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Convergence in graded ditopological texture spaces

Dedicated to the memory of Lawrence Michael Brown

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Abstract

Graded ditopological texture spaces have been presented and discussed in categorical aspects by Lawrence M. Brown and Alexander Šostak in [6]. In this paper, the authors generalize the structure of difilters in ditopological texture spaces defined in [11] to the graded ditopological texture spaces and compare the properties of difilters and graded difilters.

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1. Introduction

The concept of fuzzy topological space was defined in 1968 by C.Chang as an ordinary subset of the family of all fuzzy subsets of a given set[7]. As a more suitable approach to the idea of fuzzyness, in 1985, Šostak and Kubiak independently redefined fuzzy topology where a fuzzy subset has a degree of openness rather than being open or not [12, 10] (for historical developments and basic ideas of the theory of fuzzy topology see [13]).

In classical topology the notion of open set is usually taken as primitive with that of closed set being auxiliary. However, since closed sets are easily obtained

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as the complements of open sets they often play an important, sometimes dominating role in topological arguments. A similar situation holds for topologies on lattices where the role of set complement is played by an order reversing involution. It is the case, however, that there may be no order reversing involution available, or that the presence of such an involution is irrelevant to the topic under consideration. To deal with such cases it is natural to consider a topological structure consisting of *a priori* unrelated families of open sets and of closed sets. This was the approach adapted from the beginning for the topological structures on textures, originally introduced as a point-based representation for fuzzy sets [1, 2]. Such topological structures were given the name of a dichotomous topology, or ditopology for short. They consist of a family τ of open sets and a generally unrelated family κ of closed sets. Hence, both the open and the closed sets are regarded as primitive concepts for a ditopology.

A ditopology (τ, κ) on the discrete texture $(X, \mathcal{P}(X))$ gives rise to a bitopological space (X, τ, κ^c). This link with bitopological spaces has had a powerful influence on the development of the theory of ditopological texture spaces, but it should be emphasized that a ditopology and a bitopology are conceptually different. Indeed, a bitopology consists of two separate topological structures (complete with their open and closed sets) whose interrelations we wish to study, whereas a ditopology represents a single topological structure.

Ditopological texture spaces were introduced by L. M. Brown as a natural extention of the work of the second author on the representation of lattice-valued topologies by bitopologies in [9]. The concept of ditopology is more general than general topology, bitopology and fuzzy topology in Chang's sense. An adequate introduction to the theory of texture spaces and ditopological texture spaces may be obtained from [1, 2, 3, 4, 5].

Recently, L. M. Brown and A. Šostak have presented the concept "graded ditopology" on textures as an extention of the concept of ditopology to the case where openness and closedness are given in terms of a priori unrelated grading functions [6]. The concept of graded ditopology is more general than ditopology and smooth topology. Two sorts of neighbohood structure on graded ditopological texture spaces are presented and investigated by the authors in [8].

The aim of this work is to generalize the structure of difilters in ditopological texture spaces defined by S. Özçağ, F. Yıldız and L. M. Brown in [11] to the graded ditopological texture spaces which is introduced by L. M. Brown and A. Šostak in [6]. Furthermore we compare the properties of difilters and graded difilters. The material in this work forms a part of the first named author's Ph.D. Thesis, currently being written under the supervision of the second name author Dr. Rıza Ertürk.

2. Preliminaries

We recall various concepts and properties from [3, 4, 5] under the following subtitle.

Ditopological Texture Spaces: Let S be a set. A texturing $\mathcal{S}$ on S is a subset of $\mathcal{P}(S)$ which is a point separating, complete, completely distributive lattice with respect to inclusion which contains S, $\varnothing$ and for which meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\bigvee$ with unions $\bigcup$. The pair $(S,\mathcal{S})$ is then called a texture or a texture space.

In general, a texturing of S need not be closed under set complementation, but there may exist a mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$ and $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$ for all $A, B \in \mathcal{S}$. In this case σ is called a complementation on $(S,\mathcal{S})$ and $(S,\mathcal{S},\sigma)$ is said to be a complemented texture.

For a texture $(S,\mathcal{S})$, most properties are conveniently defined in terms of the $p-sets$

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$$

and the $q-sets$

$$Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\} = \bigvee\{P_u \mid u \in S, s \notin P_u\}.$$

A texture $(S,\mathcal{S})$ is called a plain texture if it satisfies any of the following equivalent conditions:

(1) $P_s \not\subseteq Q_s$ for all $s \in S$
(2) $A = \bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i$ for all $A_i \in \mathcal{S}, i \in I$

Recall that $M \in \mathcal{S}$ is called a molecule in $\mathcal{S}$ if $M \neq \varnothing$ and $M \subseteq A\cup B$, $A, B \in \mathcal{S}$ implies $M \subseteq A$ or $M \subseteq B$. The sets $P_s, s \in S$ are molecules, and the texture $(S,\mathcal{S})$ is called "simple" if all molecules of $\mathcal{S}$ are in the form $\{P_s \mid s \in S\}$. For a set $A \in \mathcal{S}$, the core of A (denoted by $A^\flat$) is defined by

$$A^\flat = \bigcap\Big\{\bigcup\{A_i \mid i \in I\} \;|\{A_i \mid i \in I\} \subseteq \mathcal{S},\ A = \bigvee\{A_i \mid i \in I\}\Big\}.$$

Theorem 2.1 ([3])**.** *In any texture space $(S,\mathcal{S})$, the following statements hold:*

(1) *$s \notin A \Rightarrow A \subseteq Q_s \Rightarrow s \notin A^\flat$ for all $s \in S$, $A \in \mathcal{S}$.*
(2) *$A^\flat = \{s \mid A \not\subseteq Q_s\}$ for all $A \in \mathcal{S}$.*
(3) *For $A_j \in \mathcal{S}$, $j \in J$ we have $(\bigvee_{j\in J} A_j)^\flat = \bigcup_{j\in J} A_j^\flat$.*
(4) *A is the smallest element of $\mathcal{S}$ containing $A^\flat$ for all $A \in \mathcal{S}$.*
(5) *For $A, B \in \mathcal{S}$, if $A \not\subseteq B$ then there exists $s \in S$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq B$.*
(6) *$A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$ for all $A \in \mathcal{S}$.*
(7) *$A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$ for all $A \in \mathcal{S}$.*

Example 2.2. (1) If $\mathcal{P}(X)$ is the powerset of a set X, then $(X,\mathcal{P}(X))$ is the discrete texture on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$. The mapping $\pi_X : \mathcal{P}(X) \to \mathcal{P}(X)$, $\pi_X(Y) = X \setminus Y$ for $Y \subseteq X$ is a complementation on the texture $(X,\mathcal{P}(X))$.

(2) Setting $\mathbb{I} = [0,1]$, $\mathcal{J} = \{[0,r),[0,r] \mid r \in \mathbb{I}\}$ gives the unit interval texture

$(\mathbb{I}, \mathcal{J})$. For $r \in \mathbb{I}$, $P_r = [0, r]$ and $Q_r = [0, r)$. And the mapping $\iota : \mathcal{J} \to \mathcal{J}$, $\iota[0, r] = [0, 1 - r)$, $\iota[0, r) = [0, 1 - r]$ is a complementation on this texture.
(3) The texture $(L, \mathcal{L}, \lambda)$ is defined by $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$, $\lambda((0, r]) = (0, 1 - r]$. For $r \in L$, $P_r = (0, r] = Q_r$.
(4) $\mathcal{S} = \{\varnothing, \{a, b\}, \{b\}, \{b, c\}, S\}$ is a simple texturing of $S = \{a, b, c\}$. $P_a = \{a, b\}$, $P_b = \{b\}$, $P_c = \{b, c\}$. It is not possible to define a complementation on $(S, \mathcal{S})$.
(5) If $(S, \mathcal{S}), (V, \mathcal{V})$ are textures, the product texturing $\mathcal{S} \otimes \mathcal{V}$ of $S \times V$ consists of arbitrary intersections of sets of the form $(A \times V) \cup (S \times B)$, $A \in \mathcal{S}, B \in \mathcal{V}$, and $(S \times V, \mathcal{S} \otimes \mathcal{V})$ is called the product of $(S, \mathcal{S})$ and $(V, \mathcal{V})$. For $s \in S, v \in V$, $P_{(s,v)} = P_s \times P_v$ and $Q_{(s,v)} = (Q_s \times V) \cup (S \times Q_v)$.

A dichotomous topology, or ditopology for short, on a texture $(S, \mathcal{S})$ is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of open sets τ satisfies

(T_1) $S, \varnothing \in \tau$
(T_2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$
(T_3) $G_i \in \tau, i \in I \Rightarrow \bigvee_i G_i \in \tau$

and the set of closed sets κ satisfies

(CT_1) $S, \varnothing \in \kappa$
(CT_2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$
(CT_3) $K_i \in \kappa, i \in I \Rightarrow \bigcap_i K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no priori relation between the open and closed sets.

Let (τ, κ) be a ditopology on $(S, \mathcal{S})$.

(1) If $s \in S^\flat$, a neighborhood of s is a set $N \in \mathcal{S}$ for which there exists $G \in \tau$ satisfying $P_s \subseteq G \subseteq N \not\subseteq Q_s$.
(2) If $s \in S$, a coneighborhood of s is a set $M \in \mathcal{S}$ for which there exists $K \in \kappa$ satisfying $P_s \not\subseteq M \subseteq K \subseteq Q_s$.

If the set of nhds (conhds) of s is denoted by $\eta(s)$ ($\mu(s)$) respectively, then (η, μ) is called the dinhd system of (τ, κ).

Theorem 2.3 ([5]). *For a ditopology (τ, κ) on $(S, \mathcal{S})$ let the families $\eta(s)$, $s \in S^\flat$ and $\mu(s)$, $s \in S$ be defined as above.*
(1) For each $s \in S^\flat$ we have $\eta(s) \neq \varnothing$ and these families satisfy the following conditions:

(i) $N \in \eta(s) \Rightarrow N \not\subseteq Q_s$
(ii) $N \in \eta(s), N \subseteq N' \in \mathcal{S} \Rightarrow N' \in \eta(s)$
(iii) $N_1, N_2 \in \eta(s), N_1 \cap N_2 \not\subseteq Q_s \Rightarrow N_1 \cap N_2 \in \eta(s)$
(iv) (a) $N \in \eta(s) \Rightarrow \exists N^\star \in \mathcal{S}, P_s \subseteq N^\star \subseteq N$, *so that* $N^\star \not\subseteq Q_t \Rightarrow N^\star \in \eta(t), \forall t \in S^\flat$
(b) *For $N \in \mathcal{S}$ and $N \not\subseteq Q_s$, if there exists $N^\star \in \mathcal{S}$, $P_s \subseteq N^\star \subseteq N$, which satisfies $N^\star \not\subseteq Q_t \Rightarrow N^\star \in \eta(t), \forall t \in S^\flat$, then $N \in \eta(s)$.*

Moreover, the sets G in τ are characterized by the condition that $G \in \eta(s)$ for all s with $G \not\subseteq Q_s$.

