if and only if $\overline{A} = A$. It is easy to see that, every closed subset (with respect to τ_p) of a complete partial metric space is complete.

Recall some properties of mapping $\delta_p : C^p(X) \times C^p(X) \to [0, +\infty]$

Proposition 2.5. *Let (X, p) be a partial metric space. For any A, B, $C \in C^p(X)$, we have the following:*

(i): $\delta_p(A, A) = \sup\{p(a, a), a \in A\}$;
(ii): $\delta_p(A, A) \leqslant \delta_p(A, B)$;
(iii): $\delta_p(A, B) = 0 \Rightarrow A \subseteq B$;
(iv): $\delta_p(A, B) \leqslant \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$.

Remark 2.6. The properties mentioned above are satisfied without using the concept of boundedness for A, B and C. See the proof of [5, Proposition 2.2] for further details.

In the following, J denotes an interval on $\mathbb{R}^+$ containing 0, that is an interval of the form $[0, a[$, $[0, a]$ or $[0, +\infty[$.

Definition 2.7. Let $r \geqslant 1$. A function $\varphi : J \to J$ is said to be a gauge function of order r on J if it satisfies the following conditions:

(1) $\varphi(\lambda t) \leqslant \lambda^r \varphi(t)$ for all $\lambda \in]0, 1[$ and $t \in J$;
(2) $\varphi(t) < t$ for all $t \in J \setminus \{0\}$.

We consider some examples of gauge functions of order $r \geqslant 1$.

Example 2.8.   (1) $\varphi(t) = \lambda t$ $(0 < \lambda < 1)$ is a gauge function of the first order on $J = [0, 1[$;
(2) $\varphi(t) = c\, t^r$ $(c > 0, r > 1)$ is a gauge function of order r on $J = [0, R[$, where $R = (1/c)^{1/(r-1)}$;
(3) Every convex function φ on an interval J such that $\varphi(0) = 0$ and $\varphi(t) < t$ for all $t \in J \setminus \{0\}$ is a gauge function of the first order on J.

Definition 2.9 (Bianchini-Grandolfi Gauge Functions). A nondecreasing function $\varphi : J \to J$ is said to be a Bianchini-Grandolfi gauge function on J if

$$(2.2) \qquad s(t) := \sum_{n=0}^{\infty} \varphi^n(t) \text{ is convergent for all } t \in J$$

where φ^n denotes the n-th iteration of the function φ and $\varphi^0(t) = t$ i.e.

$$\varphi^0(t) = t, \ \varphi^1(t) = \varphi(t), \ \varphi^2(t) = \varphi(\varphi(t)), \ldots, \varphi^n(t) = \varphi(\varphi^{n-1}(t)).$$

These functions are known in the literature as (c)-comparison functions in some sources (see e.g. [8, 33]) and as rate of convergence in some other sources (see e.g. [27, 28]). The sum (2.2) is called the corresponding estimate function and noticed that φ satisfies the following functional equation

$$(2.3) \qquad s(t) = t + s(\varphi(t))$$

and (with the exception of pathological cases) we have :

$$(2.4) \qquad \varphi(t) = s^{-1}(s(t) - t).$$

Lemma 2.10 ([25]). *Every gauge function of order $r \geqslant 1$ on J is a Bianchini-Grandolfi gauge function on J.*

3. The main result

Before giving our main result, we need the following lemma.

Lemma 3.1. *Let (X, p) be a partial metric space. Let $x \in X$ and $A \in C^p(X)$. If $p(x, A) < \mu$ ($\mu > 0$) then there exists $a \in A$ such that $p(x, a) < \mu$.*

Proof. We argue by contradiction. Let $x \in X$ and $A \in C^p(X)$ such that $p(x, A) < \mu$. We suppose that $p(x, a) \geqslant \mu$ for all $a \in A$. Then, we have

$$p(x, A) = \inf\{p(x, a) : a \in A\} \geqslant \mu,$$

which is a contradiction. Hence, there exists $a \in A$ such that $p(x, a) < \mu$. $\square$

Now, we are ready to state and prove our main result.