(2) For each $s \in S$ we have $\mu(s) \neq \varnothing$ and these families satisfy the following conditions:

(i) $M \in \mu(s) \Rightarrow P_s \not\subseteq M$

(ii) $M \in \mu(s), M \supseteq M' \in \mathcal{S} \Rightarrow M' \in \mu(s)$

(iii) $M_1, M_2 \in \mu(s) \Rightarrow M_1 \cup M_2 \in \mu(s)$

(iv) (a) $M \in \mu(s) \Rightarrow \exists M^\star \in \mathcal{S}, M \subseteq M^\star \subseteq Q_s$, *so that* $P_t \not\subseteq M^\star \Rightarrow M^\star \in \mu(t), \forall t \in S$

(b) *For* $M \in \mathcal{S}$ *and* $P_s \not\subseteq M$, *if there exists* $M^\star \in \mathcal{S}$, $M \subseteq M^\star \subseteq Q_s$, *which satisfies* $P_t \not\subseteq M^\star \Rightarrow M^\star \in \mu(t), \forall t \in S$, *then* $M \in \mu(s)$.

Moreover, the sets K in κ are characterized by the condition that $K \in \mu(s)$ for all s with $P_s \not\subseteq K$.

Conversely, if $\eta(s), s \in S^\flat$ and $\mu(s), s \in S$ are non-empty families of sets in $\mathcal{S}$ which satisfy conditions (1) and (2) above, respectively, then there exists a ditopology (τ, κ) on $(S, \mathcal{S})$ for which $\eta(s)$ $(\mu(s))$ are the families of nhds (resp., conhds) of the ditopology (τ, κ).

Difilters on Textures: [11] Let $(S, \mathcal{S})$ be a texture.

(1) $\mathcal{F} \subseteq \mathcal{S}$ is called a $\mathcal{S}$-filter or a filter on $(S, \mathcal{S})$, if $\mathcal{F} \neq \varnothing$ and satisfies:
(F_1) $\varnothing \notin \mathcal{F}$
(F_2) $F \in \mathcal{F},\ F \subseteq F' \in \mathcal{S} \Rightarrow F' \in \mathcal{F}$, and
(F_3) $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$.

(2) $\mathcal{G} \subseteq \mathcal{S}$ is called a $\mathcal{S}$-cofilter or a cofilter on $(S, \mathcal{S})$, if $\mathcal{G} \neq \varnothing$ and satisfies:
(CF_1) $S \notin \mathcal{G}$
(CF_2) $G \in \mathcal{G},\ G \supseteq G' \in \mathcal{S} \Rightarrow G' \in \mathcal{G}$, and
(CF_3) $G_1, G_2 \in \mathcal{G} \Rightarrow G_1 \cup G_2 \in \mathcal{G}$.

(3) If $\mathcal{F}$ is a $\mathcal{S}$-filter and $\mathcal{G}$ is a $\mathcal{S}$-cofilter then $\mathcal{F} \times \mathcal{G}$ is called a $\mathcal{S}$-difilter or a difilter on $(S, \mathcal{S})$.

A difilter $\mathcal{F} \times \mathcal{G}$ on $(S, \mathcal{S})$ is called regular if it satisfies following equivalent conditions:

(1) $\mathcal{F} \cap \mathcal{G} = \varnothing$.

(2) $(F_i, G_i) \in \mathcal{F} \times \mathcal{G},\ i = 1, 2, ..., n \Rightarrow \bigcap_{i=1}^n F_i \not\subseteq \bigcup_{i=1}^n G_i$.

(3) $A \not\subseteq B$ for all $A \in \mathcal{F}$ and $B \in \mathcal{G}$.

Example 2.4. (1) For a plain texture $(S, \mathcal{S})$ and ditopology (τ, κ) on $(S, \mathcal{S})$, then $\eta(s) \times \mu(s)$ is a regular $\mathcal{S}$-difilter for all $s \in S^\flat = S$ on $(S, \mathcal{S})$.

(2) Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space. Then the families

$$\eta^*(s) = \{A \in \mathcal{S} \mid \exists G_k \in \tau\ :\ G_k \not\subseteq Q_s,\ 1 \leq k \leq n \textit{ and } G_1 \cap ... \cap G_n \subseteq A\},\ s \in S^\flat$$

$$\mu^*(s) = \{A \in \mathcal{S} \mid \exists F_k \in \kappa\ :\ P_s \not\subseteq F_k,\ 1 \leq k \leq n \textit{ and } A \subseteq F_1 \cup ... \cup F_n\},\ s \in S$$

form a regular difilter $\eta^*(s) \times \mu^*(s)$ on $(S, \mathcal{S})$.

Definition 2.5 ([11])**.** Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space, $\mathcal{F}$ a $\mathcal{S}$-filter and $\mathcal{G}$ a $\mathcal{S}$-cofilter. (1) We say $\mathcal{F}$ converges to a point $s \in S^\flat$, and write $\mathcal{F} \to s$ if $\eta^*(s) \subseteq \mathcal{F}$; $\mathcal{G}$ converges to a point $s \in S$, and write $\mathcal{G} \to s$ if

$\mu^*(s) \subseteq \mathcal{G}$. The difilter $\mathcal{F} \times \mathcal{G}$ is said to be diconvergent if $\mathcal{F} \to s$ and $\mathcal{G} \to s'$ for some $s, s' \in S$ satisfying $P_{s'} \not\subseteq Q_s$.

(2) $\mathcal{F}$ is called prime if $A_1, A_2 \in \mathcal{S}, A_1 \cup A_2 \in \mathcal{F} \Rightarrow A_1 \in \mathcal{F}$ or $A_2 \in \mathcal{F}$. $\mathcal{G}$ is called prime if $B_1, B_2 \in \mathcal{S}, B_1 \cap B_2 \in \mathcal{G} \Rightarrow B_1 \in \mathcal{G}$ or $B_2 \in \mathcal{G}$.

Proposition 2.6 ([11])**.** *The following are equivalent for a regular difilter $\mathcal{F} \times \mathcal{G}$ on $(S, \mathcal{S})$.*

(1) *$\mathcal{F} \times \mathcal{G}$ is maximal.*
(2) *$\mathcal{F} \cup \mathcal{G} = \mathcal{S}$.*
(3) *$\mathcal{F}$ is a prime $\mathcal{S}$-filter and $\mathcal{G} = \mathcal{S} \setminus \mathcal{F}$.*
(4) *$\mathcal{G}$ is a prime $\mathcal{S}$-cofilter and $\mathcal{F} = \mathcal{S} \setminus \mathcal{G}$.*

It is obtained in [11] that there exist one to one correspondences among the set of maximal regular difilters on $(S, \mathcal{S})$, the set of prime filters on $(S, \mathcal{S})$ and the set of prime cofilters on $(S, \mathcal{S})$.

Graded Ditopological Texture Spaces: [6] Let $(S, \mathcal{S})$, $(V, \mathcal{V})$ be textures and consider $\mathcal{T}, \mathcal{K} : \mathcal{S} \to \mathcal{V}$ satisfying

(GT_1) $\mathcal{T}(S) = \mathcal{T}(\varnothing) = V$
(GT_2) $\mathcal{T}(A_1) \cap \mathcal{T}(A_2) \subseteq \mathcal{T}(A_1 \cap A_2)$ $\forall A_1, A_2 \in \mathcal{S}$
(GT_3) $\bigcap_{j \in J} \mathcal{T}(A_j) \subseteq \mathcal{T}(\bigvee_{j \in J} A_j)$ $\forall A_j \in \mathcal{S}, j \in J$

and

(GCT_1) $\mathcal{K}(S) = \mathcal{K}(\varnothing) = V$
(GCT_2) $\mathcal{K}(A_1) \cap \mathcal{K}(A_2) \subseteq \mathcal{K}(A_1 \cup A_2)$ $\forall A_1, A_2 \in \mathcal{S}$
(GCT_3) $\bigcap_{j \in J} \mathcal{K}(A_j) \subseteq \mathcal{K}(\bigcap_{j \in J} A_j)$ $\forall A_j \in \mathcal{S}, j \in J$.

Then $\mathcal{T}$ is called a $(V, \mathcal{V})$-graded topology, $\mathcal{K}$ a $(V, \mathcal{V})$-graded cotopology and $(\mathcal{T}, \mathcal{K})$ a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$. The tuple $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ is called a graded ditopological texture space. For $v \in V$ we define

$$\mathcal{T}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathcal{T}(A)\}, \ \mathcal{K}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathcal{K}(A)\}.$$

Then $(\mathcal{T}^v, \mathcal{K}^v)$ is a ditopology on $(S, \mathcal{S})$ for each $v \in V$. That is, if $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ is any graded ditopological texture space, then there exists a ditopology $(\mathcal{T}^v, \mathcal{K}^v)$ on the texture space $(S, \mathcal{S})$ for each $v \in V$

If $(S, \mathcal{S}, \sigma)$ is a complemented texture and $(\mathcal{T}, \mathcal{K})$ is a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$, then $(\mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma)$ is also a $(V, \mathcal{V})$-graded ditopology on $(S, \mathcal{S})$. $(\mathcal{T}, \mathcal{K})$ is called complemented if $(\mathcal{T}, \mathcal{K}) = (\mathcal{K} \circ \sigma, \mathcal{T} \circ \sigma)$.

Example 2.7. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $(V, \mathcal{V})$ the discrete texture on a singleton. Take $(V, \mathcal{V}) = (1, \mathcal{P}(1))$ (The notation 1 denotes the set $\{0\}$) and define $\tau^g : \mathcal{S} \to \mathcal{P}(1)$ by $\tau^g(A) = 1 \Leftrightarrow A \in \tau$. Then τ^g is a $(V, \mathcal{V})$-graded topology on $(S, \mathcal{S})$. Likewise, κ^g defined by $\kappa^g(A) = 1 \Leftrightarrow A \in \kappa$ is a $(V, \mathcal{V})$-graded cotopology on $(S, \mathcal{S})$. Then (τ^g, κ^g) is called the graded ditopology on $(S, \mathcal{S})$ corresponding to ditopology (τ, κ).

Therefore graded ditopological texture spaces are more general than ditopological texture spaces.

Graded Dineighborhood Systems: [8] Let $(S, \mathcal{S})$ and $(V, \mathcal{V})$ be two texture spaces. For any mapping $K : \mathcal{S} \to \mathcal{V}$, we use the notation ${}^vK$ to denote the family $\{A \in \mathcal{S} : K(A) \not\subseteq Q_v\}$ for all $v \in V$, and thus for each $v \in V$, we have ${}^vK \subseteq K^v$.

Definition 2.8 ([8])**.** Let $(\mathcal{T}, \mathcal{K})$ be a $(V, \mathcal{V})$-graded ditopology on texture $(S, \mathcal{S})$ and $N : S^\flat \to \mathcal{V}^\mathcal{S}$, $M : S \to \mathcal{V}^\mathcal{S}$ mappings where $N(s) = N_s : \mathcal{S} \to \mathcal{V}$ for each $s \in S^\flat$ and $M(s) = M_s : \mathcal{S} \to \mathcal{V}$ for each $s \in S$. Then the mapping N_s is called a "graded neighborhood system of s" if

$$(2.1) \quad {}^vN_s = \{A \in \mathcal{S} : \mathcal{T}(B) \not\subseteq Q_v \text{ and } P_s \subseteq B \subseteq A \not\subseteq Q_s \text{ for some } B \in \mathcal{S}\}$$

for each $v \in V^\flat$. The mapping M_s is called a "graded coneighborhood system of s" if

$$(2.2) \quad {}^vM_s = \{A \in \mathcal{S} : \mathcal{K}(B) \not\subseteq Q_v \text{ and } P_s \not\subseteq A \subseteq B \subseteq Q_s \text{ for some } B \in \mathcal{S}\}$$

for each $v \in V^\flat$. The mapping N (M) is called a "graded neighborhood system" ("graded coneighborhood system") of graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ if N_s (M_s) is a graded neighborhood system for each $s \in S^\flat$ (graded coneighborhood system for each $s \in S$) respectively. (N, M) is called a "graded dineighborhood system" of graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ if N is a graded neighborhood system and M is a graded coneighborhood system of graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$.

Proposition 2.9 ([8])**.** *For the above notations, (N, M) is a graded dinhd system of a graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ iff*

$$(2.3) \qquad N_s(A) = \begin{cases} sup\{\mathcal{T}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\}, & A \not\subseteq Q_s \\ \varnothing, & A \subseteq Q_s \end{cases}$$

for each $s \in S^\flat$, $A \in \mathcal{S}$ and

$$(2.4) \qquad M_s(A) = \begin{cases} sup\{\mathcal{K}(B) : P_s \not\subseteq A \subseteq B \subseteq Q_s, B \in \mathcal{S}\}, & P_s \not\subseteq A \\ \varnothing, & P_s \subseteq A \end{cases}$$

for each $s \in S$, $A \in \mathcal{S}$.

Theorem 2.10 ([8])**.** *Let $(\mathcal{T}, \mathcal{K})$ be a $(V, \mathcal{V})$-graded ditopology on a texture $(S, \mathcal{S})$. If (N, M) is the graded dinhd system of the graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$, then the following properties hold for all $A, A_1, A_2 \in \mathcal{S}$:*
(1) *For each $s \in S^\flat$;*
- **(N1)** $N_s(A) \neq \varnothing \Rightarrow A \not\subseteq Q_s$
- **(N2)** $N_s(\varnothing) = \varnothing$ *and* $N_s(S) = V$
- **(N3)** $A_1 \subseteq A_2 \Rightarrow N_s(A_1) \subseteq N_s(A_2)$
- **(N4)** $A_1 \cap A_2 \not\subseteq Q_s \Rightarrow N_s(A_1) \wedge N_s(A_2) \subseteq N_s(A_1 \cap A_2)$
- **(N5)** $N_s(A) \subseteq sup\{\bigwedge_{s' \in B^\flat} N_{s'}(B) : P_s \subseteq B \subseteq A \not\subseteq Q_s, B \in \mathcal{S}\}$

(2) *For each $s \in S$;*
- **(M1)** $M_s(A) \neq \varnothing \Rightarrow P_s \not\subseteq A$
- **(M2)** $M_s(S) = \varnothing$ *and* $M_s(\varnothing) = V$

(**M3**) $A_1 \subseteq A_2 \Rightarrow M_s(A_2) \subseteq M_s(A_1)$
(**M4**) $M_s(A_1) \wedge M_s(A_2) \subseteq M_s(A_1 \cup A_2)$
(**M5**) $M_s(A) \subseteq sup\{\bigwedge_{s' \in (S \setminus B)} M_{s'}(B) : P_s \nsubseteq A \subseteq B \subseteq Q_s, B \in \mathcal{S}\}$

3. Graded Difilters and Convergence

Definition 3.1. Let $(S, \mathcal{S})$ and $(V, \mathcal{V})$ be textures.

(1) A mapping $\mathfrak{F} : \mathcal{S} \to \mathcal{V}$ is called a $(V, \mathcal{V})$-graded filter on $(S, \mathcal{S})$ if $\mathfrak{F}$ satisfies:
(GF1) $\mathfrak{F}(\varnothing) = \varnothing$
(GF2) $A_1 \subseteq A_2 \Rightarrow \mathfrak{F}(A_1) \subseteq \mathfrak{F}(A_2)$
(GF3) $\mathfrak{F}(A_1) \wedge \mathfrak{F}(A_2) \subseteq \mathfrak{F}(A_1 \cap A_2)$
(2) A mapping $\mathfrak{G} : \mathcal{S} \to \mathcal{V}$ is called a $(V, \mathcal{V})$-graded cofilter on $(S, \mathcal{S})$ if $\mathfrak{G}$ satisfies:
(GCF1) $\mathfrak{G}(S) = \varnothing$
(GCF2) $A_1 \subseteq A_2 \Rightarrow \mathfrak{G}(A_2) \subseteq \mathfrak{G}(A_1)$
(GCF3) $\mathfrak{G}(A_1) \wedge \mathfrak{G}(A_2) \subseteq \mathfrak{G}(A_1 \cup A_2)$
(3) If $\mathfrak{F}$ is a $(V, \mathcal{V})$-graded filter and $\mathfrak{G}$ $(V, \mathcal{V})$-graded cofilter on $(S, \mathcal{S})$ then the pair $(\mathfrak{F}, \mathfrak{G})$ is called a $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$.

Proposition 3.2. *The following are equivalent for a $(V, \mathcal{V})$-graded difilter $(\mathfrak{F}, \mathfrak{G})$ on $(S, \mathcal{S})$.*

(1) $\mathfrak{F} \wedge \mathfrak{G} = \varnothing$ *i.e.* $\mathfrak{F}(A) \wedge \mathfrak{G}(A) = \varnothing$ *for all* $A \in \mathcal{S}$.
(2) $\forall n \in \mathbb{N}$, $\bigwedge_{i=1}^{n}(\mathfrak{F}(A_i) \wedge \mathfrak{G}(B_i)) \neq \varnothing \Rightarrow \bigcap_{i=1}^{n} A_i \nsubseteq \bigcup_{i=1}^{n} B_i$, *for all* $A_i, B_i \in \mathcal{S}$
(3) $\mathfrak{F}(A) \wedge \mathfrak{G}(B) \neq \varnothing \Rightarrow A \nsubseteq B$, *for all* $A, B \in \mathcal{S}$

Proof. (1) $\Rightarrow$ (2) : Let $n \in \mathbb{N}$, $\bigwedge_{i=1}^{n}(\mathfrak{F}(A_i) \wedge \mathfrak{G}(B_i)) \neq \varnothing$ for all $A_i, B_i \in \mathcal{S}$ and suppose that $\bigcap_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} B_i$. Then, from (1) we get

$$\begin{aligned}\varnothing = \mathfrak{F}(\bigcap_{i=1}^{n} A_i) \wedge \mathfrak{G}(\bigcap_{i=1}^{n} A_i) &\supseteq \mathfrak{F}(\bigcap_{i=1}^{n} A_i) \wedge \mathfrak{G}(\bigcup_{i=1}^{n} B_i) \\ &\supseteq \bigwedge_{i=1}^{n} \mathfrak{F}(A_i) \wedge \bigwedge_{i=1}^{n} \mathfrak{G}(B_i) = \bigwedge_{i=1}^{n} (\mathfrak{F}(A_i) \wedge \mathfrak{G}(B_i))\end{aligned}$$

which contradicts with $\bigwedge_{i=1}^{n}(\mathfrak{F}(A_i) \wedge \mathfrak{G}(B_i)) \neq \varnothing$.
(2) $\Rightarrow$ (3) : Clear.
(3) $\Rightarrow$ (1) : If we assume that $\mathfrak{F} \wedge \mathfrak{G} \neq \varnothing$ then $\mathfrak{F}(A) \wedge \mathfrak{G}(A) \neq \varnothing$ for some $A \in \mathcal{S}$. Thus we obtain that $A \nsubseteq A$ by (3) but this is a contradiction. So we have $\mathfrak{F} \wedge \mathfrak{G} = \varnothing$. □

Definition 3.3. A $(V, \mathcal{V})$-graded difilter $(\mathfrak{F}, \mathfrak{G})$ on $(S, \mathcal{S})$ is called regular if it satisfies the equivalent conditions in the previous proposition.

Example 3.4. (1) Let $(S, \mathcal{S})$ be a texture space and $\mathcal{F} \times \mathcal{G}$ a (regular) difilter on it. Then the mappings $\mathfrak{F}, \mathfrak{G} : \mathcal{S} \to P(1)$ defined by

$$\mathfrak{F}(A) = \begin{cases} \{0\}, & A \in \mathcal{F} \\ \varnothing, & A \notin \mathcal{F} \end{cases}$$

and

$$\mathfrak{G}(A) = \begin{cases} \{0\}, & A \in \mathcal{G} \\ \varnothing, & A \notin \mathcal{G} \end{cases}$$

for all $A \in \mathcal{S}$, form a (regular) $(1, P(1))$-graded difilter on $(S, \mathcal{S})$.

On the other hand, if $(\mathfrak{F}, \mathfrak{G})$ is a (regular) $(1, P(1))$-graded difilter on $(S, \mathcal{S})$ then the families defined by

$$\mathcal{F} = \{A \in \mathcal{S} : \mathfrak{F}(A) = \{0\}\ \}, \mathcal{G} = \{A \in \mathcal{S} : \mathfrak{G}(A) = \{0\}\ \}$$

form a (regular) difilter $\mathcal{F} \times \mathcal{G}$ on $(S, \mathcal{S})$.

Besides, if $(\mathfrak{F}, \mathfrak{G})$ be a (regular) $(V, \mathcal{V})$-graded difilter on a texture $(S, \mathcal{S})$ then the families

$$\mathfrak{F}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathfrak{F}(A)\}, \quad \mathfrak{G}^v = \{A \in \mathcal{S} \mid P_v \subseteq \mathfrak{G}(A)\}$$

form a (regular) difilter $\mathfrak{F}^v \times \mathfrak{G}^v$ on $(S, \mathcal{S})$ for each $v \in V$.

(2)Analogous with the dinhd-difilter situation; if (N, M) is a graded dinhd system of a graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ then M_s is a $(V, \mathcal{V})$-graded cofilter on $(S, \mathcal{S})$ but in general for $s \in S^\flat$, N_s is not a $(V, \mathcal{V})$-graded filter on $(S, \mathcal{S})$. If $(S, \mathcal{S})$ is plain then N_s is a $(V, \mathcal{V})$-graded filter on $(S, \mathcal{S})$ for each $s \in S^\flat = S$. As a result, if $(S, \mathcal{S})$ is plain then (N_s, M_s) is a $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$ for each $s \in S^\flat = S$.

(3) Let $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ be a graded ditopological texture space. If the mappings $\forall s \in S^\flat$, $N_s^* : \mathcal{S} \to \mathcal{V}$, $\forall s \in S$, $M_s^* : \mathcal{S} \to \mathcal{V}$ are defined by

$$N_s^*(A) = \begin{cases} sup\{\bigcap_{k=1}^n \mathcal{T}(B_k) : B_k \not\subseteq Q_s, B_1 \cap ... \cap B_n \subseteq A\ for B_i \in \mathcal{S}\}, & A \not\subseteq Q_s \\ \varnothing, & A \subseteq Q_s \end{cases}$$

and

$$M_s^*(A) = \begin{cases} sup\{\bigcap_{k=1}^n \mathcal{K}(B_k) : P_s \not\subseteq B_k, A \subseteq B_1 \cup ... \cup B_n\ for B_i \in \mathcal{S}\}, & P_s \not\subseteq A \\ \varnothing, & P_s \subseteq A \end{cases}$$

for all $A \in \mathcal{S}$ then (N_s^*, M_s^*) is a regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$ for each $s \in S^\flat$.