Theorem 3.2. *Let (X, p) be a complete partial metric space, and consider a point $\overline{x} \in X$, nonnegative scalar $r > 0$ and a set-valued mapping $\phi : \overline{B}_p(\overline{x}, r) \to C^p(X)$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing and continuous function such that φ is a Bianchini-Grandolfi gauge function on interval J and $\lim_{t \downarrow 0} \varphi(t) = 0$. If there exists $\alpha \in J$ such that the following two conditions hold:*

(a): $p(\overline{x}, \phi(\overline{x})) < \alpha$ where $s(\alpha) \leqslant p(\overline{x}, \overline{x}) + r$
(b): $\delta_p(\phi(x) \cap \overline{B}_p(\overline{x}, r), \phi(y)) \leqslant \varphi(p(x, y)) \quad \forall x, y \in \overline{B}_p(\overline{x}, r)$,

then ϕ has a fixed point x^* in $\overline{B}_p(\overline{x}, r)$. If ϕ is a single-valued mapping and $p(\overline{x}, \overline{x}) + 2r \in J$, then x^* is the unique fixed point of ϕ in $\overline{B}_p(\overline{x}, r)$.

Proof. If $\varphi \equiv 0$ then, by Proposition 2.5 (iii) and assumption (b), we have for all $x_1, x_2 \in \overline{B}_p(\overline{x}, r)$

(3.1)
$$\begin{cases} \phi(x_1) \cap \overline{B}_p(\overline{x}, r) \subseteq \phi(x_2) \\ \phi(x_2) \cap \overline{B}_p(\overline{x}, r) \subseteq \phi(x_1) \end{cases} \Rightarrow \phi(x_1) \cap \overline{B}_p(\overline{x}, r) = \phi(x_2) \cap \overline{B}_p(\overline{x}, r) \neq \varnothing;$$

according to assumption (a), relation (3.1) and Lemma 3.1, there exists $x \in \overline{B}_p(\overline{x}, r)$ such that $x \in \phi(\overline{x}) \cap \overline{B}_p(\overline{x}, r) = \phi(x) \cap \overline{B}_p(\overline{x}, r)$ which completes the proof.

Assume now $\varphi \not\equiv 0$, by assumption (a) and Lemma 3.1, there exists

$$x_1 \in \phi(\overline{x}) \cap \overline{B}_p(\overline{x}, r)$$

such that

(3.2)
$$p(x_1, \overline{x}) < \alpha = \varphi^0(\alpha).$$

Denoting $x_0 = \overline{x}$ and, by hypothesis (b), we have

(3.3)
$$\begin{aligned} p(x_1, \phi(x_1)) &\leqslant \delta_p(\phi(x_0) \cap \overline{B}_p(x_0, r), \phi(x_1)) \\ &\leqslant \varphi(p(x_0, x_1)) \\ &< \varphi(\alpha) = \varphi^1(\alpha). \end{aligned}$$

This implies that there exists $x_2 \in \phi(x_1)$ such that

(3.4)
$$p(x_2, x_1) < \varphi^1(\alpha)$$

and, by the property (P_4) of partial metric, we have

(3.5)
$$\begin{aligned} p(x_2, x_0) &\leqslant p(x_2, x_1) + p(x_1, x_0) - p(x_1, x_1) \\ &< \varphi^1(\alpha) + \varphi^0(\alpha) \\ &< s(\alpha) \leqslant p(x_0, x_0) + r. \end{aligned}$$

Hence $x_2 \in \phi(x_1) \cap \overline{B}_p(x_0, r)$. Proceeding by induction and suppose we have constructed, for $k \in \mathbb{N}$ (where $\mathbb{N}$ denotes the set of nonnegative integers), an element x_{k+1} such that

$$x_{k+1} \in \phi(x_k) \cap \overline{B}_p(x_0, r)$$

and

(3.6)
$$p(x_{k+1}, x_k) < \varphi^k(\alpha).$$

By hypothesis (b), we have

(3.7)
$$\begin{aligned} p(x_{k+1}, \phi(x_{k+1})) &\leqslant \delta_p(\phi(x_k) \cap \overline{B}_p(x_0, r), \phi(x_{k+1})) \\ &\leqslant \varphi(p(x_{k+1}, x_k)) \\ &< \varphi(\varphi^k(\alpha)) = \varphi^{k+1}(\alpha). \end{aligned}$$

Thus, there exists $x_{k+2} \in \phi(x_{k+1})$ such that

(3.8)
$$p(x_{k+2}, x_{k+1}) < \varphi^{k+1}(\alpha).$$

Moreover, the use of property (P_4) of partial metric gives

$$p(x_{k+2}, x_0) \leqslant \sum_{j=0}^{k+1} p(x_{j+1}, x_j) - \sum_{j=1}^{k} p(x_j, x_j)$$