Proposition 3.5. *For the above notations, if the texture $(S, \mathcal{S})$ is plain then $N_s = N_s^*$ and $M_s = M_s^*$ for each $s \in S^\flat = S$.*

Proof. Let $A \in \mathcal{S}$, $s \in S$. $A \subseteq Q_s$ implies $N_s^*(A) = N_s(A) = \varnothing$ so, let it be $A \not\subseteq Q_s$, and suppose that $N_s(A) \not\subseteq N_s^*(A)$ for some $A \in \mathcal{S}$. Then there exists $v \in V$ such that $N_s(A) \not\subseteq Q_v$ and $P_v \not\subseteq N_s^*(A)$. Considering $N_s(A) \not\subseteq Q_v$, we have $P_s \subseteq B \subseteq A \not\subseteq Q_s$ and $\mathcal{T}(B) \not\subseteq Q_v$ for some $B \in \mathcal{S}$. Since $(S, \mathcal{S})$ is plain, we have $B \not\subseteq Q_s$. So, $\mathcal{T}(B) \subseteq N_s^*(A)$ and considering $\mathcal{T}(B) \not\subseteq Q_v$ we get $N_s^*(A) \not\subseteq Q_v$ and $P_v \subseteq N_s^*(A)$ which contradicts with $P_v \not\subseteq N_s^*(A)$.

Thus, $N_s(A) \subseteq N_s^*(A)$. Now we assume that $N_s^*(A) \nsubseteq N_s(A)$ for some $A \in \mathcal{S}$. Then there exists $v \in V$ such that $N_s^*(A) \nsubseteq Q_v$ and $P_v \nsubseteq N_s(A)$. Considering $N_s^*(A) \nsubseteq Q_v$, there exist $B_1, B_2, ..., B_n \in \mathcal{S}$ such that $B_k \nsubseteq Q_s$ for $1 \leq k \leq n$, $B_1 \cap B_2 \cap ... \cap B_n \subseteq A$ and $\bigcap_{k=1}^n \mathcal{T}(B_k) \nsubseteq Q_v$. Since $\bigcap_{k=1}^n \mathcal{T}(B_k) \subseteq \mathcal{T}(\bigcap_{k=1}^n B_k)$ we have $\mathcal{T}(\bigcap_{k=1}^n B_k) \nsubseteq Q_v$ and so, $P_v \subseteq \mathcal{T}(\bigcap_{k=1}^n B_k)$. Moreover, since $B_k \nsubseteq Q_s$ for $1 \leq k \leq n$ and $(S, \mathcal{S})$ is plain, we get $P_s \subseteq (B_1 \cap B_2 \cap ... \cap B_n) \subseteq A \nsubseteq Q_s$. Thus, it is obtained that $P_v \subseteq \mathcal{T}(\bigcap_{k=1}^n B_k) \subseteq N_s(A)$ which is a contradiction. Therefore we get $N_s^*(A) \subseteq N_s(A)$ and so, $N_s = N_s^*$. Similarly it can be shown that $M_s = M_s^*$. □

Definition 3.6. Let $\mathfrak{F}$ be a $(V, \mathcal{V})$-graded filter and $\mathfrak{G}$ a $(V, \mathcal{V})$-graded co-filter on a graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$. We say that $\mathfrak{F}$ converges to s and write that $\mathfrak{F} \to s$ if $N_s^* \subseteq \mathfrak{F}$. Also we say that $\mathfrak{G}$ converges to s and write that $\mathfrak{G} \to s$ if $M_s^* \subseteq \mathfrak{G}$.

For $s, s' \in S$, the graded difilter $(\mathfrak{F}, \mathfrak{G})$ is called diconvergent if $P_{s'} \nsubseteq Q_s$, $\mathfrak{F} \to s$ and $\mathfrak{G} \to s'$. In this case, s (s') is called (co-)limit of $(\mathfrak{F}, \mathfrak{G})$.

Proposition 3.7. *If $(\mathfrak{F}, \mathfrak{G})$ is a $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ then*

(a) $\mathfrak{F} \to s \Leftrightarrow$ "$A \nsubseteq Q_s \Rightarrow \mathcal{T}(A) \subseteq \mathfrak{F}(A)$"
(b) $\mathfrak{G} \to s \Leftrightarrow$ "$P_s \nsubseteq A \Rightarrow \mathcal{K}(A) \subseteq \mathfrak{G}(A)$"

Proof. (a) Let $\mathfrak{F} \to s$ and $A \nsubseteq Q_s$. Since $\mathfrak{F} \to s$, we have $N_s^* \subseteq \mathfrak{F}$ and $N_s^*(A) \subseteq \mathfrak{F}(A)$. Considering $A \nsubseteq Q_s$ we obtain that $\mathcal{T}(A) \subseteq N_s^*(A)$ and so $\mathcal{T}(A) \subseteq \mathfrak{F}(A)$. On the other hand, if we suppose that "$A \nsubseteq Q_s \Rightarrow \mathcal{T}(A) \subseteq \mathfrak{F}(A)$" then $N_s^*(A) \subseteq \mathfrak{F}(A)$ and so we get $\mathfrak{F} \to s$.

The proof of (b) is similar. □

Proposition 3.8. *Let the texture $(S, \mathcal{S})$ be plain. If $(\mathfrak{F}, \mathfrak{G})$ is a graded difilter on $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ then the following are equivalent:*

(a) *$(\mathfrak{F}, \mathfrak{G})$ is diconvergent.*
(b) $\exists s \in S : (N_s, M_s) \subseteq (\mathfrak{F}, \mathfrak{G})$

Proof. $(a) \Rightarrow (b)$: If $(\mathfrak{F}, \mathfrak{G})$ is diconvergent then there exists $s, s' \in S$ such that $P_{s'} \nsubseteq Q_s$ and $(N_s^*, M_{s'}^*) \subseteq (\mathfrak{F}, \mathfrak{G})$. Since $(S, \mathcal{S})$ is plain, from Proposition 3.5 we have $(N_s, M_{s'}) = (N_s^*, M_{s'}^*)$. Besides, since $P_{s'} \nsubseteq Q_s$, $P_s \nsubseteq A \subseteq B \subseteq Q_s \Rightarrow P_{s'} \nsubseteq A \subseteq B \subseteq Q_{s'}$ for all $A,\ B \in \mathcal{S}$ and so we have $M_s \subseteq M_{s'}$. Thus we get $N_s = N_s^* \subseteq \mathfrak{F}$, $M_s \subseteq M_{s'} = M_{s'}^* \subseteq \mathfrak{G}$ and hence $(N_s, M_s) \subseteq (\mathfrak{F}, \mathfrak{G})$.
$(b) \Rightarrow (a)$: Since $(S, \mathcal{S})$ is plain we get $P_s \nsubseteq Q_s$ and by Proposition 3.5 we have $(N_s, M_s) = (N_s^*, M_s^*)$. Therefore $(\mathfrak{F}, \mathfrak{G})$ is diconvergent. □

Definition 3.9. Let $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ be a graded ditopological texture space, $A \in \mathcal{S}$ and $v \in V$. The set

$$\bigcap \{B \in \mathcal{S} | A \subseteq B,\ P_v \subseteq \mathcal{K}(B)\} \in \mathcal{S}$$

is called v-closure of A and denoted by $[A]^v$. The set

$$\bigvee \{B \in \mathcal{S} | B \subseteq A,\ P_v \subseteq \mathcal{T}(B)\} \in \mathcal{S}$$

is called v-interior of A and denoted by $]A[^v$.

Note that for each $v \in V$, $[A]^v$ ($]A[^v$) is the closure (the interior) of A in the ditopological texture space $(S, \mathcal{S}, \mathcal{T}^v, \mathcal{K}^v)$.

Proposition 3.10. *Let* $(\mathfrak{F}, \mathfrak{G})$ *be a regular graded difilter on a graded ditopological texture space* $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ *and* $s \in S$.

(a) $\mathfrak{F} \to s \Rightarrow$ "$\forall A \in \mathcal{S}, v \in \mathfrak{G}(A) \Rightarrow]A[^v \subseteq Q_s$"
(b) $\mathfrak{G} \to s \Rightarrow$ "$\forall A \in \mathcal{S}, v \in \mathfrak{F}(A) \Rightarrow P_s \subseteq [A]^v$"

Proof. (a) Let $\mathfrak{F} \to s$, and suppose that there exists $v \in \mathfrak{G}(A)$ such that $]A[^v \not\subseteq Q_s$ for a set $A \in \mathcal{S}$. Then $B \subseteq A$, $P_v \subseteq \mathcal{T}(B)$ and $B \not\subseteq Q_s$ for some $B \in \mathcal{S}$. Considering $\mathfrak{F} \to s$ and $B \not\subseteq Q_s$, from proposition 3.7 we get $\mathcal{T}(B) \subseteq \mathfrak{F}(B)$. So, $P_v \subseteq \mathcal{T}(B) \subseteq \mathfrak{F}(B) \subseteq \mathfrak{F}(A)$ and considering $v \in \mathfrak{G}(A)$ we get $\mathfrak{F}(A) \cap \mathfrak{G}(A) \neq \varnothing$ which contradicts with the regularity of $(\mathfrak{F}, \mathfrak{G})$.

The proof of (b) is similar. □

Definition 3.11. Let $(\mathfrak{F}, \mathfrak{G})$ be a regular graded difilter on a graded ditopological texture space $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$.

(1) $s \in S$ is called a cluster point of $\mathfrak{F}$ if for all $A \in \mathcal{S}$, $v \in \mathfrak{F}(A) \Rightarrow P_s \subseteq [A]^v$
(2) $s \in S$ is called a cluster point of $\mathfrak{G}$ if for all $A \in \mathcal{S}$, $v \in \mathfrak{G}(A) \Rightarrow]A[^v \subseteq Q_s$
(3) For $s, s' \in S$, $P_s \not\subseteq Q_{s'}$, if s is a cluster point of $\mathfrak{F}$ and s' is a cluster point of $\mathfrak{G}$ then $(\mathfrak{F}, \mathfrak{G})$ is called diclustering in $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$.

Corollary 3.12. *On a graded ditopological texture space* $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$, *each diconvergent regular graded difilter is diclustering.*

Proof. If $(\mathfrak{F}, \mathfrak{G})$ is a diconvergent regular graded difilter on $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ then there exist $s, s' \in S$ such that $\mathfrak{F} \to s$, $\mathfrak{G} \to s'$ and $P_{s'} \not\subseteq Q_s$. Considering Proposition 3.10., s is a cluster point of $\mathfrak{G}$ and s' a cluster point of $\mathfrak{F}$. Since $P_{s'} \not\subseteq Q_s$, $(\mathfrak{F}, \mathfrak{G})$ is diclustering. □

Definition 3.13. Let $(\mathfrak{F}, \mathfrak{G})$ be a $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$. $\mathfrak{F}$ is called prime if $\mathfrak{F}(A_1 \cup A_2) \subseteq \mathfrak{F}(A_1) \cup \mathfrak{F}(A_2)$ and $\mathfrak{G}$ is called prime if $\mathfrak{G}(A_1 \cap A_2) \subseteq \mathfrak{G}(A_1) \cup \mathfrak{G}(A_2)$ for all $A_1, A_2 \in \mathcal{S}$.

Example 3.14. Let $S = \{1, 2\}$, $\mathcal{S} = \{\varnothing, \{1\}, \{2\}, S\}$, $V = \{a, b, c\}$ and $\mathcal{V} = \mathcal{P}(V)$. In this case, $(S, \mathcal{S})$ and $(V, \mathcal{V})$ are texture spaces. If the mappings $\mathfrak{F}, \mathfrak{G} : \mathcal{S} \to \mathcal{V}$ are defined by

$\mathfrak{F}(\varnothing) = \varnothing$, $\mathfrak{F}(\{1\}) = \{a\}$, $\mathfrak{F}(\{2\}) = \{b\}$, $\mathfrak{F}(S) = \{a, b\}$
$\mathfrak{G}(\varnothing) = \{a, b\}$, $\mathfrak{G}(\{1\}) = \{b\}$, $\mathfrak{G}(\{2\}) = \{a\}$, $\mathfrak{G}(S) = \varnothing$

then $(\mathfrak{F}, \mathfrak{G})$ is a regular $(V, \mathcal{V})$-graded difilter. Moreover, $\mathfrak{F}$ and $\mathfrak{G}$ are prime.

The structure of graded difilter is more general than the structure of difilter. Most of the properties of difilters can be generalized to the graded case and it can be expected that graded difilters satisfy these generalized properties. But this is not possible in each case. For instance, the statements (1) – (4) in Proposition 2.6. are equivalent for difilters however the generalizations of these statements are not always equivalent for graded difilters as in the next example.

Definition 3.15. Let $(\mathfrak{F}, \mathfrak{G})$ be a $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$. $(\mathfrak{F}, \mathfrak{G})$ is called a maximal $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$ if whenever $(\mathfrak{F}', \mathfrak{G}')$ is a $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$ and $(\mathfrak{F}, \mathfrak{G}) \subseteq (\mathfrak{F}', \mathfrak{G}')$ then we have $(\mathfrak{F}, \mathfrak{G}) = (\mathfrak{F}', \mathfrak{G}')$ is hold.

Example 3.16. Let $S = \{1, 2\}$, $\mathcal{S} = \{\varnothing, \{1\}, \{2\}, S\}$, $V = \{a, b, c\}$ and $\mathcal{V} = \{\varnothing, \{b\}, \{c\}, \{b, c\}, V\}$. In this case, $(S, \mathcal{S})$ and $(V, \mathcal{V})$ are plain texture spaces. If the mappings $\mathfrak{F}, \mathfrak{G} : \mathcal{S} \to \mathcal{V}$ are defined by

$\mathfrak{F}(\varnothing) = \varnothing$, $\mathfrak{F}(\{1\}) = \{b\}$, $\mathfrak{F}(\{2\}) = \{c\}$, $\mathfrak{F}(S) = V$
$\mathfrak{G}(\varnothing) = V$, $\mathfrak{G}(\{1\}) = \{c\}$, $\mathfrak{G}(\{2\}) = \{b\}$, $\mathfrak{G}(S) = \varnothing$

then $(\mathfrak{F}, \mathfrak{G})$ is a regular $(V, \mathcal{V})$-graded difilter. Moreover, $(\mathfrak{F}, \mathfrak{G})$ is a maximal regular $(V, \mathcal{V})$-graded difilter but $\mathfrak{F} \vee \mathfrak{G} \neq V$. (Example for (1) $\not\Rightarrow$ (2) in Proposition 3.17.)

Proposition 3.17. *Let $(\mathfrak{F}, \mathfrak{G})$ be a regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$. For the statements*

(1) $(\mathfrak{F}, \mathfrak{G})$ *is a maximal regular $(V, \mathcal{V})$-graded difilter*
(2) $\mathfrak{F} \vee \mathfrak{G} = V$ *(i.e. $\forall A \in \mathcal{S}, \mathfrak{F}(A) \vee \mathfrak{G}(A) = \mathfrak{F}(A) \cup \mathfrak{G}(A) = V$)*
(3) $\mathfrak{F}$ *is prime and* $\mathfrak{G} = V \setminus \mathfrak{F}$
(4) $\mathfrak{G}$ *is prime and* $\mathfrak{F} = V \setminus \mathfrak{G}$

the following implications are hold:
(1) $\Leftarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4), (1) $\not\Rightarrow$ (2)

Proof. (1) $\Leftarrow$ (2): Let $(\mathfrak{F}, \mathfrak{G})$ be a regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$ and $\mathfrak{F} \vee \mathfrak{G} = V$. From regularity of $(\mathfrak{F}, \mathfrak{G})$ we have $\mathfrak{F} \cap \mathfrak{G} = \varnothing$. Considering $\mathfrak{F} \vee \mathfrak{G} = \mathfrak{F} \cup \mathfrak{G} = V$ we get $\mathfrak{G} = V \setminus \mathfrak{F}$ and $\mathfrak{F} = V \setminus \mathfrak{G}$. If $(\mathfrak{F}', \mathfrak{G}')$ is a regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$ and $(\mathfrak{F}, \mathfrak{G}) \subseteq (\mathfrak{F}', \mathfrak{G}')$ then considering $\mathfrak{F}' \cap \mathfrak{G}' = \varnothing$ we get

$$\mathfrak{F}(A) = V \setminus \mathfrak{G}(A) \supseteq V \setminus \mathfrak{G}'(A) \supseteq \mathfrak{F}'(A) \supseteq \mathfrak{F}(A)$$

$$\mathfrak{G}(A) = V \setminus \mathfrak{F}(A) \supseteq V \setminus \mathfrak{F}'(A) \supseteq \mathfrak{G}'(A) \supseteq \mathfrak{G}(A).$$

Thus, $(\mathfrak{F}, \mathfrak{G}) = (\mathfrak{F}', \mathfrak{G}')$ is obtained. So, $(\mathfrak{F}, \mathfrak{G})$ is a maximal regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$.
(2) $\Leftrightarrow$ (3): If $\mathfrak{F} \vee \mathfrak{G} = V$ then from regularity of $(\mathfrak{F}, \mathfrak{G})$, we have $\mathfrak{F} \cap \mathfrak{G} = \varnothing$ and so $\mathfrak{G} = V \setminus \mathfrak{F}$. Suppose that $\mathfrak{F}$ is not prime. Then, there exist $A_1, A_2 \in \mathcal{S}$ such that $\mathfrak{F}(A_1 \cup A_2) \not\subseteq \mathfrak{F}(A_1) \cup \mathfrak{F}(A_2)$. Thus it is obtained that

$$\begin{aligned}\mathfrak{G}(A_1) \cap \mathfrak{G}(A_2) = V \setminus \mathfrak{F}(A_1) \cap V \setminus \mathfrak{F}(A_2) &= V \setminus (\mathfrak{F}(A_1) \cup \mathfrak{F}(A_2)) \\ &\not\subseteq V \setminus \mathfrak{F}(A_1 \cup A_2) = \mathfrak{G}(A_1 \cup A_2)\end{aligned}$$

which contradicts with (GCF3). Therefore $\mathfrak{F}$ is prime.

On the other hand, if $\mathfrak{G} = V \setminus \mathfrak{F}$ then $\mathfrak{F} \vee \mathfrak{G} = \mathfrak{F} \cup \mathfrak{G} = V$ is obtained.
(2) $\Leftrightarrow$ (4): Similar as (2) $\Leftrightarrow$ (3).
(1) $\not\Rightarrow$ (2): Example 3.16. □

Proposition 3.18. *A texture space* $(V, \mathcal{V})$ *is discrete if and only if* $P_v \not\subseteq A \Rightarrow P_v \cap A = \varnothing$ *for all* $v \in V$, $A \in \mathcal{V}$.

Proof. Let $P_v \not\subseteq A \Rightarrow P_v \cap A = \varnothing$ for all $v \in V$, $A \in \mathcal{V}$ and suppose that $(V, \mathcal{V})$ is not discrete. Then there exists $v \in V$ such that $P_v \neq \{v\}$. So, $t \in P_v$, $t \neq v$ for some $t \in V$. Since textures are point separating lattices, we have $v \notin P_t$ and $P_v \not\subseteq P_t$. Considering $\{v\} \subseteq P_v \cap P_t \neq \varnothing$, if we take $A = P_t$ in the hypothesis then we get a contradiction. Thus, $(V, \mathcal{V})$ is discrete. On the other hand, if $(V, \mathcal{V})$ is discrete then $P_v = \{v\}$ and so $P_v \not\subseteq A \Rightarrow P_v \cap A = \varnothing$ for all $v \in V$, $A \in \mathcal{V}$. □

The generalizations of the equivalent statements in Proposition 2.6. to the graded case are equivalent if $(V, \mathcal{V})$ is discrete as in the next theorem. Hence the concepts studied and results obtained in this paper are more general than those in [11].

Theorem 3.19. *If* $(\mathfrak{F}, \mathfrak{G})$ *is a regular* $(V, \mathcal{V})$*-graded difilter on* $(S, \mathcal{S})$ *and* $(V, \mathcal{V})$ *is discrete then the statements* $(1) - (4)$ *in Proposition 3.17 are equivalent.*

Proof. It is sufficient to show the implication (1) $\Rightarrow$ (2). Let $(\mathfrak{F}, \mathfrak{G})$ be a maximal regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$ and suppose that $\mathfrak{F} \vee \mathfrak{G} \neq V$. Then there exists $A_0 \in \mathcal{S}$ such that $\mathfrak{F}(A_0) \cup \mathfrak{G}(A_0) \neq V$ and so there exists $v \in V$ such that $P_v = \{v\} \not\subseteq \mathfrak{F}(A_0) \cup \mathfrak{G}(A_0)$. There are two cases: "$\forall A, B \in \mathcal{S},\ P_v \not\subseteq \mathfrak{F}(A) \cap \mathfrak{G}(B)$" or "$\exists A, B \in \mathcal{S} :\ P_v \subseteq \mathfrak{F}(A) \cap \mathfrak{G}(B)$".

Case 1: "$\forall A, B \in \mathcal{S},\ P_v \not\subseteq \mathfrak{F}(A) \cap \mathfrak{G}(B)$". In particular, if we take $A = S$ and $B = \varnothing$ then $P_v \not\subseteq \mathfrak{F}(S) \cap \mathfrak{G}(\varnothing)$. So, there are two subcases: $P_v \not\subseteq \mathfrak{F}(S)$ or $P_v \not\subseteq \mathfrak{G}(\varnothing)$.

Case 1.1: "$P_v \not\subseteq \mathfrak{F}(S)$". Since from (GF2) $\mathfrak{F}(A) \subseteq \mathfrak{F}(S)$ for all $A \in \mathcal{S}$, we have $P_v \not\subseteq \mathfrak{F}(A)$ for all $A \in \mathcal{S}$. If we define a mapping $\mathfrak{F}^* : \mathcal{S} \to \mathcal{V}$ by

$$\mathfrak{F}^*(B) = \begin{cases} \varnothing, & B = \varnothing \\ \mathfrak{F}(B), & A_0 \not\subseteq B \\ \mathfrak{F}(B) \cup P_v, & A_0 \subseteq B \end{cases}$$

then $(\mathfrak{F}^*, \mathfrak{G})$ is a regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$:
(GF1) $\mathfrak{F}^*(\varnothing) = \varnothing$
(GF2) Let $B_1, B_2 \in \mathcal{S}$ and $B_1 \subseteq B_2$. If $B_1 = \varnothing$ then we have $\mathfrak{F}^*(B_1) = \varnothing \subseteq \mathfrak{F}^*(B_2)$. If $B_1 \neq \varnothing$ then $B_2 \neq \varnothing$ and so,

(a) If $A_0 \subseteq B_1$ then $A_0 \subseteq B_2$ and so $\mathfrak{F}^*(B_1) = P_v \cup \mathfrak{F}(B_1) \subseteq P_v \cup \mathfrak{F}(B_2) = \mathfrak{F}^*(B_2)$.
(b) If $A_0 \not\subseteq B_1$ then $\mathfrak{F}^*(B_1) = \mathfrak{F}(B_1) \subseteq \mathfrak{F}(B_2) \subseteq \mathfrak{F}^*(B_2)$.