(3.9)
$$< \sum_{j=0}^{+\infty} \varphi^j(\alpha) = s(\alpha) \leqslant p(x_0, x_0) + r.$$

Hence $x_{k+2} \in \phi(x_{k+1}) \cap \overline{B}_p(x_0, r)$ and the induction step is complete. On the other hand, we have

$$\max\{p(x_{k+1}, x_{k+1}), p(x_k, x_k)\} \leqslant p(x_{k+1}, x_k)$$

which implies that

(3.10)
$$\max\{p(x_{k+1}, x_{k+1}), p(x_k, x_k)\} < \varphi^k(\alpha).$$

Consider now

$$p^s(x_{k+1}, x_k) = 2p(x_{k+1}, x_k) - p(x_{k+1}, x_{k+1}) - p(x_k, x_k)$$

(3.11)
$$\leqslant 2p(x_{k+1}, x_k)$$
$$< 2\varphi^k(\alpha).$$

From the conditions on φ, it is clear that $\lim\limits_{k \to +\infty} \varphi^k(t) = 0$ for $t \in J \setminus \{0\}$ and $\varphi(t) < t$ (see[8, 26, 33]). Hence, we have $\lim\limits_{n \to +\infty} p^s(x_{n+1}, x_n) = 0$. Moreover, for all integers n and m such that $n > m$, we have

$$p^s(x_n, x_m) \leqslant \sum_{k=m}^{n-1} p^s(x_{k+1}, x_k)$$

(3.12)
$$\leqslant 2 \sum_{k=m}^{n-1} \varphi^k(\alpha)$$
$$\leqslant 2s(\alpha).$$

Since $s(t)$ is convergent for each $t \in J$, we obtain that $\{x_n\}$ is a Cauchy sequence in (X, p^s). Since (X, p) is complete, by Lemma 2.2, (X, p^s) is complete and the sequence $\{x_n\}$ is convergent in (X, p^s) to $x \in X$. Again by Lemma 2.2, we have

(3.13)
$$p(x, x) = \lim_{n \to +\infty} p(x_n, x) = \lim_{n,m \to +\infty} p(x_n, x_m).$$

Moreover, since $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s), we have $\lim\limits_{n,m \to +\infty} p^s(x_n, x_m) = 0$, and, from (3.10), we have $\lim\limits_{n \to +\infty} p(x_n, x_n) = 0$, thus, from definition of p^s, we have $\lim\limits_{n,m \to +\infty} p(x_n, x_m) = 0$. Therefore, from (3.13), we have

(3.14)
$$p(x, x) = \lim_{n \to +\infty} p(x_n, x) = \lim_{n,m \to +\infty} p(x_n, x_m) = 0.$$

Furthermore, since $\{x_n\}$ is a sequence in the closed p-ball $\overline{B}_p(x_0, r)$ which is complete and according to Lemma 2.3, using (3.14), we have for $y = x_0 \in X$

$$(3.15) \qquad p(x, x_0) = \lim_{n \to +\infty} p(x_n, x_0) \leqslant p(x_0, x_0) + r.$$

i.e. $x \in \overline{B}_p(x_0, r)$.

We assert now that $x \in \phi(x)$. The modified triangle inequality and assumption (b) give

$$
\begin{aligned}
(3.16) \qquad p(x, \phi(x)) &\leqslant p(x, x_n) + p(x_n, \phi(x)) - p(x_n, x_n) \\
&\leqslant p(x, x_n) + \delta_p(\phi(x_{n-1}) \cap \overline{B}_p(x_0, r), \phi(x)) \\
&\leqslant p(x, x_n) + \varphi(p(x_{n-1}, x))
\end{aligned}
$$

Taking limit as $n \to +\infty$ and using (3.14) and the continuity of φ, we obtain $p(x, \phi(x)) = 0$. Therefore, from (3.14) $(p(x, x) = 0)$, we obtain $p(x, \phi(x)) = p(x, x)$ which from Lemma 2.4 implies that

$$x \in \overline{\phi(x)} = \phi(x).$$

If ϕ is a single-valued mapping and $p(\overline{x}, \overline{x}) + 2r \in J$, we suppose that there exist two fixed points $x^*, x^{**} \in \overline{B}_p(x_0, r)$. Then, we have