(GF3) Let $B_1, B_2 \in \mathcal{S}$. If $B_1 = \varnothing$ or $B_2 = \varnothing$ then we have $\mathfrak{F}^*(B_1) \wedge \mathfrak{F}^*(B_2) = \varnothing \subseteq \mathfrak{F}^*(B_1 \cap B_2)$. If $B_1, B_2 \neq \varnothing$ then

(a) If $A_0 \nsubseteq B_1, B_2$ then $A_0 \nsubseteq B_1 \cap B_2$ and $\mathfrak{F}^*(B_1) \wedge \mathfrak{F}^*(B_2) = \mathfrak{F}(B_1) \cap \mathfrak{F}(B_2) \subseteq \mathfrak{F}(B_1 \cap B_2) = \mathfrak{F}^*(B_1 \cap B_2)$.

(b) If $A_0 \subseteq B_1, B_2$ then $A_0 \subseteq B_1 \cap B_2$ and $\mathfrak{F}^*(B_1) \wedge \mathfrak{F}^*(B_2) = (P_v \cup \mathfrak{F}(B_1)) \cap (P_v \cup \mathfrak{F}(B_2)) = P_v \cup (\mathfrak{F}(B_1) \cap \mathfrak{F}(B_2)) \subseteq P_v \cup \mathfrak{F}(B_1 \cap B_2) = \mathfrak{F}^*(B_1 \cap B_2)$.

(c) Without loss of generality let $A_0 \nsubseteq B_1$ and $A_0 \subseteq B_2$. Then $A_0 \nsubseteq B_1 \cap B_2$. Considering that $P_v \nsubseteq \mathfrak{F}(A)$ for all $A \in \mathcal{S}$ and $\mathcal{V}$ is discrete we have $\mathfrak{F}(B_1) \cap P_v = \varnothing$. Thus we get

$$\begin{aligned} \mathfrak{F}^*(B_1) \wedge \mathfrak{F}^*(B_2) &= \mathfrak{F}(B_1) \cap (P_v \cup \mathfrak{F}(B_2)) = (\mathfrak{F}(B_1) \cap P_v) \cup (\mathfrak{F}(B_1) \cap \mathfrak{F}(B_2)) \\ &= \mathfrak{F}(B_1) \cap \mathfrak{F}(B_2) \subseteq \mathfrak{F}(B_1 \cap B_2) = \mathfrak{F}^*(B_1 \cap B_2). \end{aligned}$$

Let $B \in \mathcal{S}$. Then since $(\mathfrak{F}, \mathfrak{G})$ is regular;

(a) If $A_0 \subseteq B$ then, since $\mathcal{V}$ is discrete, $\mathfrak{G}(B) \subseteq \mathfrak{G}(A_0)$ and $P_v \nsubseteq \mathfrak{F}(A_0) \cup \mathfrak{G}(A_0)$ it is obtained that $P_v \nsubseteq \mathfrak{G}(B)$ and so we have $P_v \cap \mathfrak{G}(B) = \varnothing$. Therefore we get $\mathfrak{F}^*(B) \cap \mathfrak{G}(B) = (P_v \cup \mathfrak{F}(B)) \cap \mathfrak{G}(B) = (P_v \cap \mathfrak{G}(B)) \cup (\mathfrak{F}(B) \cap \mathfrak{G}(B)) = \varnothing$, i.e. $\mathfrak{F}^*(B) \cap \mathfrak{G}(B) = \varnothing$.

(b) If $A_0 \nsubseteq B$ then we get $\mathfrak{F}^*(B) \cap \mathfrak{G}(B) = \mathfrak{F}(B) \cap \mathfrak{G}(B) = \varnothing$.

Therefore $(\mathfrak{F}^*, \mathfrak{G})$ is a regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$. However, $(\mathfrak{F}, \mathfrak{G}) \subsetneqq (\mathfrak{F}^*, \mathfrak{G})$ (at least $\mathfrak{F}^*(A_0) = P_v \cup \mathfrak{F}(A_0) \neq \mathfrak{F}(A_0)$) and this contradicts with the maximality of $(\mathfrak{F}, \mathfrak{G})$. Hence, the implication (1) $\Rightarrow$ (2) is satisfied.

Case 1.2: "$P_v \nsubseteq \mathfrak{G}(\varnothing)$". Since from (GCF2) $\mathfrak{G}(A) \subseteq \mathfrak{G}(\varnothing)$ for all $A \in \mathcal{S}$, we have $P_v \nsubseteq \mathfrak{G}(A)$ for all $A \in \mathcal{S}$. If we define a mapping $\mathfrak{G}^* : \mathcal{S} \to \mathcal{V}$ by

$$\mathfrak{G}^*(B) = \begin{cases} \varnothing, & B = S \\ \mathfrak{G}(B), & B \nsubseteq A_0 \\ \mathfrak{G}(B) \cup P_v, & B \subseteq A_0 \end{cases}$$

then $(\mathfrak{F}, \mathfrak{G}^*)$ is a regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$ (it can be shown like in case 1.1).

However, $(\mathfrak{F}, \mathfrak{G}) \subsetneqq (\mathfrak{F}, \mathfrak{G}^*)$ (at least $\mathfrak{G}^*(A_0) = P_v \cup \mathfrak{G}(A_0) \neq \mathfrak{G}(A_0)$) and this contradicts with the maximality of $(\mathfrak{F}, \mathfrak{G})$. Hence, the implication (1) $\Rightarrow$ (2) is satisfied.

Case 2: "$\exists C, D \in \mathcal{S},\ P_v \subseteq \mathfrak{F}(C) \cap \mathfrak{G}(D)$". If we suppose that $A_0 = S$ then since $C \subseteq A_0$ we have $P_v \subseteq \mathfrak{F}(C) \subseteq \mathfrak{F}(A_0)$ which contradicts with $P_v \nsubseteq \mathfrak{F}(A_0)$. Similarly if we suppose that $A_0 = \varnothing$ then since $A_0 \subseteq D$ we have $P_v \subseteq \mathfrak{G}(D) \subseteq \mathfrak{G}(A_0)$ which contradicts with $P_v \nsubseteq \mathfrak{G}(A_0)$. Thus we get that $A_0 \neq \varnothing, S$.

Now, we show that

$$\forall A, B \in \mathcal{S} \ : \ P_v \subseteq \mathfrak{F}(A) \cap \mathfrak{G}(B) \Rightarrow A_0 \cap A \nsubseteq B$$

or

$$\forall A, B \in \mathcal{S} \ : \ P_v \subseteq \mathfrak{F}(A) \cap \mathfrak{G}(B) \Rightarrow A \nsubseteq A_0 \cup B.$$

Contrary, if we assume that there exist $A_1, B_1, A_2, B_2 \in \mathcal{S}$ such that "$P_v \subseteq \mathfrak{F}(A_1) \cap \mathfrak{G}(B_1)$, $A_0 \cap A_1 \subseteq B_1$" and "$P_v \subseteq \mathfrak{F}(A_2) \cap \mathfrak{G}(B_2)$, $A_2 \subseteq A_0 \cup B_2$" then we have $P_v \subseteq \mathfrak{F}(A_1) \cap \mathfrak{F}(A_2) \subseteq \mathfrak{F}(A_1 \cap A_2)$ and $P_v \subseteq \mathfrak{G}(B_1) \cap \mathfrak{G}(B_2) \subseteq \mathfrak{G}(B_1 \cup B_2)$ and so $\mathfrak{F}(A_1 \cap A_2) \cap \mathfrak{G}(B_1 \cup B_2) \neq \varnothing$. Since $(\mathfrak{F}, \mathfrak{G})$ is regular, from proposition 3.2. (3) we get $A_1 \cap A_2 \nsubseteq B_1 \cup B_2$. Hence there exists $s \in A_1 \cap A_2$ such that

$s \notin B_1 \cup B_2$. Considering $s \in A_2$, $s \notin B_2$ and $A_2 \subseteq A_0 \cup B_2$ we have $s \in A_0$. Now, considering $s \in A_0$, $s \in A_1$ and $A_0 \cap A_1 \subseteq B_1$ we have $s \in B_1$ which contradicts with $s \notin B_1 \cup B_2$. Therefore we have again two subcases.

Case 2.1:"$\forall A, B \in \mathcal{S} \,:\, P_v \subseteq \mathfrak{F}(A) \cap \mathfrak{G}(B) \Rightarrow A_0 \cap A \nsubseteq B$". If we define a mapping $\mathfrak{F}' : \mathcal{S} \to \mathcal{V}$ by

$$\mathfrak{F}'(B) = \begin{cases} \varnothing, & B = \varnothing \\ \mathfrak{F}(B), & B \neq \varnothing \text{ and } "P_v \subseteq \mathfrak{F}(A) \text{ for all } A \in \mathcal{S} \Rightarrow A_0 \cap A \nsubseteq B" \\ \mathfrak{F}(B) \cup P_v, & B \neq \varnothing \text{ and } "\exists A \in \mathcal{S} \,:\, P_v \subseteq \mathfrak{F}(A) \text{ and } A_0 \cap A \subseteq B" \end{cases}$$

then $(\mathfrak{F}', \mathfrak{G})$ is a regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$:
(GF1) $\mathfrak{F}'(\varnothing) = \varnothing$
(GF2) Let $B_1, B_2 \in \mathcal{S}$ and $B_1 \subseteq B_2$. If $B_1 = \varnothing$ then we have $\mathfrak{F}'(B_1) = \varnothing \subseteq \mathfrak{F}'(B_2)$. If $B_1 \neq \varnothing$ then $B_2 \neq \varnothing$ and so,

- (a) Let "$\exists A \in \mathcal{S} \,:\, P_v \subseteq \mathfrak{F}(A)$ and $A_0 \cap A \subseteq B_1$". Then $A_0 \cap A \subseteq B_1 \subseteq B_2$ and so $\mathfrak{F}'(B_1) = P_v \cup \mathfrak{F}(B_1) \subseteq P_v \cup \mathfrak{F}(B_2) = \mathfrak{F}'(B_2)$.
- (b) Let "$P_v \subseteq \mathfrak{F}(A) \Rightarrow A_0 \cap A \nsubseteq B_1$ for all $A \in \mathcal{S}$". Then $\mathfrak{F}'(B_1) = \mathfrak{F}(B_1) \subseteq \mathfrak{F}(B_2) \subseteq \mathfrak{F}'(B_2)$.

(GF3) Let $B_1, B_2 \in \mathcal{S}$. If $B_1 = \varnothing$ or $B_2 = \varnothing$ then we have $\mathfrak{F}'(B_1) \wedge \mathfrak{F}'(B_2) = \varnothing \subseteq \mathfrak{F}'(B_1 \cap B_2)$. If $B_1, B_2 \neq \varnothing$ then

- (a) Let "$\exists A_1 \in \mathcal{S} \,:\, P_v \subseteq \mathfrak{F}(A_1)$, $A_0 \cap A_1 \subseteq B_1$" and "$\exists A_2 \in \mathcal{S} \,:\, P_v \subseteq \mathfrak{F}(A_2)$, $A_0 \cap A_2 \subseteq B_2$". Then $A_1 \cap A_2 \in \mathcal{S}$, $A_0 \cap (A_1 \cap A_2) \subseteq B_1 \cap B_2$ and so, $\mathfrak{F}'(B_1) \wedge \mathfrak{F}'(B_2) = (P_v \cup \mathfrak{F}(B_1)) \cap (P_v \cup \mathfrak{F}(B_2)) = P_v \cup (\mathfrak{F}(B_1) \cap \mathfrak{F}(B_2)) \subseteq P_v \cup \mathfrak{F}(B_1 \cap B_2) = \mathfrak{F}'(B_1 \cap B_2)$.
- (b) Let "$P_v \subseteq \mathfrak{F}(A) \Rightarrow A_0 \cap A \nsubseteq B_1$" and "$P_v \subseteq \mathfrak{F}(A) \Rightarrow A_0 \cap A \nsubseteq B_2$". Then $\mathfrak{F}'(B_1) \wedge \mathfrak{F}'(B_2) = \mathfrak{F}(B_1) \cap \mathfrak{F}(B_2) \subseteq \mathfrak{F}(B_1 \cap B_2) \subseteq \mathfrak{F}'(B_1 \cap B_2)$.
- (c) Without loss of generality let "$\exists A_1 \in \mathcal{S} \,:\, P_v \subseteq \mathfrak{F}(A_1)$, $A_0 \cap A_1 \subseteq B_1$" and "$P_v \subseteq \mathfrak{F}(A) \Rightarrow A_0 \cap A \nsubseteq B_2$". Since $P_v \subseteq \mathfrak{F}(B_2)$ implies the contradiction $A_0 \cap B_2 \nsubseteq B_2$ we have $P_v \nsubseteq \mathfrak{F}(B_2)$. Then $P_v \cap \mathfrak{F}(B_2) = \varnothing$ because $\mathcal{V}$ is discrete. Moreover, since "$P_v \subseteq \mathfrak{F}(A) \Rightarrow A_0 \cap A \nsubseteq B_2$ we have "$P_v \subseteq \mathfrak{F}(A) \Rightarrow A_0 \cap A \nsubseteq B_1 \cap B_2$ and so $\mathfrak{F}'(B_1 \cap B_2) = \mathfrak{F}(B_1 \cap B_2)$. Thus we get

$$\begin{aligned} \mathfrak{F}'(B_1) \wedge \mathfrak{F}'(B_2) = (P_v \cup \mathfrak{F}(B_1)) \cap \mathfrak{F}(B_2) &= (P_v \cap \mathfrak{F}(B_2)) \cup (\mathfrak{F}(B_1) \cap \mathfrak{F}(B_2)) \\ &= \mathfrak{F}(B_1) \cap \mathfrak{F}(B_2) \subseteq \mathfrak{F}(B_1 \cap B_2) = \mathfrak{F}'(B_1 \cap B_2). \end{aligned}$$

Let $B \in \mathcal{S}$. If $B = \varnothing$ then $\mathfrak{F}'(B) \cap \mathfrak{G}(B) = \varnothing$. So, assume that $B \neq \varnothing$. Then since $(\mathfrak{F}, \mathfrak{G})$ is regular;

- (a) If "$\exists A \in \mathcal{S} \,:\, P_v \subseteq \mathfrak{F}(A)$ and $A_0 \cap A \subseteq B$" then, because of the implication of case 2.1. we have $P_v \nsubseteq \mathfrak{G}(B)$. Since $\mathcal{V}$ is discrete we get $P_v \cap \mathfrak{G}(B) = \varnothing$. Therefore we get $\mathfrak{F}'(B) \cap \mathfrak{G}(B) = (P_v \cup \mathfrak{F}(B)) \cap \mathfrak{G}(B) = (P_v \cap \mathfrak{G}(B)) \cup (\mathfrak{F}(B) \cap \mathfrak{G}(B)) = \varnothing \cup (\mathfrak{F}(B) \cap \mathfrak{G}(B)) = \mathfrak{F}(B) \cap \mathfrak{G}(B) = \varnothing$. That is $\mathfrak{F}'(B) \cap \mathfrak{G}(B) = \varnothing$.
- (b) If "$P_v \subseteq \mathfrak{F}(A) \Rightarrow A_0 \cap A \nsubseteq B$" then we get $\mathfrak{F}'(B) \cap \mathfrak{G}(B) = \mathfrak{F}(B) \cap \mathfrak{G}(B) = \varnothing$.

Therefore $(\mathfrak{F}', \mathfrak{G})$ is a regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$. However, since $P_v \subseteq \mathfrak{F}(C)$ and $A_0 \cap C \subseteq A_0$ we have $\mathfrak{F}'(A_0) = P_v \cup \mathfrak{F}(A_0)$ and since $P_v \nsubseteq \mathfrak{F}(A_0)$ we get $\mathfrak{F}' \neq \mathfrak{F}$. Thus $(\mathfrak{F}, \mathfrak{G}) \subsetneqq (\mathfrak{F}', \mathfrak{G})$ and this contradicts with the maximality of $(\mathfrak{F}, \mathfrak{G})$. Hence, the implication $(1) \Rightarrow (2)$ is satisfied.

Case 2.2:"$\forall A, B \in \mathcal{S} \ : \ P_v \subseteq \mathfrak{F}(A) \cap \mathfrak{G}(B) \Rightarrow A \nsubseteq A_0 \cup B$". If we define a mapping $\mathfrak{G}' : \mathcal{S} \to \mathcal{V}$ by

$$\mathfrak{G}'(A) = \begin{cases} \varnothing, & A = S \\ \mathfrak{G}(A), & A \neq S \text{ and } "P_v \subseteq \mathfrak{G}(B) \text{ for all } B \in \mathcal{S} \Rightarrow A \nsubseteq A_0 \cup B" \\ \mathfrak{G}(A) \cup P_v, & A \neq S \text{ and } "\exists B \in \mathcal{S} \ : \ P_v \subseteq \mathfrak{G}(B) \text{ and } A \subseteq A_0 \cup B" \end{cases}$$

then $(\mathfrak{F}, \mathfrak{G}')$ is a regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$ (it can be shown like in case 2.1). However, $(\mathfrak{F}, \mathfrak{G}) \subsetneqq (\mathfrak{F}, \mathfrak{G}')$ and this contradicts with the maximality of $(\mathfrak{F}, \mathfrak{G})$. Hence, the implication $(1) \Rightarrow (2)$ is satisfied. □

Corollary 3.20. *If $(\mathfrak{F}, \mathfrak{G})$ is a maximal regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$ and $(V, \mathcal{V})$ is discrete then $\mathfrak{F}(S) = \mathfrak{G}(\varnothing) = V$.*

Proof. Considering theorem 3.19., since $(\mathfrak{F}, \mathfrak{G})$ is maximal regular we get $\mathfrak{F} \vee \mathfrak{G} = V$. Suppose that $\mathfrak{F}(S) \neq V$ or $\mathfrak{G}(\varnothing) \neq V$. Then we have $\mathfrak{F}(S) \cup \mathfrak{G}(S) \neq V$ or $\mathfrak{F}(\varnothing) \cup \mathfrak{G}(\varnothing) \neq V$ which contradicts with $\mathfrak{F} \vee \mathfrak{G} = V$. □

It is obtained in [11] that if $\mathcal{F}$ is a prime filter on a texture $(S, \mathcal{S})$ then $\mathcal{S} \setminus \mathcal{F}$ is a prime cofilter and $\mathcal{F} \times (\mathcal{S} \setminus \mathcal{F})$ is a maximal regular difilter on the same texture. However the generalization of this statement is not true for graded difilters. In example 3.14., $\mathfrak{F}$ is a prime $(V, \mathcal{V})$-graded filter on $(S, \mathcal{S})$ and the texture $(V, \mathcal{V})$ is discrete but $V \setminus \mathfrak{F}$ isn't a prime $(V, \mathcal{V})$-graded cofilter on $(S, \mathcal{S})$ since $\mathfrak{F}(S) \neq V$ and $(V \setminus \mathfrak{F})(S) \neq \varnothing$. Since $(V, \mathcal{V})$ is discrete, by corollary 3.20. we get that there is no maximal regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$ whose first component is $\mathfrak{F}$.

Proposition 3.21. *Let $(\mathfrak{F}, \mathfrak{G})$ be a regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$. Then there exists a maximal regular $(V, \mathcal{V})$-graded difilter $(\mathfrak{F}^M, \mathfrak{G}^M)$ on $(S, \mathcal{S})$ such that $(\mathfrak{F}, \mathfrak{G}) \subseteq (\mathfrak{F}^M, \mathfrak{G}^M)$.*

Proof. Let $(\mathfrak{F}_j, \mathfrak{G}_j)_{j \in J}$ be a chain of regular $(V, \mathcal{V})$-graded difilters on $(S, \mathcal{S})$ which satisfies $(\mathfrak{F}, \mathfrak{G}) \subseteq (\mathfrak{F}_j, \mathfrak{G}_j)$ for all $j \in J$. If the mappings $\mathfrak{F}', \mathfrak{G}' : \mathcal{S} \to \mathcal{V}$ are defined by $\mathfrak{F}' = \bigvee_{j \in J} \mathfrak{F}_j$ and $\mathfrak{G}' = \bigvee_{j \in J} \mathfrak{G}_j$ then $(\mathfrak{F}', \mathfrak{G}')$ is a regular $(V, \mathcal{V})$-graded difilter on $(S, \mathcal{S})$:
(GF1) $\mathfrak{F}'(\varnothing) = \bigvee_{j \in J} \mathfrak{F}_j(\varnothing) = \varnothing$
(GF2) Let $A_1, A_2 \in \mathcal{S}$ and $A_1 \subseteq A_2$. Since $\mathfrak{F}_j(A_1) \subseteq \mathfrak{F}_j(A_2)$ for each $j \in J$ we get $\mathfrak{F}'(A_1) = \bigvee_{j \in J} \mathfrak{F}_j(A_1) \subseteq \bigvee_{j \in J} \mathfrak{F}_j(A_2) = \mathfrak{F}'(A_2)$.
(GF3) Let $A_1, A_2 \in \mathcal{S}$. Consider the sets $J_{i^1} = \{j \in J \mid (\mathfrak{F}_i, \mathfrak{G}_i) \subseteq (\mathfrak{F}_j, \mathfrak{G}_j)\}$ and $J_{i^2} = \{j \in J \mid (\mathfrak{F}_i, \mathfrak{G}_i) \supsetneqq (\mathfrak{F}_j, \mathfrak{G}_j)\}$ for each $i \in J$. Since $(\mathfrak{F}_j, \mathfrak{G}_j)_{j \in J}$ is a

chain we have $J_{i^1} \cup J_{i^2} = J$ for each $i \in J$ and so we get

$$\begin{aligned}
\mathfrak{F}'(A_1) \wedge \mathfrak{F}'(A_2) &= \bigvee_{i\in J} \mathfrak{F}_i(A_1) \wedge \bigvee_{j\in J} \mathfrak{F}_j(A_2) = \bigvee_{i\in J}(\mathfrak{F}_i(A_1) \wedge \bigvee_{j\in J} \mathfrak{F}_j(A_2)) \\
&= \bigvee_{i\in J}(\bigvee_{j\in J}(\mathfrak{F}_i(A_1) \wedge \mathfrak{F}_j(A_2))) \\
&= \bigvee_{i\in J}(\bigvee_{j\in J_{i^1}}(\mathfrak{F}_i(A_1) \wedge \mathfrak{F}_j(A_2)) \vee \bigvee_{j\in J_{i^2}}(\mathfrak{F}_i(A_1) \wedge \mathfrak{F}_j(A_2))) \\
&\subseteq \bigvee_{i\in J}(\bigvee_{j\in J_{i^1}}(\mathfrak{F}_j(A_1) \wedge \mathfrak{F}_j(A_2)) \vee \bigvee_{j\in J_{i^2}}(\mathfrak{F}_i(A_1) \wedge \mathfrak{F}_i(A_2))) \\
&\subseteq \bigvee_{i\in J}(\bigvee_{j\in J_{i^1}}(\mathfrak{F}_j(A_1 \cap A_2)) \vee \bigvee_{j\in J_{i^2}}(\mathfrak{F}_i(A_1 \cap A_2))) \\
&= \bigvee_{i\in J} \mathfrak{F}_i(A_1 \cap A_2) = \mathfrak{F}'(A_1 \cap A_2).
\end{aligned}$$