$$
\begin{aligned}
p(x^*, x^{**}) &\leqslant p(x^*, x_0) + p(x_0, x^{**}) - p(x_0, x_0) \\
&\leqslant p(x_0, x_0) + 2r \in J
\end{aligned}
$$

and

$$
\begin{aligned}
(3.17) \qquad p(x^*, x^{**}) = p(x^*, \phi(x^{**})) &\leqslant \delta_p(\phi(x^*) \cap \overline{B}_p(x_0, r), \phi(x^{**})) \\
&\leqslant \varphi(p(x^*, x^{**})) \\
&< p(x^*, x^{**})
\end{aligned}
$$

which is a contradiction and the proof is completed. $\qquad\qquad\square$

The following example shows the usage of Theorem 3.2.

Example 3.3. Let $X = \mathbb{R}^+ = [0, +\infty[$ be endowed with the partial metric

$$
p(x, y) = \begin{cases} 0, & x = y \in \left[0, \dfrac{361}{900}\right]; \\ \max\{x, y\}, & \text{otherwise.} \end{cases}
$$

Note that $p(x, y)$ is a metric on $\left[0, \dfrac{361}{900}\right]$ and

$$
p^s(x, y) = \begin{cases} 2p(x, y), & x, y \in \left[0, \dfrac{361}{900}\right]; \\ |x - y|, & \text{otherwise.} \end{cases}
$$

First, observe that, for $a \geqslant \dfrac{361}{900}$, $[a, +\infty[$ is closed with respect to the partial metric p.

Let $a \geqslant \dfrac{361}{900}$, then we have

$$
\begin{aligned}
y \in \overline{[a, +\infty[} \quad &\Leftrightarrow \quad p(y, [a, +\infty[) = p(y, y) \\
&\Leftrightarrow \quad \inf_{z \in [a, +\infty[} p(y, z) = p(y, y) \\
&\Leftrightarrow \quad y \geqslant z \geqslant a \\
&\Leftrightarrow \quad y \in [a, +\infty[
\end{aligned}
$$

Hence, $[a, +\infty[$ is closed for any $a \geqslant \dfrac{361}{900}$.

Take

$$
\varphi(t) = \begin{cases} \dfrac{2}{3} t, & 0 \leqslant t \leqslant \dfrac{1}{2}; \\[2mm] 5t - \dfrac{13}{6}, & t \geqslant \dfrac{1}{2}. \end{cases}
$$

which is a Bianchini-Grundolfi gauge function on $J = \left[0, \dfrac{1}{2}\right]$ such that $s(t) = 3t$ and $\lim\limits_{t \downarrow 0} \varphi(t) = 0$.

We set $\phi : [0, 1] \to C^p(X)$ defined by

$$
\phi(x) = \begin{cases} \{x^2\}, & x \in \left[0, \dfrac{19}{30}\right]; \\[2mm] [1, +\infty[, & x \in \left]\dfrac{19}{30}, 1\right]. \end{cases}
$$

We apply Theorem 3.2 with the following specifications

$$
\overline{x} = \frac{1}{4}, \quad r = 1, \quad \alpha = \frac{1}{3} \in J, \quad \overline{B}_p(\overline{x}, r) = [0, 1].
$$

First, observe that,

$$
p\left(\frac{1}{4}, \phi\left(\frac{1}{4}\right)\right) = p\left(\frac{1}{4}, \left\{\frac{1}{16}\right\}\right) = \frac{1}{4} < \frac{1}{3} = \alpha
$$

and $s(\alpha) = 3 * \dfrac{1}{3} = 1 \leqslant p\left(\dfrac{1}{4}, \dfrac{1}{4}\right) + 1$; that is, condition (a) of Theorem 3.2 holds.

To see that condition (b) of Theorem 3.2 holds it is sufficient to consider the following cases:

(1) If $x = y \in \left[0, \dfrac{19}{30}\right]$ then

$$
\delta_p(\phi(x) \cap [0, 1], \phi(y)) = \delta_p(\{x^2\}, \{x^2\}) = 0 \leqslant \begin{cases} \dfrac{2}{3} p(x, x) = \varphi\left(p(x, y)\right), & x \in \left[0, \dfrac{1}{2}\right]; \\[2mm] 5x - \dfrac{13}{6} = \varphi\left(p(x, y)\right), & \text{otherwise.} \end{cases}
$$