Thus $\mathfrak{F}'$ is a $(V, \mathcal{V})$-graded filter on $(S, \mathcal{S})$ and similarly it can be shown that $\mathfrak{G}'$ is a $(V, \mathcal{V})$-graded cofilter on $(S, \mathcal{S})$.

Now, we use the similar method as in (GF3) to show that $(\mathfrak{F}', \mathfrak{G}')$ is regular. So, consider the sets J_{i^1}, J_{i^2} for each $i \in J$ as above. Let $A \in \mathcal{S}$. Since $(\mathfrak{F}_j, \mathfrak{G}_j)$ is regular for each $j \in J$ we obtain that

$$\begin{aligned}
\mathfrak{F}'(A) \cap \mathfrak{G}'(A) &= \bigvee_{i\in J} \mathfrak{F}_i(A) \cap \bigvee_{j\in J} \mathfrak{G}_j(A) = \bigvee_{i\in J}(\mathfrak{F}_i(A) \cap \bigvee_{j\in J} \mathfrak{G}_j(A)) \\
&= \bigvee_{i\in J}(\bigvee_{j\in J}(\mathfrak{F}_i(A) \cap \mathfrak{G}_j(A))) \\
&= \bigvee_{i\in J}(\bigvee_{j\in J_{i^1}}(\mathfrak{F}_i(A) \cap \mathfrak{G}_j(A)) \vee \bigvee_{j\in J_{i^2}}(\mathfrak{F}_i(A) \cap \mathfrak{G}_j(A))) \\
&\subseteq \bigvee_{i\in J}(\bigvee_{j\in J_{i^1}}(\mathfrak{F}_j(A) \cap \mathfrak{G}_j(A)) \vee \bigvee_{j\in J_{i^2}}(\mathfrak{F}_i(A) \wedge \mathfrak{G}_i(A))) = \varnothing.
\end{aligned}$$

Therefore $(\mathfrak{F}', \mathfrak{G}')$ is an upper bound for the chain $(\mathfrak{F}_j, \mathfrak{G}_j)_{j\in J}$ in the set $\mathfrak{Z} = \{(\mathfrak{U}, \mathfrak{R}) \mid (\mathfrak{U}, \mathfrak{R})\ \textit{is a regular}\ (V, \mathcal{V})-\textit{graded difilter on}\ (S, \mathcal{S})\ \textit{and}\ (\mathfrak{F}, \mathfrak{G}) \subseteq (\mathfrak{U}, \mathfrak{R})\}$. By Zorn's Lemma the set $\mathfrak{Z}$ has a maximal element $(\mathfrak{F}^M, \mathfrak{G}^M)$. Hence $(\mathfrak{F}, \mathfrak{G}) \subseteq (\mathfrak{F}^M, \mathfrak{G}^M)$ and we obtain that $(\mathfrak{F}^M, \mathfrak{G}^M)$ is maximal in the set of all regular $(V, \mathcal{V})$-graded difilters on $(S, \mathcal{S})$. □

Proposition 3.22. *Let $(V, \mathcal{V})$ be a discrete texture space and $(\mathfrak{F}, \mathfrak{G})$ a maximal regular graded difilter on $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$. Then $(\mathfrak{F}, \mathfrak{G})$ is diconvergent if and only if it is diclustering.*

Proof. If $(\mathfrak{F}, \mathfrak{G})$ is diconvergent then it is diclustering by Corollary 3.12. On the other hand, let $(\mathfrak{F}, \mathfrak{G})$ be diclustering. Namely let s be cluster point of $\mathfrak{F}$, s' cluster point of $\mathfrak{G}$ where $P_s \not\subseteq Q_{s'}$. Then $v \in \mathfrak{F}(A) \Rightarrow P_s \subseteq [A]^v$ and $v \in \mathfrak{G}(A) \Rightarrow]A[^v \subseteq Q_{s'}$ for all $A \in \mathcal{S}$.

Let $A \in \mathcal{S}$ with $A \not\subseteq Q_{s'}$ and $v \notin \mathfrak{F}(A)$. By Theorem 3.19. we have $\mathfrak{F}(A) \cup \mathfrak{G}(A) = V$. So, considering regularity of $(\mathfrak{F}, \mathfrak{G})$ we get $v \in \mathfrak{G}(A)$ and so $]A[^v \subseteq Q_{s'}$. Hence $v \notin \mathcal{T}(A)$ because $v \in \mathcal{T}(A)$ implies $A =]A[^v \subseteq Q_{s'}$ which contradicts with $A \not\subseteq Q_{s'}$. Thus we have $v \notin \mathfrak{F}(A) \Rightarrow v \notin \mathcal{T}(A)$, i.e. $\mathcal{T}(A) \subseteq \mathfrak{F}(A)$. Therefore, considering Proposition 3.7. we get $\mathfrak{F} \to s'$. Using similar method, it can be obtained that $\mathfrak{G} \to s$. So $(\mathfrak{F}, \mathfrak{G})$ is diconvergent. □

Proposition 3.23. *Let $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ be a graded ditopological texture space. Then, for the staements*

(a) *Every regular graded difilter on $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ is diclustering.*
(b) *Every maximal regular graded difilter on $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ is diconvergent.*

the implication $(b) \Rightarrow (a)$ and in case of $(V, \mathcal{V})$ is discrete, $(a) \Rightarrow (b)$ are hold.

Proof. $(a) \Rightarrow (b)$: Let $(\mathfrak{F}, \mathfrak{G})$ be a maximal regular graded difilter on $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$. From (a), $(\mathfrak{F}, \mathfrak{G})$ is diclustering. Considering that $(V, \mathcal{V})$ is discrete and Proposition 3.22. $(\mathfrak{F}, \mathfrak{G})$ is diconvergent.
$(b) \Rightarrow (a)$: Let $(\mathfrak{F}, \mathfrak{G})$ be a regular graded difilter on $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$. Considering Proposition 3.21., there exists a maximal regular graded difilter $(\mathfrak{F}^M, \mathfrak{G}^M)$ on $(S, \mathcal{S}, \mathcal{T}, \mathcal{K}, V, \mathcal{V})$ with $(\mathfrak{F}, \mathfrak{G}) \subseteq (\mathfrak{F}^M, \mathfrak{G}^M)$. From (b) we have $\mathfrak{F}^M \to s$, $\mathfrak{G}^M \to s'$ and $P_{s'} \not\subseteq Q_s$ for some $s, s' \in S$. Considering $\mathfrak{G}^M \to s'$ and Proposition 3.10. we have $v \in \mathfrak{F}(A) \Rightarrow v \in \mathfrak{F}^M(A) \Rightarrow P_{s'} \subseteq [A]^v$ and so we get that s' is a cluster point of $\mathfrak{F}$. Similarly it can be obtained that s is a cluster point of $\mathfrak{G}$. Thus $(\mathfrak{F}, \mathfrak{G})$ is diclustering. □

4. Conclusion

Filters and their convergence are convenient tools for topological spaces as filter convergence can describe some of topological concepts. In this work, graded difilters are introduced and their convergence which characterizes interior, closure of sets, etc. is investigated. This new sturucture is helpful to deal with the theory of graded ditopological texture spaces. Moreover, the relations between difilters and graded difilters are studied.

As expected, graded difilters are based on graded dinhd systems. Graded dinhd systems are not graded difilters in general so, the method used in [11] is used to define the convergence of graded difilters. Obviously, graded difilters are more general than difilters and naturally some properties of difilters are not valid for graded difilters in general (see Prop. 3.17. and Theorem 3.19.).

Clearly, graded difilters and their convergence can be useful to define and investigate the concept of compactness on graded ditopological texture spaces.

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References

[1] L. M. Brown and M. Diker, Ditopological texture spaces and intuitionistic sets, Fuzzy Sets and Systems 98 (1998), 217–224.

[2] L. M. Brown and R. Ertürk, Fuzzy sets as texture spaces, I. Representation theorems, Fuzzy Sets and Systems 110, no. 2 (2000), 227–236.

[3] L. M. Brown, R. Ertürk and Ş. Dost, Ditopological texture spaces and fuzzy topology, I. Basic concepts, Fuzzy Sets and Systems147, no. 2 (2004), 171–199.

[4] L. M. Brown, R. Ertürk and Ş. Dost, Ditopological texture spaces and fuzzy topology, II. Topological considerations, Fuzzy Sets and Systems 147, no. 2 (2004), 201–231.

[5] L. M. Brown, R. Ertürk and Ş. Dost, Ditopological texture spaces and fuzzy topology, III. Separation Axioms, Fuzzy Sets and Systems 157, no. 14 (2006), 1886–1912.

[6] L. M. Brown and A. Šostak, Categories of fuzzy topology in the context of graded ditopologies on textures, Iranian Journal of Fuzzy Systems 11, no. 6 (2014), 1–20.

[7] C. L. Chang, Fuzzy topological spaces, J. Math. Anal. Appl. 24 (1968), 182–190.

[8] R. Ekmekçi and R. Ertürk, Neighborhood structures of graded ditopological texture spaces, Filomat 29, no. 7 (2015), 1445–1459.

[9] R. Ertürk, Separation axioms in fuzzy topology characterized by bitopologies, Fuzzy Sets and Systems 58 (1993), 206–209.

[10] T. Kubiak, On fuzzy topologies, Ph.D. Thesis, A. Mickiewicz University Poznan, Poland, 1985.

[11] S. Özçağ, F. Yıldız and L. M. Brown, Convergence of regular difilters and the completeness of di-uniformities, Hacettepe Journal of Mathematics and Statistics 34S (2005), 53–68.

[12] A. Šostak, On a fuzzy topological structure, Rend. Circ. Matem. Palermo, Ser. II, 11 (1985), 89–103.

[13] A. Šostak, Two decades of fuzzy topology: basic ideas, notions and results, Russian Mathematical Surveys 44, no. 6 (1989) 125–186.