(2) If $x, y \in \left[0, \dfrac{19}{30}\right]$ and $x \neq y$ then

$$\delta_p(\phi(x) \cap [0,1], \phi(y)) = \delta_p(\{x^2\}, \{y^2\}) \leqslant \begin{cases} \dfrac{1}{2} \max\{x, y\}, & x, y \in \left[0, \dfrac{1}{2}\right]; \\ 5 \max\{x, y\} - \dfrac{9}{4}, & \text{otherwise.} \end{cases}$$

$$\leqslant \begin{cases} \dfrac{2}{3} \max\{x, y\} = \varphi\left(p(x, y)\right), & x, y \in \left[0, \dfrac{1}{2}\right]; \\ 5 \max\{x, y\} - \dfrac{13}{6} = \varphi\left(p(x, y)\right), & \text{otherwise.} \end{cases}$$

(3) If $x, y \in \left]\dfrac{19}{30}, 1\right]$ then

$$\delta_p(\phi(x) \cap [0,1], \phi(y)) = \delta_p(\{1\}, [1, +\infty[) = 1 \leqslant 5 \max\{x, y\} - \dfrac{13}{6} = \varphi\left(p(x, y)\right)$$

(4) If $x \in \left[0, \dfrac{19}{30}\right]$ and $y \in \left]\dfrac{19}{30}, 1\right]$ then

$$\delta_p(\phi(x) \cap [0,1], \phi(y)) = \delta_p(\{x^2\}, [1, +\infty[) = 1 \leqslant 5y - \dfrac{13}{6} = \varphi\left(p(x, y)\right)$$

and

$$\delta_p(\phi(y) \cap [0,1], \phi(x)) = \delta_p(\{1\}, \{x^2\}) = 1 \leqslant 5y - \dfrac{13}{6} = \varphi\left(p(x, y)\right)$$

Hence, all conditions of Theorem 3.2 are satisfied and $x^* \in \{0, 1\} \subset \overline{B}_p(\overline{x}, r)$ are the required points.

On the other hand, it is easy to show that Theorem 1.1 is not applicable in this case. Indeed, for $x \in \left[0, \dfrac{19}{30}\right]$ and $y = 1$, we have

$$e_{p^s}(\phi(x) \cap [0,1], \phi(y)) = e_{p^s}(\{x^2\}, [1, +\infty[) = |1 - x^2| \geqslant |1 - x| = p^s(x, y)$$

Hence, no constant λ, $0 \leqslant \lambda < 1$ can be chosen in a way that $e_{p^s}(\phi(x) \cap [0,1], \phi(y)) \leqslant \lambda p^s(x, y)$ for all $x, y \in [0, 1]$.

We can obtain the following corollaries from Theorem 3.2.

Corollary 3.4. *Let (X, p) be a complete partial metric space, and consider a point $\overline{x} \in X$, nonnegative scalars $r > 0$ and $0 \leqslant \lambda < 1$ and a set-valued mapping $\phi : \overline{B}_p(\overline{x}, r) \to C^p(X)$. Let the following two conditions hold:*

    (a): $p(\overline{x}, \phi(\overline{x})) < (p(\overline{x}, \overline{x}) + r)(1 - \lambda)$,
    (b): $\delta_p(\phi(x) \cap \overline{B}_p(\overline{x}, r), \phi(y)) \leqslant \lambda p(x, y) \quad \forall x, y \in \overline{B}_p(\overline{x}, r)$,

then ϕ has a fixed point x^* in $\overline{B}_p(\overline{x}, r)$. If ϕ is a single-valued mapping, then x^* is the unique fixed point of ϕ in $\overline{B}_p(\overline{x}, r)$.

Proof. We apply Theorem 3.2 for $\varphi(t) = \lambda t$ which is a Bianchini-Grandolfi gauge function on $J = [0, p(\overline{x}, \overline{x}) + 2r]$ and the corresponding estimate function $s(t) = \dfrac{t}{1 - \lambda}$. Take $\alpha = (p(\overline{x}, \overline{x}) + r)(1 - \lambda) \in J$. $\qquad\square$

Remark 3.5. Corollary 3.4 extends Theorem 1.1 on partial metric spaces.

The next example demonstrates the usage of Theorem 3.2 and Corollary 3.4.

Example 3.6 ([5, Example 3.3])**.** Let $X = \{0, 1, 4\}$ be endowed with the partial metric $p(x, y) = \dfrac{1}{4}|x - y| + \dfrac{1}{2}\max\{x, y\}$ for all $x, y \in X$. Define the mapping $\phi : X \to C^p(X)$ by $\phi(0) = \phi(1) = \{0\}$ and $\phi(4) = \{0, 1\}$.

We apply Corollary 3.4 for $\overline{x} = 1$, $r = 2$ and $\lambda = \dfrac{1}{2}$. First, we have

$$p(1, \phi(1)) = p(1, \{0\}) = \frac{3}{4} < (p(1, 1) + 2)(1 - \frac{1}{2}).$$

On the other hand, we get

$$\delta_p(\phi(x) \cap \overline{B}_p(1, 2), \phi(y)) \leqslant \frac{1}{2}p(x, y) \quad \forall x, y \in \overline{B}_p(1, 2) = \{0, 1\}.$$

Thus, all the hypotheses are satisfied and the fixed point of ϕ is $x = 0 \in \overline{B}_p(1, 2)$.

Corollary 3.7. *Let (X, p) be a complete partial metric space, and consider a point $\overline{x} \in X$, nonnegative scalar $r > 0$ and a set-valued mapping $\phi : \overline{B}_p(\overline{x}, r) \to C^p(X)$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing and continuous function such that φ is a Bianchini-Grandolfi gauge function on interval J and $\lim\limits_{t \downarrow 0} \varphi(t) = 0$. If there exists $\alpha \in J$ such that the following two conditions hold:*

(a): $p(\overline{x}, \phi(\overline{x})) < \alpha$ *where* $s(\alpha) \leqslant p(\overline{x}, \overline{x}) + r$
(b): $H_p(\phi(x), \phi(y)) \leqslant \varphi(p(x, y)) \quad \forall x, y \in \overline{B}_p(\overline{x}, r),$

then ϕ has a fixed point x^* in $\overline{B}_p(\overline{x}, r)$. If ϕ is a single-valued mapping and $p(\overline{x}, \overline{x}) + 2r \in J$, then x^* is the unique fixed point of ϕ in $\overline{B}_p(\overline{x}, r)$.

Proof. We must assert that the second condition of Theorem 3.2 is satisfied. For any $x_1, x_2 \in \overline{B}_p(\overline{x}, r)$ we obtain

$$\begin{aligned}
\delta_p(\phi(x_1) \cap \overline{B}_p(\overline{x}, r), \phi(x_2)) &\leqslant \delta_p(\phi(x_1), \phi(x_2)) \\
&\leqslant H_p(\phi(x_1), \phi(x_2)) \\
&\leqslant \varphi(p(x_1, x_2)).
\end{aligned}$$

We complete the proof by applying Theorem 3.2. $\qquad\square$

Corollary 3.8. *Let (X, p) be a complete partial metric space, and consider a point $\overline{x} \in X$, nonnegative scalars $r > 0$ and λ be such that $0 \leqslant \lambda < 1$, and a set-valued mapping $\phi : \overline{B}_p(\overline{x}, r) \to C^p(X)$. Let the following two conditions hold:*

(a): $p(\overline{x}, \phi(\overline{x})) < (p(\overline{x}, \overline{x}) + r)(1 - \lambda),$
(b): $H_p(\phi(x_1), \phi(x_2)) \leqslant \lambda p(x_1, x_2) \quad \forall x_1, x_2 \in \overline{B}_p(\overline{x}, r),$

then ϕ has a fixed point x^* in $\overline{B}_p(\overline{x}, r)$. If ϕ is a single-valued mapping, then x^* is the unique fixed point of ϕ in $\overline{B}_p(\overline{x}, r)$.

Remark 3.9. Corollary 3.8 extends Theorem 9.1 in [3] and Lemma 1 in [19] on the partial metric space.

The Nadler's fixed point theorem ([5, Theorem 3.2]) on partial metric spaces follows readily from Corrollary 3.8. Observe that no boundedness assumption on the values is required.

Corollary 3.10. *Let* (X, p) *be a complete partial metric space. If* $\phi : X \to C^p(X)$ *is a set-valued mapping such that*

$$H_p(\phi(x), \phi(y)) \leqslant \lambda p(x, y)$$

for all $x, y \in X$ *where* $0 \leqslant \lambda < 1$. *Then* ϕ *has a fixed point.*

Proof. Let $\overline{x} \in X$. Choose $r > 0$ with $p(\overline{x}, \phi(\overline{x})) < (p(\overline{x}, \overline{x}) + r)(1 - \lambda)$. The result now follows from Corollary 3.8. □

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